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Byeng D. Youn a; Kyung K. Choi a; Kiyoung Yi a
a Center for Computer-Aided Design and Department of Mechanical & Industrial Engineering, College of Engineering, The University of Iowa, Iowa City, Iowa, USA

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Performance Moment Integration (PMI) Method for Quality Assessment in Reliability-Based Robust Design Optimization

Byeng D. Youn, Kyung K. Choi, and Kiyounge Yi

Center for Computer-Aided Design and Department of Mechanical & Industrial Engineering, College of Engineering, The University of Iowa, Iowa City, Iowa, USA

Abstract: The reliability-based robust design optimization deals with two objectives of structural design methodologies subject to various uncertainties: reliability and robustness. The reliability constraints deal with the probability of failures, while the robustness minimizes the product quality loss. In general, the product quality loss is described by using the first two statistical moments: mean and standard deviation. In this paper, a performance moment integration (PMI) method is proposed by using a three-level numerical integration on the output range to estimate the product quality loss. For the reliability part of the reliability-based robust design optimization, the enriched performance measure approach (PMA+) and its numerical method, enhanced hybrid-mean value (HMV+) method, are used. New formulations of reliability-based robust design optimization are presented for three different types of robustness objectives, such as smaller-the-better, larger-the-better, and nominal-the-best types. Examples that include an engine rubber gasket are used to demonstrate the effectiveness of reliability-based robust design optimization using the proposed PMI method for different types of robust objective.
Keywords: Performance moment integration (PMI); Reliability-based robust design; Robustness.

1. INTRODUCTION

Due to the competitive market, manufacturers are trying to improve the reliability and quality of their product designs. Incorporation of various uncertainties in developing a product design has become increasingly important to produce reliable and robust product design. Reliability-based robust design optimization methodology addresses two objectives of product design subject to various uncertainties: reliability (Enevoldsen and Sorensen, 1994; Tu and Choi, 1999; Youn et al., 2003; Youn and Choi, 2004a,b; Yu et al., 1998) and robustness (Chen et al., 2000; Gupta and Li, 2000; Kalsi et al., 2001; Parkinson, 1997; Renaud and Su, 1997). A reliability-based design optimization (RBDO) achieves the target confidence of product reliability, while the robust design optimization minimizes the quality loss of the product. The product quality is commonly defined using the quality loss function (Chandra, 2001; Taguchi et al., 1989), and the confidence of product performances is addressed by assessing their reliability (or a probability of failure) (Liu and Kiureghian, 1991; Youn et al., 2003; Youn and Choi, 2004a; Wu and Wirsching, 1987).

In RBDO, recent advances have been made with a performance measure approach (PMA) (Tu and Choi, 1999; Youn et al., 2003; Youn and Choi, 2004a,b) and its numerical method, hybrid-mean value (HMV) method (Youn et al., 2003; Youn and Choi, 2004a,b). A new response surface methodology was proposed for RBDO to extend its applicability to the area that does not have analytical design sensitivity (Youn and Choi, 2004b). Even with the success of PMA, there were great demands to improve numerical efficiency in RBDO methodologies. Several different RBDO methods were recently proposed: single-loop RBDO (Chen et al., 1997; Wang and Kodiylam, 2003), sequential optimization and reliability analysis (Du and Chen, 2002), safety factor approach (Wu et al., 2001), and enriched PMA (PMA+) (Youn and Choi, 2005).

Various methods have been developed to estimate the quality loss of a product for robust design, such as the worst-case method (arithmetic sum) (Bennett and Gupta, 1969; Forouraghi, 2000), root sum square (RSS) using a Taylor series (Jung and Lee, 2002; McAllister and Simpson, 2003; Renaud and Su, 1997), Monte Carlo Simulation (MCS) (Lin et al., 1997; Varghese et al., 1996), experimental design techniques (or Taguchi’s method) (Lee and Park, 2002; Reddy et al., 1997; Taguchi, 1978), a variability function method (Parkinson, 1997), etc. In general, the quality loss of product is described as the first two statistical moments: mean and standard deviation. It has been
reported that these methods have difficulties in estimating the quality loss accurately and efficiently. The worst-case method tends to produce a conservative design (Du and Chen, 2000; Greenwood and Chase, 1987; Liu and Hu, 1998). The moment approximation method using a Taylor series is highly sensitive to the nonlinearity of the structural response, thus yielding inaccurate estimation of statistical moments for nonlinear systems (Du and Chen, 2000; Greenwood and Chase, 1987). In addition to such difficulties, both the worst-case method and RSS using a Taylor series require a second-order sensitivity analysis in the design optimization, which is very expensive and complicated (Chen and Choi, 1996; Belegundu and Zhang, 1992). Experimental design techniques appear to be more appropriate for tolerance design than for parameter design of the product (Taguchi et al., 1989). Moreover, experimental design techniques are not good for robust design optimization with multiple robust responses (Reddy et al., 1997). A variability function method (Parkinson, 1997) and similar ideas (Du et al., 2003) were proposed for the robust design optimization, but they do not provide accurate measure of the product quality loss.

There is no formulation available for reliability-based robust design with different types of robustness, such as smaller-the-better type (S-Type), larger-the-better type (L-Type), and nominal-the-best type (N-Type) (Chandra, 2001; Parkinson, 1997). This paper presents new formulations of reliability-based robust design optimization for these types by proposing a performance moment integration (PMI) method to effectively assess the quality loss of the product and perform reliability-based robust design optimization effectively. The proposed method resolves the burden of a second-order sensitivity required for design optimization in the worst-case method or RSS using a Taylor series for statistical moment calculation.

Several numerical examples are used to show the effectiveness of the proposed PMI method for the reliability-based robust design optimization without requiring a second-order design sensitivity.

2. RELIABILITY-BASED ROBUST DESIGN OPTIMIZATION

2.1. General Formulation of Reliability-Based Robust Design Optimization

In general parameter design, the reliability-based robust design optimization (Enevoldsen and Sorensen, 1994; Tu and Choi, 1999; Yu et al., 1998) can be formulated as

\[
\begin{align*}
\text{minimize} & \quad C(X; \mathbf{d}) \\
\text{subject to} & \quad P(G_i(X; \mathbf{d}) \leq 0) \geq \Phi(\beta_i), & i = 1, \ldots, np \\
& \quad \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U, & \mathbf{d} \in R^{ndv} \quad \text{and} \quad X \in R^{nrv}
\end{align*}
\]
where \( C(\mathbf{X}; \mathbf{d}) \) is the objective function; \( \mathbf{d} = \mu(\mathbf{X}) \) is the design vector; \( \mathbf{X} \) is the random vector; and the probabilistic constraint is described by the performance function \( G_i(\mathbf{X}) \), its probability distributions, and its prescribed reliability target \( \beta_i \), while \( np, ndv, \) and \( nrv \) are the number of probabilistic constraints, design variables, and random variables, respectively.

