A New Fuzzy Analysis Method for Possibility-Based Design Optimization

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Structural analysis and design optimization have recently been extended to stochastic approach to take various uncertainties into account. However in areas when it is not possible to produce accurate statistical information, the probabilistic methods are not appropriate for stochastic structural analysis and design optimization, since improper modeling of uncertainty could cause greater degree of statistical uncertainty than those of physical uncertainty. To handle uncertainty when modeling physical uncertainty with insufficient information, possibility-based (or fuzzy set) methods have recently been introduced in stochastic structural analysis and design optimization. The main advantage of the fuzzy analysis compared to other methods is that it preserves the intrinsic random nature of physical variables through their membership functions and yields more conservative results in analysis and thus optimum design than those from the probabilistic methods. There are two computational aspects in the fuzzy analysis compared to the probability analysis. First, the fuzzy variables can be defined easier than the random variables when no or few statistical data are available. Secondly, extended fuzzy operations are much simpler than those required to use probability, especially when a number of variables are involved. For possibility-based design optimization (PBDO), like for reliability-based design optimization (RBDO), the performance measure approach (PMA) is more appropriate than other approaches, such as a possibility index approach. This paper proposes a new formulation of PBDO using PMA. For fuzzy analysis, the maximal possibility search (MPS) method is proposed to improve numerical efficiency, stability, and accuracy comparing with the vertex method and recently developed multilevel-cut method. Examples, including a non-monotonic response, are used to demonstrate the proposed fuzzy analysis and PBDO.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>Random fuzzy parameter; $X = [X_1, X_2, \ldots, X_n]^T$</td>
</tr>
<tr>
<td>x</td>
<td>Realization of $X$; $x = [x_1, x_2, \ldots, x_n]^T$</td>
</tr>
<tr>
<td>U</td>
<td>Random fuzzy parameter with non-interactive triangular membership function; $U = [U_1, U_2, \ldots, U_n]^T$</td>
</tr>
<tr>
<td>u</td>
<td>Realization of $U$; $u = [u_1, u_2, \ldots, u_n]^T$</td>
</tr>
<tr>
<td>$\Pi(\cdot)$</td>
<td>Possibility of the event</td>
</tr>
<tr>
<td>$\Pi_x(\cdot)$</td>
<td>Membership function of the fuzzy parameter $X$</td>
</tr>
<tr>
<td>i, l</td>
<td>Index and the index set</td>
</tr>
<tr>
<td>d</td>
<td>The design vector; $d = [d_1, d_2, \ldots, d_n]^T$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Possibility index, mostly used in the term “$\alpha$-cut”</td>
</tr>
<tr>
<td>$\alpha_t$</td>
<td>Target possibility</td>
</tr>
<tr>
<td>$G(X)$</td>
<td>Performance function; the design is considered “fail” if $G(X) &gt; 0$</td>
</tr>
<tr>
<td>$G_{it}(\cdot)$</td>
<td>The $i^{th}$ possibility constraint using PMA approach</td>
</tr>
<tr>
<td>$u^*$</td>
<td>Most Possible Point</td>
</tr>
<tr>
<td>t</td>
<td>Parametric coordinate along the line in the n-dimensional cube</td>
</tr>
</tbody>
</table>

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I. Introduction

STRUCTURAL analysis and design optimization have recently been extended to stochastic approach to take various uncertainties into account. Probabilistic methods are prevailing in stochastic structural analysis and design optimization by assuming the amount of raw data sufficient enough to create accurate statistical input. However, in many practical applications, sufficient raw data may not be available due to restrictions of time, human and facility resources, money, etc. In areas where it is not possible to produce accurate statistical information, the probabilistic methods are not appropriate for stochastic structural analysis and design optimization, since improper modeling of uncertainty could cause greater degree of statistical uncertainty than those of physical uncertainty [1]. To handle uncertainty when modeling physical uncertainty with insufficient information, possibility-based (or fuzzy set) methods have recently been introduced in stochastic structural analysis and design. This paper presents a new numerical method of fuzzy (or possibility) analysis for stochastic structural analysis and design optimization.

There are two different types of uncertainties: aleatory and epistemic uncertainties [2]. Aleatory uncertainty is classified as objective and irreducible uncertainty, whereas epistemic uncertainty is a subjective and reducible uncertainty that stems from lack of knowledge on data. To deal with epistemic uncertainty, three main methodologies have been addressed: (1) interval analysis, (2) convex modeling, and (3) possibility theory using a fuzzy set. It is shown in [3] that first two methodologies have their drawbacks on numerical accuracy. Thus, a possibility approach is used in this paper when dealing with epistemic uncertainty, since this approach is able to define the upper and lower bounds for probability measures compatible with the available data [3,4]. A fuzzy analysis represents a very useful tool to perform operations in the framework of possibility theory [3-8]. The main advantage of the fuzzy analysis compared to other methods is that it preserves the intrinsic random nature of physical variables through their membership functions. Moreover, extended fuzzy operations are simpler than those required to use probability, especially when a number of variables are involved. It has been pointed out that, when little information is available for input data, the possibility-based method is better since it is easier to identify the most conservative possibilistic design than the probabilistic design that is consistent with the limited available information [8,9]. This is a desirable merit, since a conservative optimum design is preferred when accurate statistical information is not available.