For robustness of the design, the cost function in Eq. (1) is defined as

\[
C(\mathbf{X}; \mathbf{d}) = C_m(\mathbf{X}; \mathbf{d}) + C_{ql}(\mathbf{H}(\mathbf{X}; \mathbf{d}))
\]

(2)

where \( \mathbf{H}(\mathbf{X}; \mathbf{d}) \) is the robustness that is associated with a product quality, \( C_m(\mathbf{X}; \mathbf{d}) \) is the material cost, and \( C_{ql}(\mathbf{H}(\mathbf{X}; \mathbf{d})) \) is the quality loss cost defined as the loss that the product costs society from the time the product is released for shipment (e.g., rework cost, scrap cost, maintenance cost). The quality loss function can be defined using different types of robustness: nominal-the-best type (N-Type), smaller-the-better type (S-Type), and larger-the-better type (L-Type).

For reliability of the design, the constraint in Eq. (1) can be redefined using PMA as (Youn et al., 2003; Youn and Choi, 2004a; Yu et al., 1998)

\[
\begin{align*}
\text{minimize} & \quad C(\mathbf{X}; \mathbf{d}) = C_m(\mathbf{X}; \mathbf{d}) + C_{ql}(\mathbf{H}(\mathbf{X}; \mathbf{d})) \\
\text{subject to} & \quad G_{pi}(\mathbf{X}; \mathbf{d}) \leq 0, \quad i = 1, \ldots, np \\
& \quad \mathbf{d}^l \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in R^{ndv}, \quad \text{and} \quad \mathbf{X} \in R^{nrv}
\end{align*}
\]

(3)

where \( G_{pi} \) is the \( i \)th probabilistic constraint. Using the first-order reliability method (FORM) in PMA, the first-order probabilistic performance measure \( G_{p,\text{FORM}} \) can be obtained in \( U \)-space from

\[
\begin{align*}
\text{maximize} & \quad G(U) \\
\text{subject to} & \quad ||U||_2 = \beta_i
\end{align*}
\]

(4)

where the optimum point on the target reliability surface is identified as the most probable point (MPP) \( \mathbf{u}_{p=\beta_i}^* \) with a prescribed reliability \( \beta_i = ||\mathbf{u}_{p=\beta_i}^*||_2 \) and \( G_{p,\text{FORM}} = G(\mathbf{u}_{p=\beta_i}^*) \). In this paper, the enhanced hybrid-mean value (HMV+) method (Youn et al., 2003; Youn and Choi, 2004a) is used to perform the inverse reliability analysis in Eq. (4).

2.2. Quality Loss Function, \( C_{ql}(\mathbf{H}(\mathbf{X}; \mathbf{d})) \) (Taguchi et al., 1989)

The quality loss function is defined as the loss that the product costs society from the time the product is released for shipment. The quality loss function developed by Taguchi et al. (1989) is simply the cost of deviating from the target nominal value, which can be approximated in a quadratic form as

\[
C_{ql}(\mathbf{H}(\mathbf{X}; \mathbf{d})) = k'(||\mathbf{H} - \mathbf{h}_i||_2)^2
\]

(5)
where \( k' \) is a proportionality constant and \( h_t \) is the target nominal value of the robust response vector \( H \). Different quality loss functions are defined for different types of robustness to describe different quality characteristics. In this paper, different formulations of reliability-based robust design optimization are developed for different types of robustness requirement. For convenience, the quality loss function is derived for one robust response, \( H \), and the derivation can be easily extended to the robust response vector \( H \) by adding all robustness requirements as shown in Section 2.6.

2.3. Reliability-Based Robust Design for Nominal-the-Best Type (N-Type)

For the N-Type robustness with the target nominal value \( h_t \) of \( H \), the expected value of the loss function in Eq. (5) is defined as (Chandra, 2001; Taguchi et al., 1989)

\[
E[C_{ql}(H)] = k'[\frac{(\mu_H - h_t)^2 + \sigma_H^2}{\mu_H^2}]
\]

where \( \mu_H \) and \( \sigma_H \) are the mean and standard deviation of robust response \( H \) and \( E[\bullet] \) is the statistical expectation of the function. Thus, the product quality can be deteriorated due to a deviation of the mean of robust response from its target, \( (\mu_H - h_t)^2 \), and a deviation of uncertain robust response from its mean, \( \sigma_H^2 \). The reliability-based robust design optimization using PMA is formulated for the N-Type as

\[
\begin{align*}
\text{minimize} & \quad \left( \frac{\mu_H - h_t}{\mu_{H_0} - h_{t_0}} \right)^2 + \left( \frac{\sigma_H}{\sigma_{H_0}} \right)^2 \\
\text{subject to} & \quad G_{\rho_i}(X; d) \leq 0, \quad i = 1, \ldots, np \\
& \quad d^L \leq d \leq d^U, \quad d \in R^{ndv} \quad \text{and} \quad X \in R^{nrv}
\end{align*}
\]

It is recommended that the mean, the target nominal value, and the standard deviation of robust response, \( H \) in the robust objective be respectively normalized by their values at the initial design, \( \mu_{H_0} \) and \( \sigma_{H_0} \), as shown in Eq. (7).

2.4. Reliability-Based Robust Design for Smaller-the-Better Type (S-Type)

The expected quality loss for the S-Type is defined as

\[
E[C_{ql}(H)] = k'E[H^2]
\]

\[
= k'E[(H - \mu_H + \mu_H)^2]
\]

\[
= k'E[(H - \mu_H)^2 + 2\mu_H(H - \mu_H) + \mu_H^2]
\]
\[ k' E[(H - \mu_H)^2] + k' \mu_H^2 \]
\[ = k' \left[ \mu_H^2 + \sigma_H^2 \right] \]  

The reliability-based robust design optimization using PMA is formulated for the S-Type as

\[
\begin{align*}
\text{minimize} & \quad \text{sgn}(\mu_H) \cdot \left( \frac{\mu_H}{\mu_{H0}} \right)^2 + \left( \frac{\sigma_H}{\sigma_{H0}} \right)^2 \\
\text{subject to} & \quad G_{pi}(X; d) \leq 0, \quad i = 1, \ldots, np \\
& \quad d^L \leq d \leq d^U, \quad d \in R^{ndv} \text{ and } X \in R^{nrv}
\end{align*}
\]

(9)

Again, the robust objective is normalized, as shown in Eq. (9). For the negative robust response, the signum function of \( \mu_H \) (= 1 or −1) is multiplied to properly minimize the S-Type robust objective.

2.5. Reliability-Based Robust Design for Larger-the-Better Type (L-Type)

For the L-Type robustness, the inverse response, \( Q = \frac{1}{H} \), can be introduced to formulate as an S-Type. Thus, the expected quality loss for the L-Type robustness can be defined as

\[
E\left[C_{ql}\left(\frac{1}{H}\right)\right] = E[C_{ql}(Q)] = k' \left[ \mu_Q^2 + \sigma_Q^2 \right] = k' \left[ \mu_{Q/H}^2 + \sigma_{Q/H}^2 \right]
\]

(10)

The reliability-based robust design optimization using PMA is formulated for the L-Type as

\[
\begin{align*}
\text{minimize} & \quad \text{sgn}(\mu_Q) \cdot \left( \frac{\mu_Q}{\mu_{Q0}} \right)^2 + \left( \frac{\sigma_Q}{\sigma_{Q0}} \right)^2 \\
\text{subject to} & \quad G_{pi}(X; d) \leq 0, \quad i = 1, \ldots, np \\
& \quad d^L \leq d \leq d^U, \quad d \in R^{ndv} \text{ and } X \in R^{nrv}
\end{align*}
\]

(11)

The robust objective for the L-Type is normalized as shown in Eq. (11). For the negative robust response, the signum of \( \mu_Q \) is multiplied to properly maximize the L-Type robust objective.
2.6. Reliability-Based Robust Design for a General Case

For a general case that contains N-, S-, and L-Type robustness, the objective function in Eq. (3) can be formulated as

\[
\begin{align*}
\sum_{i=1}^{nn} w_{ni} \cdot & \left[ \left( \frac{\mu_H - h_t}{\mu_{H0} - h_t} \right)^2_i + \left( \frac{\sigma_H}{\sigma_{H0}} \right)^2_i \right] \\
+ \sum_{j=1}^{ns} w_{sj} \cdot & \left[ \text{sgn}(\mu_H) \cdot \left( \frac{\mu_H}{\mu_{H0}} \right)^2_j + \left( \frac{\sigma_H}{\sigma_{H0}} \right)^2_j \right] \\
+ \sum_{k=1}^{nl} w_{lk} \cdot & \left[ \text{sgn}(\mu_Q) \cdot \left( \frac{\mu_Q}{\mu_{Q0}} \right)^2_k + \left( \frac{\sigma_Q}{\sigma_{Q0}} \right)^2_k \right]
\end{align*}
\]

where \(nn\), \(ns\), and \(nl\) are the number of the N-Type, S-Type, and L-Type robustness. The optimum design depends on the designer’s preference to determine these weights. A Pareto set can be obtained along with diverse designer’s preference (Das and Dennis, 1997). It is also suggested in (Messac, 2000) that the physical programming method yields a well-distributed set of Pareto solutions by building a preference spectrum. However, this paper is not focused on the topic of designer’s preference to determine these weights in the reliability-based robust design optimization.

2.7. Refined Hybrid Reliability Method for Reliability-Based Robust Design

Even with the HMV method (Tu and Choi, 1999; Yu et al., 1998), highly nonlinear probabilistic constraints (e.g., nonmonotonic constraints) could yield numerical inefficiency and/or instability in the inverse reliability analysis. Such numerical difficulty can be resolved by checking whether numerical iterations monotonically increase (or decrease) the objective function. If that is the case, the HMV method is used. Otherwise, a numerical procedure to enforce monotonicity in the iteration of the inverse reliability analysis is integrated to the HMV method. This method is called the enhanced HMV (HMV+) method (Youn et al., 2004a), which is used for evaluating both the robustness and reliability of product responses in reliability-based robust design optimization in this paper. The method is summarized as

Step 1. Set the iteration counter, \(k = 0\), with a target reliability, \(\beta_t\), and the convergence parameter, \(\varepsilon = 10^{-4}\). Let \(u_{HMV+}^{(0)} = 0\).

Step 2. Calculate the performance function, \(g(u_{HMV+}^{(k)})\), and its sensitivity, \(\nabla g(u_{HMV+}^{(k)})\).
Step 3. Check the Karush-Kuhn-Tucker condition: 
\[ \left| \text{sgn}(\beta_i) \cdot \frac{u^{(k)}_{\text{HMV}+}}{\|u^{(k)}_{\text{HMV}+}\|} - 1 \right| \leq \varepsilon \text{ for } k \geq 2, \text{ where } u^{(k)}_{\text{HMV}+} \text{ is the normalized steepest ascent direction of } G(U) \text{ at } u^{(k)}_{\text{HMV}+} ; \text{ and } \text{sgn}(\beta_i) \text{ is the signum function, such that it is } +1 \text{ if } \beta_i > 0 \text{ and } -1 \text{ if } \beta_i < 0. \text{ If it is satisfied, then stop.} \]

Step 4. If \( k \geq 2 \) and \[ \text{sgn}(\beta_i) \cdot \left[ g(u^{(k)}_{\text{HMV}+}) - g(u^{(k-1)}_{\text{HMV}+}) \right] < 0, \]
interpolate the performance function along the arc region between these two search points and obtain a new search point, \( u^{(k+1)}_{\text{HMV}+} \), by maximizing the approximate performance function (Youn et al., 2004a). Otherwise, use the HMV method (Youn et al., 2003) to obtain a new searching point \( u^{(k+1)}_{\text{HMV}+} \). Let \( k = k + 1 \) and go to Step 2.