In the probability-based methods for structural design, significant achievement has been made through a performance measure approach (PMA), which transforms probabilistic design formulation into non-probabilistic one so as to simplify its numerical process [10,11]. This paper discusses some existing numerical methods and their difficulties in possibility-based analysis and design optimization, and proposes an alternative approach. It is found that PMA is more appropriate than other approaches in possibility-based design optimization (PBDO). Therefore, this paper proposes a new formulation of PBDO, and a numerical procedure for fuzzy analysis using PMA to overcome numerical challenges. For numerical methods of fuzzy analysis, the vertex method is a popular method but could be extremely expensive for large-scale engineering applications. Moreover, the vertex method could yield inaccurate results of fuzzy analysis when an output response has a maximum or minimum within the range of input fuzzy parameters. Thus a level-cuts (α-cuts) method has been used, where the nonlinear problem was solved with various design levels a [8]. Recently, a multilevel-cut method [12] has been developed to overcome the inaccuracy of the vertex method for nonlinear structural design, but it could be too expensive to carry out PBDO.

Using PMA, a new formulation of PBDO is formulated in this paper to improve numerical efficiency, stability, and accuracy. To resolve disadvantages of the vertex method and the multilevel-cut method, this paper proposes a new maximal possibility search (MPS) fuzzy analysis method, such that it evaluates possibility constraints efficiently and accurately for nonlinear structural applications. Monotonic and non-monotonic response examples are used to demonstrate the proposed MGI fuzzy analysis and PBDO methods.

II. Possibility Theories and Definition of Membership Function

2.1 Possibility and Fuzzy Set Theories

Possibility is a subjective measure that expresses the degree to which a person considers that an event can occur. The possibility measure Π should satisfy the following axioms:

1. Boundary requirement: Π(∅) = 0, Π(Ω) = 1.
2. Monotonicity: if \( A_i \subseteq A_j \), then \( \Pi(A_i) \leq \Pi(A_j) \).
3. Union measure: \( \Pi\left(\bigcup_{i \in I} A_i\right) = \max_{i \in I} \{\Pi(A_i)\} \), \( \forall A_i, i \in I \).

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where \( \{A_i : i \in I\} \) is a partition of the universal event \( \Omega \).

One basis of the possibility theories is the theory of the fuzzy sets. The membership function of the fuzzy set represents the “grade of membership”. According to Zadeh [13], the membership function \( \Pi_x(x) \) should have the properties:

1. Bounded: \( 0 \leq \Pi_x(x) \leq 1 \).
2. Possibility of an event: \( \Pi(A) = \max_{x \in A} \{ \Pi_x(x) \} \).

A fuzzy parameter \( X \) with the membership function \( \Pi_x(x) \) is called to satisfy the \textit{unity} if and only if there exists unique \( x \) such that \( \Pi_x(x) = 1 \).

A fuzzy parameter \( X \) with the membership function \( \Pi_x(x) \) is \textit{strongly convex} if and only if the event \( \{x \mid \Pi_x(x) \geq \alpha \} \) is strongly convex \( \forall \alpha \in (0,1] \). An alternative definition of strong convexity is \( \Pi_x(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\Pi_x(x_1), \Pi_x(x_2)\}, \forall x_1, x_2 \text{ and } \forall \lambda \in (0,1) \).

A fuzzy parameter \( X \) with the membership function \( \Pi_x(x) \) is \textit{bounded} if and only if the event \( \{x \mid \Pi_x(x) \geq \alpha \} \) is bounded \( \forall \alpha \in (0,1] \).

The fuzzy parameter \( X_1 \) and \( X_2 \) are \textit{non-interactive} [9] if the joint membership function \( \Pi_{x_1,x_2}(x_1, x_2) \) satisfies \( \Pi_{x_1,x_2}(x_1, x_2) = \min\{\Pi_{x_1}(x_1), \Pi_{x_2}(x_2)\} \). The non-interactive joint membership function is more conservative than the other types of the joint membership function.

### 2.2 Generation of the Input Membership Functions

The generation of the input membership functions of the random fuzzy parameters using the available limited set of data is a very important step of the possibility analysis and PBDO. Several methods have been proposed depending on the number and the kind of the data available. If the fuzzy parameters have a random nature but the available data are not sufficient to assign the probability of elementary events, the membership function can be estimated from histogram of a finite number of samples [16]. If the only available information is the judgment of experts (subjective) with the most likely value and the interval corresponding to the certain confidence levels, the membership function can be generated using the framework of Program-Evaluation and Review Technique (PERT) analysis [17]. Savoia [18] proposed a method to generate the membership function if the probability assignment to a set of nested focal elements is given. Thus, there are several methods to define the membership function, but the generated random fuzzy parameters should have the properties of the unity, boundedness and the strong convexity. In order to get conservative analysis and design, the non-interactive property is reasonable to assume.