3. STATISTICAL ESTIMATION OF QUALITY LOSS

3.1. Output Statistical Moment Modeling: Numerical Integration on Input Domain

One purpose of statistical moment estimation stems from the robust design optimization, which requires minimization of the quality loss (Chandra, 2001; Taguchi et al., 1989). It is a function of the statistical mean and standard deviation. Several methods are proposed to estimate the first two statistical moments of the output response. Analytically, the statistical moments are expressed in an integration form as

\[ E[H] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H(x) f_X(x) dx \]
\[ E[(H(x) - \mu_H)^k] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (H(x) - \mu_H)^k f_X(x) dx \]

It is assumed that random variables are statistically independent. Thus, a joint probability density function (PDF) is defined as a multiplication of all probability density functions, such as \( f_X(x) = \prod_{i=1}^{m} f_{X_i}(x_i) \). The statistical moments of output response are obtained through numerical integrations on the input domain as

\[ E[H] \cong \bar{\mu}_H = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H(x) \prod_{i=1}^{m} f_{X_i}(x_i) dx_1, \ldots, dx_n \]
\[ \cong \sum_{j_1=1}^{m} w_{j_1} \cdots \sum_{j_m=1}^{m} w_{j_m} H(\mu_1 + \alpha_1, \ldots, \mu_n + \alpha_n) \]

\[ E[H(x) - \mu_H]^k \cong \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (H(x) - \mu_H)^k \prod_{i=1}^{m} f_{X_i}(x_i) dx_1, \ldots, dx_n \]
\[ \cong \sum_{j_1=1}^{m} w_{j_1} \cdots \sum_{j_m=1}^{m} w_{j_m} [H(\mu_1 + \alpha_1, \ldots, \mu_n + \alpha_n) - \bar{\mu}_H]^k \]
For a practical method, Taguchi et al. (1989), Taguchi (1978) proposed an experimental design approach for statistical tolerance design with a three-level \((m = 3)\) factorial experiment, which is composed of low, center, and high levels as

\[
\{w_1, w_2, w_3, x_1, x_2, x_3\} = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}} \right\}
\]

The three-level factorial experiment is modified by Youn et al. (2004a) by employing distinctive weights at different levels as

\[
\{w_1, w_2, w_3, x_1, x_2, x_3\} = \left\{ \frac{1}{6}, \frac{4}{6}, \frac{1}{6}, -\sqrt{3}, 0, \sqrt{3} \right\}
\]

The modified three-level factorial experiment improved numerical accuracy in estimating the statistical moments of output response. In numerical integration, three weights for \(X_i\) are used to approximate the probability density of \(X_i\) at three different probability levels. From the statistical point of view, the modified three-level factorial experiment is meaningful, since many random input variables follows the pattern of high density near the mean and low density away from the mean, as shown in Fig. 1.

### 3.2. Output Statistical Moment Modeling: Numerical Integration on the Output Range

In Section 3.1, the statistical moments of output response are estimated through numerical integration on the input domain, making it very expensive for reliability-based robust design optimization since the number of function evaluations or experiments required is \(N = 3^n\).
where \( N \) is a number of design and random parameters. In this paper, the proposed method directly identifies uncertainty propagation using numerical integration on the output range. Unlike Eq. (13), the statistical moment calculation is carried out by

\[
E[H]^1 = \int_{-\infty}^{\infty} hf_H(h)dh = \mu_H
\]

\[
E[H - \mu_H]^k = \int_{-\infty}^{\infty} (h - \mu_H)^k f_H(h)dh
\]

where \( f_H(h) \) is a probability density function (PDF) of \( H \). To approximate the statistical moments of \( H \) accurately, \( N \)-point numerical quadrature technique can be used as

\[
E[H]^1 = \mu_H \approx \sum_{i=1}^{N} w_i h_i \quad \text{and}
\]

\[
E[H - \mu_H]^k \approx \sum_{i=1}^{N} w_i (h_i - \mu_H)^k \quad \text{for} \quad 2 \leq k \leq 5
\]

At minimum, a three-point integration (\( N = 3 \)) is required to maintain a good accuracy in estimating first two statistical moments. By solving Eq. (16), three levels and weights on the output range are obtained as \( \{h_1, h_2, h_3\} = \{h_{\beta=\sqrt{3}}, h(\mu_X), h_{\beta=+\sqrt{3}}\} \) and \( \{w_1, w_2, w_3\} = \{1/6, 4/6, 1/6\} \), respectively, as shown in Fig. 2. In general, upper and lower levels are not symmetrically located, as shown in Fig. 3.

Using the three-level numerical integration on the output range, the first two statistical moments in Eq. (15), the mean and standard variation of the output response are approximated to be

\[
E[H]^1 = \mu_H \approx \frac{1}{6} h_{\beta=-\sqrt{3}} + \frac{4}{6} h(\mu_X) + \frac{1}{6} h_{\beta=+\sqrt{3}}
\]

\[
E[H - \mu_H]^2 = \sigma^2_H = \int_{-\infty}^{\infty} (h - \mu_H)^2 f_H(h)dh
\]

\[
\approx \frac{1}{6} (h_{\beta=-\sqrt{3}} - \mu_H)^2 + \frac{1}{6} (h_{\beta=+\sqrt{3}} - \mu_H)^2
\]

**Figure 2.** Three-level numerical integration on the output range.
Since the statistical moments of output response are estimated through a numerical integration on the output (or performance) range, this method is called a performance moment integration (PMI) method. In the PMI method, \( h_{\beta=-\sqrt{3}} \) and \( h_{\beta=+\sqrt{3}} \) are obtained through reliability analyses (Du and Chen, 2002; Liu and Kiureghian, 1991; Youn et al., 2003; Youn and Choi, 2005, 2004a,b) at \( \beta = \pm \sqrt{3} \) confidence levels. In this paper, the HMV+ method is used for the inverse reliability analysis (Youn et al., 2003; Youn and Choi, 2004a,b).

3.3. Numerical Results of Statistical Moment Estimation of Output Response

Using the PMI method, the mean and standard deviation of output response can be approximated as

\[
\mu_H \approx \frac{1}{6} h_{\beta=-\sqrt{3}} + \frac{4}{6} h(\mu_X) + \frac{1}{6} h_{\beta=+\sqrt{3}}
\]

\[
\sigma^2_H \approx \frac{1}{6} (h_{\beta=-\sqrt{3}} - \mu_H)^2 + \frac{1}{6} (h_{\beta=+\sqrt{3}} - \mu_H)^2
\]

(18)

Two nonlinear mathematical examples and one vehicle crashworthiness for side-impact are used to demonstrate the effectiveness of the PMI method in estimating the mean and standard deviation of the output response. The Monte Carlo simulation (MCS) with a million samples and the root sum square (RSS) method are used for numerical comparison. The MCS result is used as the benchmark data. A degree of
statistical nonnormality is shown with statistical skewness and kurtosis. The skewness is a measure of symmetry of PDF, and a normal distribution has a skewness value of 0. On the other hand, the kurtosis is a measure of relative peakness/flatness of PDF, and the normal distribution has a kurtosis value of 3.