### III. Possibility-Based Design Optimization (PBDO)

#### 3.1 Formulation of PBDO

For general engineering applications, the PBDO model can be formulated as

\[
\begin{align*}
\text{minimize} \quad & \text{Cost}(d) \\
\text{subject to} \quad & \Pi(G_i(d(X)) > 0) \leq \alpha_i, \ i = 1, 2, \cdots, np \\
& d^l \leq d \leq d^u
\end{align*}
\]

where \( d = [d^T_i] \in R^n \) is the design vector, \( X = [X_i] \in R^{nr} \) is the fuzzy random vector where the fuzzy random parameter \( X_i \) has the membership function \( \Pi_{x_i}(x_i) \) and the maximal grade [13] \( \max \{\Pi_{x_i}(x_i)\} = d_i \), \( \alpha_i \) is a target possibility of failure, and \( n, nr, \text{ and } np \) are the number of design parameters, fuzzy random parameters, and possibility constraints, respectively. All fuzzy random parameters considered here are assumed to be non-interactive. The possibility constraints are described by a possibility \( \Pi(\bullet) \leq \alpha_i \) for any failure event \( G_i(d(X)) > 0 \).

#### 3.2 Performance Measure Approach (PMA) in PBDO
The performance measure approach (PMA) has been successfully applied to reliability-based design optimization (RBDO), significantly improving numerical stability and efficiency [10,11]. The PMA approach, which is developed for design optimization, is applied to the PBDO model to formulate as

$$\begin{align*}
\text{minimize} & \quad \text{Cost}(\mathbf{d}) \\
\text{subject to} & \quad G_{ni}(\mathbf{d}(\mathbf{X})) \leq 0, \quad i = 1, 2, \ldots, np \\
& \quad \mathbf{d}^l \leq \mathbf{d} \leq \mathbf{d}^u
\end{align*}$$

where $G_{ni}$ is $i^{th}$ possibility constraint.

In this paper, it is assumed that the non-interactive input fuzzy parameters $X_i$ have its membership function $\Pi_{X_i}(x_i)$ satisfying three properties [3,9,13]: (1) unity, (2) strong convexity, and (3) boundedness. These three properties enable non-interactive input fuzzy parameters to be uniquely transformed to an isosceles triangular membership function as

$$\Pi_{U_i}(u_i) = \begin{cases} u_i + 1, & -1 \leq u_i \leq 0 \\ 1 - u_i, & 0 \leq u_i \leq 1 \end{cases} = 1 - |u_i|, \quad |u_i| \leq 1.$$  

The transformation can be written as

$$U_i = \begin{cases} \Pi_{X_i, L}(X_i) - 1 & X_i \leq d_i \\ 1 - \Pi_{X_i, R}(X_i) & X_i > d_i \end{cases}$$

where $\Pi_{X_i, L}(x_i)$ and $\Pi_{X_i, R}(x_i)$ are the left side and right side of the membership function of the input fuzzy parameter $X_i$, respectively, and $d_i$ is the maximal grade of this membership function.

Thus, evaluation of the possibility constraint requires a fuzzy analysis using PMA, which is formulated as

$$\begin{align*}
\text{maximize} & \quad G(U) \\
\text{subject to} & \quad \|U\|_e \leq 1 - \alpha_i
\end{align*}$$

where the optimum point on the target possibility domain $\|U\|_e \leq 1 - \alpha_i$ is identified as the most possible point (MPP) $\mathbf{u}_{\alpha_i}$ with a prescribed possibility of failure $\alpha_i$.

The difference between reliability analysis [10,11] and fuzzy analysis [3-9,12-14] is that, the most probable (MPP) in reliability analysis is based on FORM, which means the related probability is first order approximation, whereas MPP in possibility analysis is exact, along with the related possibility. Another difference is the search domain enclosed by the target confidence level. The reliability analysis has $nr$-dimensional sphere as its search domain, $\|U\|_e \leq \beta$, whereas the possibility analysis has $nr$-dimensional hyper-cube as its search domain, $\|U\|_e \leq 1 - \alpha_i$, that makes the numerical computation simpler, compared to the reliability analysis. Since the MPP search in possibility analysis is different from the one in reliability analysis, a new numerical method is proposed in the next section to solve the problem (5).

IV. Fuzzy Analysis for PBDO

For a non-monotonic response within the range of input fuzzy parameters, the fuzzy analysis could be inaccurate using the vertex method [3] or computationally expensive using the multilevel-cut method or other computational
schemes such as a global optimization method [14]. Thus, in this paper, the maximal possibility search (MPS) method is proposed for fuzzy analysis to ensure numerical efficiency and accuracy in PBDO. This method first attempts to find an MPP using the proposed maximal possibility search, since in majority of cases the MPP is likely to be on the vertex of the target possibility domain \( \|U\|_{\infty} \leq 1 - \alpha \). If the proposed maximal possibility search does not yield a solution at a vertex, then the maximal possibility search is integrated with an interpolation in search of the MPP on the edge or in the interior domain of the hyper-cube.

4.1 Maximal Possibility Search
The proposed maximal possibility search is as following:

**Step 1.** Set the iteration counter \( k = 0 \) with the convergence parameter \( \varepsilon = 10^{-3} \). Set the pointer \( j = 1 \). Let \( u^{(0)} = 0 \). Calculate the performance \( G(u^{(0)}) \) and the sensitivity \( \nabla G(u^{(0)}) \). Let \( d^{(k)} = \nabla G(u^{(k)}) \).

**Step 2.** Compute the next point as \( u^{(k+1)} = \pi_j \cdot \text{sgn}(d^{(k)}) \) where \( \pi_j = 1 - \alpha_j \) and \( \text{sgn}(X) = (\text{sgn}(X_1), \text{sgn}(X_2), \ldots, \text{sgn}(X_n)) \) if \( X = [X_j]^T \in R^n \). Let \( k = k + 1 \).

**Step 3.** Calculate the performance \( G(u^{(k)}) \) and the sensitivity \( \nabla G(u^{(k)}) \). Let \( d^{(k)} = \nabla G(u^{(k)}) + \beta d^{(k-1)} \) where \( \beta = \left( \| \nabla G(u^{(k)}) \| \right) \left( \| \nabla G(u^{(k-1)}) \| \right)^2 \). If \( \text{sgn}(\nabla G(u^{(k)})) = \text{sgn}(u^{(k)}) \), it is the maximum point and stop. If \( G(u^{(k)}) \geq G(u^{(j)}) \), let \( j = k \) and go to Step 2. Otherwise, go to Step 4.

**Step 4.** Go to Step 5, with \( u^{(j)}, G(u^{(j)}) \) and \( \nabla G(u^{(j)}) \).

4.2 Maximal Possibility Search with an Interpolation
The proposed maximal possibility search with an interpolation is as following:

**Step 5.** Let \( l = 0 \) and \( d^{(l)} = \nabla G(u^{(j)}) \). Go to Step 6.

**Step 6.** Calculate the new point \( u^{(k+1)} \) on the boundary of the domain using the start point \( u^{(j)} \) and the search direction \( d^{(l)} \). Let \( k = k + 1 \).

**Step 7.** Calculate the performance \( G(u^{(k)}) \) and the sensitivity \( \nabla G(u^{(k)}) \). If

\[
\text{sgn}\left( \frac{\partial G}{\partial x_i}(u^{(k)}) \right) = \text{sgn}(u_i), \quad \text{for } u_i = \pi_j \text{ or } u_i = -\pi_j
\]

\[
\left| \frac{\partial G}{\partial x_i}(u^{(k)}) \right| < \varepsilon, \quad \text{for } -\pi_j < u_i < \pi_j
\]

then it is the maximum point and stop. Otherwise, go to Step 8.

**Step 8.** Using \( G(u^{(j)}), G(u^{(k)}), \nabla G(u^{(j)}) \) and \( \nabla G(u^{(k)}) \) to construct the third order polynomial \( P_3(t) \) on the straight line between \( u^{(j)} \) and \( u^{(k)} \) where \( t \) is the parameter for the line. Calculate the maximum point \( t^* \) for this polynomial. Let \( u^{(k+1)} \) be the point on the line corresponding to \( t^* \). Let \( k = k + 1 \).

**Step 9.** Calculate the performance \( G(u^{(k)}) \) and the sensitivity \( \nabla G(u^{(k)}) \). Check the convergent criteria using the equation in Step 7. If convergent, stop. Otherwise, let the new conjugate direction be \( d^{(k+1)} = \nabla G(u^{(k)}) + \beta d^{(k)} \) where \( \beta = \left( \| \nabla G(u^{(k)}) \| \right) \left( \| \nabla G(u^{(k-2)}) \| \right)^2 \). Let \( j = k \), \( l = l + 1 \), and go to Step 6.

4.3 Remark
The proposed maximal possibility search is sufficient for the monotonic responses. The proposed maximal possibility search with an interpolation will be used only when the maximal possibility search fails. Since in most cases the maximal possibility search finds MPP efficiently, the MPS method will be efficient while robust for nonlinear and non-monotonic responses.

V. Numerical Examples

5.1 Mathematical Examples for Fuzzy Analysis
5.1.1 Monotonic Response Problems

For the first example, consider the input fuzzy parameters $X_1$ and $X_2$ that are non-interactive and have the same triangular membership function on the interval $[4.737,7.263]$. The performance function is

$$ G_1(X) = \exp(X_1 - 7) + X_2 - 10. $$

The target possibility $\alpha_t$ is set to 0.05 and the design point is $d = [6.0,6.0]^T$. $\alpha_t = 0.05$ is equivalent to the possibility of failure equal to 0.05. Using the proposed maximal possibility search, the MPP is obtained in one iteration, as shown in Table 1 and in Fig. 1(a).