For the first example, the mathematical model is

\[ H_1(X) = 1 - X_1^2 X_2 / 20 \]  \hspace{1cm} (19)

For this example, the input random parameters are modeled as \( X_i \sim N(5.0, 0.3) \) for \( i = 1, 2 \). As shown in Table 1 and Fig. 3, the probabilistic distribution of the first response is close to a normal distribution with a moderate amount of skewness (third statistical moment) and kurtosis (fourth statistical moment). Thus, PMI and RSS show a good accuracy overall in estimating the first two statistical moments of responses.

A more difficult problem is to estimate statistical information of other three stochastic responses with high skewness and kurtosis, as shown in Figs. 4 to 6. The second example has analytical expression

\[ H_2(X) = -e^{X_1 - 7} - X_2 + 10 \]  \hspace{1cm} (20)

with a high kurtosis (= 7.135), as depicted in Fig. 4, where the input random parameters are modeled as \( X_i \sim N(6.0, 0.8) \) for \( i = 1, 2 \). As shown in Table 1, the RSS method yields a large approximation error of 107\% for the second moment, whereas the PMI method is accurate for both the mean and standard deviation.

The last two examples employ two stochastic responses with high skewness and kurtosis, as shown in Table 1 and Figs. 5 and 6. The third response \( H_3 \) in Table 1 is generated with \( H_1 \) and \( X_i \sim Gumbel(5.0, 1.0) \), and the fourth response \( H_4 \) is the pubic force from a side impact event (D’Errico and Zaino, 1988), which is modeled with input uncertainties of

---

**Table 1. Numerical accuracy of standard deviation**

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RSS</td>
<td>PMI</td>
</tr>
<tr>
<td>( H_1 )</td>
<td>-5.2500</td>
<td>-5.2856</td>
</tr>
<tr>
<td>Error, %</td>
<td>0.415</td>
<td>0.259</td>
</tr>
<tr>
<td>( H_2 )</td>
<td>3.6321</td>
<td>3.6082</td>
</tr>
<tr>
<td>Error, %</td>
<td>3.961</td>
<td>3.277</td>
</tr>
<tr>
<td>( H_3 )</td>
<td>-5.2500</td>
<td>-5.7157</td>
</tr>
<tr>
<td>Error, %</td>
<td>4.559</td>
<td>3.907</td>
</tr>
<tr>
<td>( H_4 )</td>
<td>-1.4100</td>
<td>-1.4135</td>
</tr>
<tr>
<td>Error, %</td>
<td>1.337</td>
<td>1.092</td>
</tr>
</tbody>
</table>
Gumbel distribution and a 10% coefficient of variation. Even though the stochastic response is highly skewed with large kurtosis, the PMI method seems to predict the first two statistical moments accurately, whereas the RSS could yield larger errors, as shown in Table 1.

These examples show that the PMI method maintains a good accuracy in estimating the first two statistical moments of output responses, compared to the RSS method. In addition, PMI does not require a second-order sensitivity for design optimization. Therefore, the...
PMI method can be integrated with a response surface method (or a surrogate model) for design optimization. On the other hand, if the RSS method is used, the response surface method can not be used for design optimization because a second-order sensitivity of an approximate response surface may contain large errors.

4. NUMERICAL EXAMPLES FOR RELIABILITY-BASED ROBUST DESIGN OPTIMIZATION

In this section, three different types of robustness requirements are considered to evaluate new formulations of reliability-based robust design optimization using the proposed PMI method. Also, investigated is whether or not the deterministic design optimization can improve numerical efficiency of reliability-based robust design optimization.

4.1. Reliability-Based Robust Design Optimization with N-Type Robustness

For the N-Type robustness, a reliability-based robust design optimization is formulated as

$$\text{minimize} \quad \left( \frac{\mu_H - h_t}{\mu_{H_0} - h_t} \right)^2 + \left( \frac{\sigma_H}{\sigma_{H_0}} \right)^2$$

subject to \( P(G_i(X; d) \leq 0) \geq \Phi(\beta_i), \quad i = 1, \ldots, np \)
Table 2. Results of reliability-based robust design optimization with N-type

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$\mu_H - h_i$</th>
<th>Std. dev.</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$G_{p1}$</th>
<th>$G_{p2}$</th>
<th>$G_{p3}$</th>
<th>NFE</th>
<th>NRFE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.032</td>
<td>4.696</td>
<td>2.000</td>
<td>8.000</td>
<td>0.518</td>
<td>-1.997</td>
<td>0.015</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>0.153</td>
<td>3.219</td>
<td>2.842</td>
<td>4.446</td>
<td>0.181</td>
<td>-0.393</td>
<td>-0.384</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>0.172</td>
<td>2.389</td>
<td>4.068</td>
<td>3.589</td>
<td>-0.696</td>
<td>-0.039</td>
<td>-0.312</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>0.150</td>
<td>1.569</td>
<td>5.501</td>
<td>3.699</td>
<td>-2.610</td>
<td>-0.079</td>
<td>-0.030</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>0.163</td>
<td>1.324</td>
<td>5.853</td>
<td>3.433</td>
<td>-2.805</td>
<td>-0.004</td>
<td>0.000</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>0.163</td>
<td>1.325</td>
<td>5.847</td>
<td>3.434</td>
<td>-2.797</td>
<td>-0.003</td>
<td>0.000</td>
<td>24</td>
<td>21</td>
</tr>
<tr>
<td>Opt</td>
<td>0.163</td>
<td>1.325</td>
<td>5.847</td>
<td>3.434</td>
<td>-2.797</td>
<td>-0.003</td>
<td>0.000</td>
<td>98</td>
<td>82</td>
</tr>
</tbody>
</table>

MC simulation w/100K samples at the optimum: $\mu_H - h_i = 0.166$ and standard deviation = 1.327.

and $H(X) = (X_1 - 8)^2 - (X_2 - 3)^2$

$G_1(X) = 1 - X_1^2X_2/20$

$G_2(X) = 1 - (X_1 + X_2 - 5)^2/30 - (X_1 - X_2 - 12)^2/120$

$G_3(X) = 1 - 80/(X_1^2 + 8X_2 + 5)$

where $X_i \sim N(\mu_i, \sigma = 0.3)$ and $\beta_i = 3.0$ for $i = 1, 2$; $\mu_1 = 2.0$ and $\mu_2 = 8.0$. The nominal value $h_i$ of $H$ is obtained by $h_i = (h_1 + h_3)/2 = (h_{\beta_i = -\sqrt{3}} + h_{\beta_i = +\sqrt{3}})/2$.