Table 1. Fuzzy Analysis Result of Example with $G_1(X)$

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$G$</th>
<th>$\partial G / \partial U_1$</th>
<th>$\partial G / \partial U_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
<td>-3.632</td>
<td>0.465</td>
<td>1.263</td>
</tr>
<tr>
<td>1</td>
<td>0.95</td>
<td>0.95</td>
<td>-1.579</td>
<td>1.543</td>
<td>1.263</td>
</tr>
<tr>
<td>MPP</td>
<td>0.95</td>
<td>0.95</td>
<td>-1.579</td>
<td>1.543</td>
<td>1.263</td>
</tr>
</tbody>
</table>

![Figure 1. Possibility Analysis Process for Examples with Monotonic Responses](image)

As a second example, consider the input fuzzy parameters $X_1$ and $X_2$ that are non-interactive and have the triangular membership functions on the intervals $[2.737,5.263]$ and $[3.737,6.263]$, respectively. The performance function is

$$ G_2(X) = -[\exp(0.8X_1 - 1.2) + \exp(0.7X_2 - 0.6) - 5]/10. $$

The target possibility $\alpha_t$ is set to 0.05 and the design point is $d = [4.0,5.0]^T$. Using the proposed maximal possibility search, the solution is again obtained in one iteration, as shown in Table 2 and in Fig. 1(b).

Table 2. Fuzzy Analysis Result of Example with $G_2(X)$

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$G$</th>
<th>$\partial G / \partial U_1$</th>
<th>$\partial G / \partial U_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
<td>-2.056</td>
<td>-0.747</td>
<td>-1.601</td>
</tr>
<tr>
<td>1</td>
<td>-0.95</td>
<td>-0.95</td>
<td>-0.568</td>
<td>-0.286</td>
<td>-0.694</td>
</tr>
<tr>
<td>MPP</td>
<td>-0.95</td>
<td>-0.95</td>
<td>-0.568</td>
<td>-0.286</td>
<td>-0.694</td>
</tr>
</tbody>
</table>

5.1.2 Non-monotonic Response Problems

Assume non-interactive input fuzzy parameters $X_1$ and $X_2$ have the triangular membership functions on the interval $|X_{i}^U - X_{i}^L| = 1.737$, $i = 1,2$. A midpoint of the interval corresponds to the design point. The performance function is given as
\[ G(X) = -0.3X_1^2X_2 + X_2 - 0.8X_1 - 1. \]

The target possibility \( \alpha \) is set to 0.05 and the design point is \( \mathbf{d} = [-0.5, 2.2]^\top \). The proposed maximal possibility search fails, but the proposed maximal possibility search with an interpolation gives the solution in four iterations as shown in Table 3 and in Fig. 2(a).

### Table 3. Fuzzy Analysis Result of Non-monotonic Example with \( \mathbf{d} = [-0.5, 2.2]^\top \)

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( U_1 )</th>
<th>( U_2 )</th>
<th>( G )</th>
<th>( \partial G / \partial U_1 )</th>
<th>( \partial G / \partial U_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
<td>1.435</td>
<td>-0.122</td>
<td>0.803</td>
</tr>
<tr>
<td>1</td>
<td>-0.95</td>
<td>0.95</td>
<td>1.942</td>
<td>1.394</td>
<td>0.411</td>
</tr>
<tr>
<td>2</td>
<td>0.95</td>
<td>0.95</td>
<td>1.669</td>
<td>-1.207</td>
<td>0.841</td>
</tr>
<tr>
<td>3*</td>
<td>0.068</td>
<td>0.95</td>
<td>2.201</td>
<td>0.000</td>
<td>0.818</td>
</tr>
<tr>
<td>MPP</td>
<td>0.036</td>
<td>0.95</td>
<td>2.201</td>
<td>0.000</td>
<td>0.818</td>
</tr>
</tbody>
</table>

* Maximal possibility search with an interpolation

![Figure 2. Possibility Analysis Process for Examples with Non-monotonic Responses](image)

If the target possibility \( \alpha \) is set to 0.05 and the design point is \( \mathbf{d} = [-1.8, 0.0]^\top \). The proposed MPS method finds the solution in three iterations, as shown in Table 4 and in Fig. 2(b).

### Table 4. Fuzzy Analysis Result of Non-monotonic Example with \( \mathbf{d} = [-1.8, 0.0]^\top \)

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( U_1 )</th>
<th>( U_2 )</th>
<th>( G )</th>
<th>( \partial G / \partial U_1 )</th>
<th>( \partial G / \partial U_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
<td>0.440</td>
<td>-0.695</td>
<td>0.024</td>
</tr>
<tr>
<td>1</td>
<td>-0.95</td>
<td>0.95</td>
<td>0.220</td>
<td>0.434</td>
<td>-0.927</td>
</tr>
<tr>
<td>2*</td>
<td>-0.95</td>
<td>0.033</td>
<td>1.069</td>
<td>-0.655</td>
<td>-0.927</td>
</tr>
<tr>
<td>3*</td>
<td>-0.95</td>
<td>-0.95</td>
<td>1.980</td>
<td>-1.823</td>
<td>-0.927</td>
</tr>
<tr>
<td>MPP</td>
<td>-0.95</td>
<td>-0.95</td>
<td>1.980</td>
<td>-1.823</td>
<td>-0.927</td>
</tr>
</tbody>
</table>