As shown in Table 2 and Fig. 7, the N-Type robust objective in Eq. (21) is minimized using the PMI method, while all probabilistic constraints are satisfied using PMA. The reliability-based robust design optimization is successfully carried out using the PMI method and without requiring a second-order design sensitivity. In Table 2, NFE and NRFE refer to numbers of function evaluations for reliability and robustness analyses parts, respectively. At the optimum design, the robust objective obtained using the PMI method is compared to the one obtained using MCS, showing that the PMI method estimates the statistical information accurately.

4.2. Reliability-Based Robust Design Optimization with S-Type Robustness

The same example is used with the S-Type robustness and its reliability-based robust design optimization is formulated as

$$\text{minimize} \quad \text{sgn}(\mu_H) \cdot \left( \frac{\mu_H}{\mu_H^0} \right)^2 + \left( \frac{\sigma_H}{\sigma_H^0} \right)^2$$
subject to \[ P(G_i(X; d) \leq 0) \geq \Phi(\beta_i), \quad i = 1, \ldots, np \]

where \[ H(X) = (X_1 - 8)^2 - (X_2 - 3)^2, \quad w_1 = w_2 = 1.0 \quad (22) \]

\[ G_1(X) = 1 - X_1^2X_2/20 \]

\[ G_2(X) = 1 - (X_1 + X_2 - 5)^2/30 - (X_1 - X_2 - 12)^2/120 \]

\[ G_3(X) = 1 - 80/(X_1^2 + 8X_2 + 5) \]

where \( X_i \sim N(\mu^0_i, \sigma = 0.3) \) and \( \beta_i = 3.0 \) for \( i = 1, 2; \mu^0_1 = 2.0 \) and \( \mu^0_2 = 8.0. \)

The optimum design for the S-Type is the same as the one for the N-Type, since both N- and S-Type objectives are minimized at their optimum design. However, as depicted in Fig. 8, the design optimization path is shown to be rather different, because different sensitivities of N-
and S-Type objectives make two different paths for design optimization. As shown in Table 3, the PMI method minimizes the S-Type robust objective (mean and standard deviation) in Eq. (22), while all probabilistic constraints become feasible and active using the PMA. Compared to the MCS, the PMI method predicts statistical moments accurately.

4.3. Reliability-Based Robust Design Optimization with L-Type Robustness

The same example is used with the L-Type robustness and its reliability-based robust design optimization is formulated as

\[
\text{minimize} \quad \text{sgn}\left(\frac{\mu_{1/H}}{\mu_{1/H_0}}\right) \cdot \left(\frac{\mu_{1/H}}{\mu_{1/H_0}}\right) + \left(\frac{\sigma_{1/H}}{\sigma_{1/H_0}}\right)^2
\]
Table 3. Results of reliability-based robust design optimization with S-type

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Std. dev.</th>
<th>d₁</th>
<th>d₂</th>
<th>Gₚ₁</th>
<th>Gₚ₂</th>
<th>Gₚ₃</th>
<th>NFE</th>
<th>NRFE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11.02</td>
<td>4.696</td>
<td>2.000</td>
<td>8.000</td>
<td>0.518</td>
<td>−1.997</td>
<td>0.015</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>14.54</td>
<td>2.957</td>
<td>4.429</td>
<td>6.362</td>
<td>−2.882</td>
<td>−1.204</td>
<td>0.078</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>6.017</td>
<td>2.102</td>
<td>4.996</td>
<td>4.746</td>
<td>−2.815</td>
<td>−0.468</td>
<td>0.000</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>5.343</td>
<td>1.673</td>
<td>5.460</td>
<td>4.082</td>
<td>−2.975</td>
<td>−0.224</td>
<td>0.002</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>4.579</td>
<td>1.322</td>
<td>5.830</td>
<td>3.457</td>
<td>−2.802</td>
<td>−0.011</td>
<td>−0.001</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>4.505</td>
<td>1.322</td>
<td>5.855</td>
<td>3.423</td>
<td>−2.793</td>
<td>0.000</td>
<td>0.000</td>
<td>16</td>
</tr>
<tr>
<td>Opt</td>
<td>4.505</td>
<td>1.322</td>
<td>5.855</td>
<td>3.423</td>
<td>−2.793</td>
<td>0.000</td>
<td>0.000</td>
<td>94</td>
</tr>
</tbody>
</table>

MC simulation w/100K samples at the optimum: mean = 4.443 and standard deviation = 1.325

subject to \( P(G_i(X; d) \leq 0) \geq \Phi(\beta_i), \quad i = 1, \ldots, np \) (23)
\[
d^L \leq d \leq d^U
\]
where \( H(X) = (X_1 - 8)^2 - (X_2 - 3)^2 \), \( w_1 = w_2 = 1.0 \)
\( G_1(X) = 1 - X_1 X_2 / 20 \)
\( G_2(X) = 1 - (X_1 + X_2 - 5)^2 / 30 - (X_1 - X_2 - 12)^2 / 120 \)
\( G_3(X) = 1 - 80 / (X_1^2 + 8X_2 + 5) \)

where \( X_i \sim N(\mu_i^0, \sigma = 0.3) \) and \( \beta_i = 3.0 \) for \( i = 1, 2; \mu_1^0 = 2.0 \) and \( \mu_2^0 = 8.0 \).

The L-Type robustness provides a different optimum design from the other types, as shown in Fig. 9. As shown in Table 7, the PMI method maximizes the mean of the robust objective, while minimizing its standard deviation in the process of design optimization. Again, it is shown in Table 4 that the PMI method estimates the statistical information accurately.

4.4. Reliability-Based Robust Design Optimization for Side Impact Crashworthiness

A vehicle side impact is employed to demonstrate the effectiveness of the proposed PMI method for reliability-based robust design optimization. The problem is described in detail in Youn et al. (2004b). Nine constraints are used to determine driver safety in the event of a side impact crash. The design objective is to minimize the abdomen load and its variation during the side impact crash, which is thus identified as S-Type robustness. Other performances are defined as probabilistic
Figure 9. Six-sigma design optimization for L-type.

constraints with target confidences. Nine design and random parameters are composed of seven geometry (lognormal distribution) and two material parameters (Weibull distribution). Two additional random parameters are barrier height and hitting position (normal distribution). A total of nine design and 11 random parameters are used in this model.