* Maximal possibility search with an interpolation

### 5.2 Fuzzy Analysis of a Vehicle Side Impact

Assume all fuzzy parameters are non-interactive and fuzzy parameters \( X_i \sim X_j \) have the same triangular membership functions on the interval \([0.91, 1.09]\); \( X_8 \) and \( X_9 \) have the same triangular membership functions on the
interval [0.282,0.318]; and \( X_i \) and \( X_j \) are equal to 0. From the example of vehicle side impact, the limit state function of upper rib deflection is defined as [15]

\[ G(X) = 3.02 - 3.818X_1 + 4.2X_1X_2 - 0.0207X_5X_{10} - 6.63X_6X_9 + 7.7X_7X_8 - 0.32X_8X_{10}. \]

The target possibility \( \alpha_i \) is set to 0.05. Using the proposed MPS method, the solution is obtained in one iteration, as shown in Table 5. From the results in Table 5, it is confirmed that the search method finds the correct MPP that has a local maximum of \( G(U) \), as shown in Eq. (5), since the sign of the components of the point in U-space coincide with the sign of the component of the gradient.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>3.723</td>
</tr>
<tr>
<td>Sign((\nabla G))_i</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.95</td>
<td>0.95</td>
<td>-0.95</td>
<td>-0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>-0.95</td>
<td>6.410</td>
</tr>
<tr>
<td>Sign((\nabla G))_i</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>MPP</td>
<td>0.95</td>
<td>0.95</td>
<td>-0.95</td>
<td>-0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>-0.95</td>
<td>6.410</td>
</tr>
</tbody>
</table>

### 5.3 PBDO of a Mathematical Example

Consider the following design problem

\[
\begin{align*}
\text{minimize} & \quad d_1 + d_2 \\
\text{subject to} & \quad \Pi(G_i(X;d) > 0) \leq \alpha_i, \quad i = 1, \ldots, np \\
& \quad d^l \leq d \leq d^u
\end{align*}
\]

where

\[
\begin{align*}
G_1(X) &= 1 - X_1^2X_2 / 20 \\
G_2(X) &= 1 - (X_1 + X_2 - 5)^2 / 30 - (X_1 - X_2 - 12)^2 / 120 \\
G_3(X) &= 1 - 80/(X_1^2 + 8X_2 + 5)
\end{align*}
\]

where \( X_1 \) and \( X_2 \) are random fuzzy parameter with triangular membership function on the interval with a length of 0.902436, \( \alpha_1 = \alpha_2 = 0.1 \), \( d^l = [0,0]^T \) and \( d^u = [10,10]^T \). The initial design is \( d^{(0)} = [5.0,5.0]^T \).

The PMA approach of this PBDO model yields:

\[
\begin{align*}
\text{minimize} & \quad d_1 + d_2 \\
\text{subject to} & \quad G_{ii}(d(X)) \leq 0, \quad i = 1, \ldots, np \\
& \quad d^l \leq d \leq d^u
\end{align*}
\]

where

\[
\begin{align*}
G_1(X) &= 1 - X_1^2X_2 / 20 \\
G_2(X) &= 1 - (X_1 + X_2 - 5)^2 / 30 - (X_1 - X_2 - 12)^2 / 120 \\
G_3(X) &= 1 - 80/(X_1^2 + 8X_2 + 5)
\end{align*}
\]

where \( G_{ii}(d(X)) \) can be calculated by solving the sub-optimization problem:

\[
\begin{align*}
\text{maximize} & \quad G_i(U) \\
\text{subject to} & \quad \|U\|_\infty \leq 1 - \alpha_i
\end{align*}
\]

in the standard normalized U-space.

The PBDO history using a SQP optimizer is presented in Table 6 with graphical results given in Fig. 3(a).
Table 6. PBDO History Using PMA and SQP Optimizer

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Cost</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>No. of Analyses</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.00</td>
<td>5.000</td>
<td>5.000</td>
<td>-2.492</td>
<td>-0.529</td>
<td>0.077</td>
<td>6(2/2/2)</td>
</tr>
<tr>
<td>1</td>
<td>7.886</td>
<td>4.175</td>
<td>3.710</td>
<td>-0.534</td>
<td>-0.068</td>
<td>-0.188</td>
<td>8(2/4/2)</td>
</tr>
<tr>
<td>2</td>
<td>7.228</td>
<td>3.719</td>
<td>3.508</td>
<td>-0.058</td>
<td>-0.004</td>
<td>-0.305</td>
<td>6(2/2/2)</td>
</tr>
<tr>
<td>3</td>
<td>7.228</td>
<td>3.719</td>
<td>3.509</td>
<td>-0.058</td>
<td>-0.003</td>
<td>-0.305</td>
<td>6(2/2/2)</td>
</tr>
</tbody>
</table>

Figure 3. PBDO and RBDO Results

For comparison, consider the RBDO model:

\[
\text{minimize} \quad d_1 + d_2 \\
\text{subject to} \quad P(G_i(X; d) > 0) \geq \Phi(-\beta_i), \quad i = 1, \ldots, np \\
\quad d^l \leq d \leq d^u \\
\text{where} \quad G_i(X) = 1 - \frac{X_1^2 X_2}{20} \\
\quad G_2(X) = 1 - \frac{(X_1 + X_2 - 5)^2 - 30 - (X_1 - X_2 - 12)^2}{120} \\
\quad G_3(X) = 1 - \frac{80}{X_1^2 + 8X_2 + 5}
\]

where $X_1$ and $X_2$ are normal distributed random parameters with the standard deviation $\sigma = 0.3$, $\beta_i = \beta_i = 3.0$, $d^l = [0,0]^T$ and $d^u = [10,10]^T$. The initial design is $d^{(0)} = (5.0,5.0)^T$.