The reliability-based robust design optimization for side impact crashworthiness is formulated as

\[
\min \; \text{sgn}(\mu_H) \cdot \left( \frac{\mu_H}{\mu_{H_0}} \right)^2 + \left( \frac{\sigma_H}{\sigma_{H_0}} \right)^2 \\
\text{s.t.} \; P(\text{upper/mid/lower VC} \leq 0.32 \text{ m/s}) \geq \Phi(\beta_i) \\
P(\text{upper/mid/lower rib deflection} \leq 32 \text{ mm}) \geq \Phi(\beta_i) \\
P(\text{pubic symphysis force, F} \leq 4.0kN) \geq \Phi(\beta_i) \quad (24) \\
P(\text{velocity of B-pillar at mid-point} \leq 9.9 \text{ mm/ms}) \geq \Phi(\beta_i) 
\]
Table 4. Results of reliability-based robust design optimization with L-type

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Mean</th>
<th>Std. dev.</th>
<th>d1</th>
<th>d2</th>
<th>Gp1</th>
<th>Gp2</th>
<th>Gp3</th>
<th>NFE</th>
<th>NRFE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11.02</td>
<td>4.696</td>
<td>2.000</td>
<td>8.000</td>
<td>0.518</td>
<td>−1.997</td>
<td>0.015</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>24.59</td>
<td>3.219</td>
<td>2.842</td>
<td>4.446</td>
<td>0.181</td>
<td>−0.393</td>
<td>−0.385</td>
<td>27</td>
<td>18</td>
</tr>
<tr>
<td>2</td>
<td>22.22</td>
<td>2.872</td>
<td>3.257</td>
<td>3.605</td>
<td>0.041</td>
<td>−0.103</td>
<td>−0.479</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>20.91</td>
<td>2.751</td>
<td>3.428</td>
<td>3.291</td>
<td>0.007</td>
<td>−0.002</td>
<td>−0.511</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>21.09</td>
<td>2.770</td>
<td>3.403</td>
<td>3.370</td>
<td>0.000</td>
<td>−0.025</td>
<td>−0.499</td>
<td>27</td>
<td>18</td>
</tr>
<tr>
<td>Opt</td>
<td>21.09</td>
<td>2.770</td>
<td>3.403</td>
<td>3.370</td>
<td>0.000</td>
<td>−0.025</td>
<td>−0.499</td>
<td>98</td>
<td>67</td>
</tr>
</tbody>
</table>

MC simulation w/100K samples at the optimum: mean = 20.99 and standard deviation = 2.774

\[ P(\text{velocity of B-pillar front door} \leq 15.7\,\text{mm/ms}) \geq \Phi(\beta_i) \]
\[ d^L \leq d \leq d^U, \quad d \in \mathbb{R}^9 \quad \text{and} \quad X \in \mathbb{R}^{11} \]

where a target reliability, \( \beta_i \), is set to 3.0.

As shown in Fig. 10, the robust objective is significantly minimized (i.e., robustness is increased) by reducing the mean and standard deviation of abdomen load from −0.3564 and 0.04484 to −0.8439 and 0.02375, respectively. The MCS with a million samples is used to confirm numerical accuracy of the statistical information at the optimum design, which are −0.8432 and 0.2392. The total number of constraint evaluations and robust response evaluations are 118 and 24, respectively.

4.5. Reliability-Based Robust Design Optimization for Gasket Sealing Performance

Reliability-based robust design optimization was applied to an engine rubber gasket design problem. An engine gasket is used to prevent oil leakage. The design objective was to determine the shape of the gasket so that robustness of the sealing performance and reliability of other performances like the contact force \( F^{CT} \) and stress \( \sigma_i \) are improved when installed. Parametric spline curves are used to represent the gasket boundary, and the shapes of these curves are defined as design parameters. Nine shape design parameters are defined, as shown in Fig. 11. In order to maintain the symmetrical shape, four design parameters are linked. Both design and random parameters are defined in Table 5. All random parameters are assumed to be statistically independent.

Figure 11 shows the initial gasket geometry before installation. Since the engine block is much stiffer than the rubber gasket, only the gasket is modeled, using a meshfree method with 325 particles; it is assumed that all other parts are rigid. More details are found in Kim et al. (2002).
Figure 10. Reliability-based robust design optimization history.

Figure 11. Design parameterization of the gasket.
Table 5. Properties of design and random variables

<table>
<thead>
<tr>
<th>IDV, IRV</th>
<th>$d_L$</th>
<th>$d$, mean</th>
<th>$d_U$</th>
<th>Std. dev.</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.01$</td>
<td>0.00</td>
<td>0.01</td>
<td>0.001</td>
<td>normal</td>
</tr>
<tr>
<td>2</td>
<td>$-0.01$</td>
<td>0.00</td>
<td>0.01</td>
<td>0.001</td>
<td>normal</td>
</tr>
<tr>
<td>3</td>
<td>$-0.4$</td>
<td>0.00</td>
<td>0.4</td>
<td>0.01</td>
<td>normal</td>
</tr>
<tr>
<td>4</td>
<td>$-0.4$</td>
<td>0.00</td>
<td>0.4</td>
<td>0.01</td>
<td>normal</td>
</tr>
<tr>
<td>5</td>
<td>$-0.4$</td>
<td>0.00</td>
<td>0.4</td>
<td>0.01</td>
<td>normal</td>
</tr>
<tr>
<td>6</td>
<td>$-0.4$</td>
<td>0.00</td>
<td>0.4</td>
<td>0.01</td>
<td>normal</td>
</tr>
<tr>
<td>7</td>
<td>$-0.4$</td>
<td>0.00</td>
<td>0.4</td>
<td>0.01</td>
<td>normal</td>
</tr>
<tr>
<td>8</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.001</td>
<td>normal</td>
</tr>
<tr>
<td>9</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.001</td>
<td>normal</td>
</tr>
</tbody>
</table>

Although the sealing performance can be enhanced by increasing the gasket size or by applying a large amount of installation force, such methods can cause stress concentration and, thus, short service life. The improvement of the sealing performance is defined as the minimization of the gap. Thus, the reliability-based robust design optimization is formulated as an S-Type as

$$
\begin{align*}
\min & \quad \text{sgn}(\mu_{Gap}) \cdot \left( \frac{\mu_{Gap}}{\mu_{Gap_0}} \right)^2 + \left( \frac{\sigma_{Gap}}{\sigma_{Gap_0}} \right)^2 \\
\text{s.t.} & \quad P(G_i(X; d) \leq 0) \geq \Phi(\beta_i), \quad i = 1, \ldots, np \\
& \quad d^L \leq d \leq d^U
\end{align*}
$$

(25)

where

$$
\begin{align*}
G_i(X; d) &= F_i^{ct} - F_{i,\text{threshold}}^{ct} \leq 0, \quad i = 1, \ldots, m \\
G_i(X; d) &= \sigma_i - \sigma_{\text{bound}} \leq 0, \quad i = m + 1, \ldots, np
\end{align*}
$$

Since the contact region cannot be defined before analysis is carried out, a possible contact region is initially defined, and the square sum of the gap ($\sum \text{gap}^2$) along the region is then measured at the specified points, as shown in Fig. 12.