The RBDO history using SQP optimizer is presented in Table 7 with graphical results given in Fig. 3(b). Note the optimum cost of PBDO is larger than that of RBDO.

Table 7. RBDO History Using PMA and SQP Optimizer

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Cost</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>No. of Analyses</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.00</td>
<td>5.000</td>
<td>5.000</td>
<td>-3.044</td>
<td>-0.580</td>
<td>0.025</td>
<td>11(4/4/3)</td>
</tr>
<tr>
<td>1</td>
<td>7.823</td>
<td>3.923</td>
<td>3.900</td>
<td>-0.699</td>
<td>-0.142</td>
<td>-0.288</td>
<td>11(4/4/3)</td>
</tr>
<tr>
<td>2</td>
<td>6.814</td>
<td>3.504</td>
<td>3.402</td>
<td>-0.092</td>
<td>-0.024</td>
<td>-0.469</td>
<td>11(4/4/3)</td>
</tr>
<tr>
<td>3</td>
<td>6.728</td>
<td>3.440</td>
<td>3.288</td>
<td>-0.000</td>
<td>-0.000</td>
<td>-0.508</td>
<td>11(4/4/3)</td>
</tr>
</tbody>
</table>
5.4 PBDO of a Vehicle Side Impact

A vehicle side impact example is detail described in [15]. The design objective is to minimize vehicle weight while enhancing side impact crash performance. The mathematical model for this vehicle impact example is:

\[
\begin{align*}
\text{minimize} & \quad \text{Cost}(d) \\
\text{subject to} & \quad \Pi(G_i(X; d) > 0) \geq \alpha_i, \quad i = 1, \ldots, np \\
& \quad d^l \leq d \leq d^u
\end{align*}
\]

where the fuzzy parameters \(X_1 \sim X_7\) have the same triangular membership functions on the interval with length 0.18; \(X_8\) and \(X_9\) have the same triangular membership functions on the interval with length 0.036; and \(X_{10}\) and \(X_{11}\) are equal to 0. The number of the possibilistic constraint is \(np = 10\); and the initial design is \(X_i = 1, i = 1 \sim 7\) and \(X_8 = X_9 = 0.03\).

![Figure 4. Side Impact Model](image)

The PBDO problem cost is

\[
\text{Cost}(d) = 1.98 + 4.9d_1 + 6.67d_2 + 6.98d_3 + 4.01d_4 + 1.78d_5 + 2.73d_6
\]

where constraints are

\[
\begin{align*}
G1(x) &= 14.36 + (-9.9\times X(2) - 12.9\times X(1) \times X(8) + 0.1107\times X(3) \times X(10)) \\
G2(x) &= 1.86 + (2.95\times X(3) + 0.1792\times X(10) - 5.057\times X(1) \times X(2) - 11.\times X(2) \times X(8) - 0.215\times X(5) \times X(10) \\
&\quad - 9.9\times X(7) \times X(8) + 22.\times X(8) \times X(9)) \\
G3(x) &= -3.02 + (3.818\times X(3) - 4.2\times X(1) \times X(2) + 0.0207\times X(5) \times X(10) + 6.63\times X(6) \times X(9) - 7.7\times X(7) \times X(8) \\
&\quad + 0.32\times X(9) \times X(10)) \\
G4(x) &= -0.059 + (-0.0159\times X(1) \times X(2) - 0.188\times X(1) \times X(8) - 0.019\times X(2) \times X(7) + 0.0144\times X(3) \times X(5) \\
&\quad + 0.000875\times X(5) \times X(10) + 0.08045\times X(6) \times X(9) + 0.00139\times X(8) \times X(11) + 0.0001575\times X(10) \times X(11)) \\
G5(x) &= -0.16 + (0.00817\times X(5) - 0.131\times X(1) \times X(8) - 0.0704\times X(1) \times X(9) + 0.03099\times X(2) \times X(6) \\
&\quad - 0.018\times X(2) \times X(7) + 0.0208\times X(3) \times X(8) + 0.121\times X(3) \times X(9) - 0.00364\times X(5) \times X(6) \\
&\quad + 0.007715\times X(5) \times X(10) - 0.005354\times X(6) \times X(10) + 0.00121\times X(8) \times X(11) + 0.00184\times X(9) \times X(10) \\
&\quad - 0.018\times X(2) \times **2) \\
G6(x) &= 0.42 + (-0.61\times X(2) - 0.163\times X(3) \times X(8) + 0.001232\times X(3) \times X(10) - 0.166\times X(7) \times X(9) + 0.227\times X(2) \times **2) \\
G7(x) &= 0.72 + (-0.5\times X(4) - 0.19\times X(2) \times X(3) - 0.0122\times X(4) \times X(10) + 0.009325\times X(6) \times X(10) \\
&\quad + 0.00191\times X(11) \times **2) \\
G8(x) &= 0.68 + (-0.674\times X(1) \times X(2) - 1.95\times X(2) \times X(8) + 0.02054\times X(3) \times X(10) - 0.0198\times X(4) \times X(10) \\
&\quad + 0.028\times X(6) \times X(10)) \\
G9(x) &= 0.75 + (-0.489\times X(3) \times X(7) - 0.843\times X(5) \times X(6) + 0.0432\times X(9) \times X(10) - 0.0556\times X(9) \times X(11) \\
&\quad - 0.000786\times X(11) \times **2) \\
G10(x) &= 0.16 + (-0.371\times X(2) \times X(4) - 0.00931\times X(2) \times X(10) - 0.484\times X(3) \times X(9) + 0.01343\times X(6) \times X(10))
\end{align*}
\]

Using the proposed MPS method and the SQP optimizer, the possibility-based optimal design is obtained as shown in Figure 5, Table 8, and Table 9. For each optimization iteration and each possibility constraint, the MPP search process just take two function evaluations just like the example in Section 5.2.
Table 8. PBDO History Using PMA and SQP Optimizer (1)

<table>
<thead>
<tr>
<th>Iter.</th>
<th>Cost</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>$d_7$</th>
<th>$d_8$</th>
<th>$d_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>29.05</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.300</td>
<td>0.300</td>
<td>0.300</td>
</tr>
<tr>
<td>1</td>
<td>26.75</td>
<td>0.765</td>
<td>1.350</td>
<td>0.500</td>
<td>1.358</td>
<td>1.120</td>
<td>1.200</td>
<td>0.400</td>
<td>0.300</td>
<td>0.300</td>
</tr>
<tr>
<td>2</td>
<td>26.68</td>
<td>0.764</td>
<td>1.350</td>
<td>0.500</td>
<td>1.424</td>
<td>0.933</td>
<td>1.200</td>
<td>0.400</td>
<td>0.300</td>
<td>0.300</td>
</tr>
<tr>
<td>3</td>
<td>26.66</td>
<td>0.764</td>
<td>1.350</td>
<td>0.500</td>
<td>1.418</td>
<td>0.933</td>
<td>1.200</td>
<td>0.400</td>
<td>0.300</td>
<td>0.300</td>
</tr>
<tr>
<td>Opt.</td>
<td>26.66</td>
<td>0.764</td>
<td>1.350</td>
<td>0.500</td>
<td>1.418</td>
<td>0.933</td>
<td>1.200</td>
<td>0.400</td>
<td>0.300</td>
<td>0.300</td>
</tr>
</tbody>
</table>

Figure 5. PBDO design history

Table 9. PBDO History Using PMA and SQP Optimizer (2)

<table>
<thead>
<tr>
<th>Iter.</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>$G_4$</th>
<th>$G_5$</th>
<th>$G_6$</th>
<th>$G_7$</th>
<th>$G_8$</th>
<th>$G_9$</th>
<th>$G_{10}$</th>
<th>No. of Analyses</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.55</td>
<td>-1.60</td>
<td>-1.21</td>
<td>-0.08</td>
<td>-0.10</td>
<td>-0.02</td>
<td>0.15</td>
<td>-0.29</td>
<td>-0.23</td>
<td>-0.23</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>-0.00</td>
<td>-2.56</td>
<td>-1.46</td>
<td>-0.06</td>
<td>-0.09</td>
<td>-0.01</td>
<td>0.03</td>
<td>-0.49</td>
<td>-0.17</td>
<td>-0.43</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>0.00</td>
<td>-2.55</td>
<td>-1.46</td>
<td>-0.07</td>
<td>-0.09</td>
<td>-0.01</td>
<td>0.00</td>
<td>-0.49</td>
<td>0.00</td>
<td>-0.46</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>0.00</td>
<td>-2.55</td>
<td>-1.45</td>
<td>-0.07</td>
<td>-0.09</td>
<td>-0.01</td>
<td>0.00</td>
<td>-0.49</td>
<td>0.00</td>
<td>-0.46</td>
<td>20</td>
</tr>
</tbody>
</table>

VI. Discussions and Conclusion

When little information is available for input data, the possibility-based method is better since it is easier to identify the most conservative possibilistic design than the probabilistic design that is consistent with the limited available information, which is a desirable merit, since a conservative optimum design is preferred when accurate statistical information is not available.

This paper proposed a maximal possibility Search (MPS) method to resolve disadvantages of vertex method and multilevel-cut method, such that it evaluates possibility constraints efficiently and accurately for nonlinear structural applications. In this paper, numerical results demonstrated that the proposed MPS method successfully performs fuzzy analysis for highly nonlinear examples. A PBDO method is proposed using the new PMA formulation and proposed MPS fuzzy analysis method. Using PMA, a new formulation of PBDO was formulated in this paper. It is shown that it can provide numerical benefits, such as efficiency, stability, and accuracy.

Acknowledgments

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