![Figure 12. Sealing performance.](image)
Figure 13. Stress result at initial design.

Figure 14. Stress result at deterministic optimum design.

Figure 15. Stress result at reliability-based robust optimum design.
A reliability-based robust optimum design is successfully obtained after seven iterations of reliability-based robust design optimization. As shown in Figs. 14–16, the reliability-based robust optimum design is compared to the initial and deterministic optimum designs. The reliability-based robust optimum design has higher reliability and quality while it has a slightly smaller contact region than does the deterministic optimum design. It is interesting to note that the initial circular region of the gasket top changes to an H-shape at the optimum design in order to reduce the concentration of stress while increasing the contact region. The reliability-based robust optimum design has smaller dip than the deterministic optimum design, as shown in Fig. 16. Table 6 summarizes the result of sealing performance in terms of the gap, showing product quality improvement at the initial and optimum design. The quality of sealing performance \(1 - C_{ql}(x_{opt})/C_{ql}(x_{int})\) is improved by 39.2%.

### 4.6. Effect on Preceding Deterministic Robust Design Optimization for Reliability-Based Robust Design Optimization

It has been found that a deterministic design optimization prior to RBDO enhances its numerical efficiency by moving the design near the reliability-based optimum design (Youn and Choi, In press, 2004). In this paper, we have tested the effect of the deterministic design optimization from the viewpoint of numerical efficiency for reliability-based robust

<table>
<thead>
<tr>
<th></th>
<th>Initial</th>
<th>Reliability-based robust optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.8692</td>
<td>0.3894</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.0131</td>
<td>0.0132</td>
</tr>
</tbody>
</table>
Table 7. Effect on deterministic optimization for reliability-based robust design optimization

<table>
<thead>
<tr>
<th>Type of robust design optimization</th>
<th>Reliability-based robust design optimization</th>
<th>Deterministic opt. and then reliability-based robust design optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Response</td>
<td>Constraints (FEA, DSA)</td>
<td>Objective (FEA, DSA)</td>
</tr>
<tr>
<td>N</td>
<td>98</td>
<td>98</td>
</tr>
<tr>
<td>S</td>
<td>94</td>
<td>94</td>
</tr>
<tr>
<td>L</td>
<td>98</td>
<td>98</td>
</tr>
</tbody>
</table>

design optimization. Unlike RBDO, the reliability-based robust design optimization requires output statistical moment analysis, even during the deterministic design iteration, to minimize the robustness objective. Thus, the deterministic design optimization may not improve numerical efficiency, which is shown in Table 7. It is mainly because it requires expensive inverse reliability analyses to compute the robust objective. In fact, it turned out that having the deterministic robust design optimization made the reliability-based robust design optimization for the S-Type less efficient. Thus, it may not be better to carry out deterministic design optimization prior to the reliability-based robust design optimization. In Table 7, FEA and DSA denote number of finite element analyses and design sensitivity analyses, respectively.

5. CONCLUSIONS

This paper proposed the PMI method to estimate statistical moments accurately, thus enabling designers to perform reliability-based robust design optimization effectively. The PMI method does not require the Hessian information of the robust response, and also estimates its statistical output accurately for reliability-based robust design optimization. Thus, the PMI method will make it possible to integrate the reliability-based robust design optimization and the response surface method. It has been shown that statistical moments are accurately estimated for even nonnormally distributed statistical responses with high skewness and kurtosis. The paper also presents new formulations of reliability-based robust design optimization for different types of robustness: N-Type, S-Type, and L-Type. These new formulations are shown to be effective for reliability-based robust design optimization by minimizing the robust objective and satisfying the probabilistic constraints. Furthermore, it has been found that the deterministic design optimization may not be beneficial for reliability-based robust design
optimization, since the robustness of design must be considered through an inverse reliability analysis in the process of deterministic design optimization.

**NOMENCLATURE**

\( C(X; d) \) Cost function for reliability-based robust design optimization
\( C_m(X; d) \) Material cost at a given design
\( C_ql(H; d) \) Quality loss cost due to manufacturing variability
\( d \) Design parameter; \( d = [d_1, d_2, \ldots, d_n]^T \)
\( E^k[A] \) The \( k \)th statistical moment of the event \( A \)
\( F_G(\bullet) \) CDF of the performance function \( G(X) \);
\( F_G(g) = P(G(X) < g) \)
\( f_X(x) \) Joint Probability Density Function (PDF) of the random parameter
\( G(X) \) Performance function; the design is considered “fail” if \( G(X) < 0 \)
\( G_p, G_{p, FORM} \) Probabilistic constraint and its value from the first-order reliability method
\( H, h_t \) Robust response vector and its target nominal vector
\( np \) Number of probabilistic constraints
\( ndv, nrv \) Numbers of design variables and random variables, respectively
\( P(\bullet) \) Probability function
\( Q \) Reciprocal of \( H, 1/H \)
\( sgn(B) \) Sign of value \( B, sgn(B) = 1 \) or \(-1\)
\( X, x \) Random parameter, \( X = [X_1, \ldots, X_n]^T \), and its realization, \( x = [x_1, \ldots, x_n]^T \)
\( U, u \) Standard normal random parameter, \( U = [U_1, \ldots, U_n]^T \), and its realization
\( u^*_{\beta_t} \) The most probable point when \( \beta = \beta_t \) in PMA
\( u^*_{\beta_t, MV+} \) MPP using refined hybrid mean value method in PMA
\( w_i \) Weight at the \( i \)th integration point for moment calculation
\( w_n, w_s, w_l \) Weights for N-, S-, and L-Types, respectively
\( x_i \) The \( i \)th integration point for moment calculation
\( \beta_{\mu} \) Target reliability index
\( \Phi(\bullet) \) Standard normal Cumulative Distribution Function (CDF) \( \Phi(\bullet) \)
\( \mu \) Mean of random parameter \( X; \mu = [\mu_1, \mu_2, \ldots, \mu_n]^T \)
\( \mu_H, \sigma_H \) Mean and standard deviation of output response \( H \), respectively
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REFERENCES


