A SEQUENTIAL ADJOINT VARIABLE METHOD IN DESIGN SENSITIVITY ANALYSIS OF NVH PROBLEMS

N.H. Kim, K.K. Choi, and J. Dong
Center for Computer-Aided Design
College of Engineering
The University of Iowa
Iowa City, IA 52242
nkim@ccad.uiowa.edu

C. Pierre, N. Vlahopoulos, Z.D. Ma, M. Castanier
Automotive Research Center
College of Engineering
The University of Michigan
Ann Arbor, MI 48109

ABSTRACT
A design sensitivity analysis of a sequential structural-acoustic problem is presented. A frequency response analysis is used to obtain the dynamic behavior of an automotive structure, while the boundary element method is used to solve the pressure response of an interior, acoustic domain. For the purposes of design sensitivity analysis, a direct differentiation method and an adjoint variable method are presented. In the adjoint variable method, an adjoint load is obtained from the acoustic boundary element re-analysis, while the adjoint solution is calculated from the structural dynamic re-analysis. The evaluation of pressure sensitivity only involves a numerical integration process for the structural part. The proposed sensitivity results are compared to finite difference sensitivity results with excellent agreement.

KEYWORDS
Design Sensitivity Analysis, Structural Acoustics, Adjoint Variable Method, and Boundary Element Method.

1. INTRODUCTION
A structure-induced noise and vibration control is an important area of research for reducing the noise level generated by various structural parts. Some research results have been reported in the design sensitivity analysis (DSA) of a structural-acoustic problem. Choi et al. developed the DSA of a coupled structural-acoustic problem using a finite element method (FEM). However, their approach is limited to low frequency ranges, because the element size in an acoustic domain has to be small enough to capture the short wavelength in mid/high frequency ranges. To avoid the problems associated with a large number of elements in an acoustic domain, Salagame et al. presented an analytical sensitivity method using a Rayleigh integral. The sensitivity of a velocity is obtained by differentiating the frequency-response matrix equation, and the pressure sensitivity is then calculated by differentiating the Rayleigh integral. This approach is limited to a flat plate problem.

Compared to FEM, the boundary element method (BEM) has several advantages in the structural-acoustic problem. First, it is unnecessary to generate a complicated, 3D acoustic model. The initial structural model can be used for an acoustic analysis. In addition, a mid-frequency range can be handled in BEM, in which FEM requires a small element size within an acoustic domain. Several research studies have been conducted for DSA using BEM. Assuming that the structure’s velocity sensitivity is known, Smith and Bernhard developed a semi-analytical design sensitivity formulation, and both Kane et al. and Matsumoto et al. presented an analytical design sensitivity formulation. For the general structure-induced noise problem, however, the velocity sensitivity is unknown and has to be calculated from the structural frequency response analysis.

In this paper, a design sensitivity analysis of a sequential structural-acoustic problem is presented. In the case of a harmonic excitation, the dynamic behavior of a structure is described using a frequency response analysis. A boundary element method is used to calculate the radiated noise (pressure) from the structural response. Instead of differentiating a discrete matrix equation, a continuous variational equation is differentiated with respect to the design parameter.

While a direct differentiation method in DSA follows the same solution process as the response analysis, an adjoint variable method follows a reverse process. One of the biggest challenges of the adjoint variable method in the sequential problem is how to effectively and practically formulate this reverse process. For example, in the transient response DSA developed by Haug et al., the adjoint equation becomes a terminal-value problem, whereas the response problem is an initial-value problem. Such an opposite solution process in the adjoint problem causes a significant amount of inconvenience and ineffectiveness in the DSA. To overcome these difficulties, a sequential adjoint variable method is presented in which the adjoint load is obtained from the boundary element re-analysis, and the adjoint variable is calculated from the structural dynamic
re-analysis. To our knowledge, no research results have been reported in the development of the adjoint variable method in a sequential problem.

2. REVIEW OF A STRUCTURAL-ACOUSTIC ANALYSIS

2.1 Frequency Response Analysis

Consider a structure under dynamic load \( F(x,t) \). The differential equation that governs the behavior of this hyperbolic system can be written as

\[
\rho \ddot{y}_x(x,t) + C_y(x,t) + L_y(x,t) = F(x,t), \quad x \in \Omega^S, \quad t > 0
\]

(1)

where \( \Omega^S \) is the structure's domain, \( y(x,t) \) the displacement, \( L(x) \) the linear partial differential operator, \( \rho(x) \) the structural mass density, and \( C(x) \) the viscous damping effect. The subscribed comma denotes the derivative with respect to time, i.e., \( y, = \partial y/\partial t \) (velocity) and \( y,, = \partial^2 y/\partial t^2 \) (acceleration). The initial conditions of the dynamic problem are given by

\[
y(x,0) = y^0(x), \quad x \in \Omega^S
\]

(2)

\[
y'(x,0) = y^0_0(x), \quad x \in \Omega^S
\]

where \( y^0(x) \) is the initial displacement, and \( y^0_0(x) \) is the initial velocity.

For the steady-state response, the time-dependent terms from Eq. (1) should be removed. Since the harmonic load is being considered, \( F(x,t) \) can be expressed as

\[F(x,t) = f(x)e^{j \omega t}\]

(3)

where \( f(x) \) is the magnitude of the harmonic load and \( \omega \) is the load frequency, which is considered a constant. In contrast, the steady-state response has the same frequency as the applied load but may have a different phase angle. Using the complex variable method, the displacement \( y(x,t) \) can be expressed as

\[y(x,t) = z(x)e^{j \omega t}\]

(4)

where \( z(x) \) is the complex displacement. Thus, the structure's displacement oscillates with shape \( z(x) \) and frequency \( \omega \).

Time-dependency of the dynamic problem can be eliminated by substituting Eqs. (3) and (4) into Eq. (1), to obtain the spatial state operator equation as

\[-j \omega^2 \rho \dot{z}(x) + j \omega C \dot{z}(x) + L \phi(x) = F(x), \quad x \in \Omega^S
\]

(5)

The variational formulation of Eq. (5) is similar to the static problem. However, since complex variable \( z(x) \) is used for the state variable, complex conjugate \( \bar{z}^* \) is used for the displacement variation. By multiplying \( \bar{z}^* \) and integrating it over the domain \( \Omega^S \), the variational equation can be derived after integration by parts for differential operator \( L \) as

\[
\int_{\Omega^S} \{ -j \omega^2 \rho \bar{z}^* + j \omega C \bar{z}^* \} d\Omega^S + \int_{\partial \Omega^S} \sigma(z)^T \sigma(e(z^*)) d\Omega^S
\]

(6)

\[
= \int_{\Omega^S} f^T \bar{z}^* d\Omega^S + \int_{\partial \Omega^S} f^T \bar{z}^* d\Gamma, \quad \forall \Omega \in Z
\]

where \( \bar{z}^* \) is the complex conjugate of the kinematically admissible virtual displacement \( \bar{z} \), and \( Z \) is the complex space of kinematically admissible virtual displacements. Equation (6) provides the variational equation of the dynamic frequency response under an oscillating excitation with frequency \( \omega \). For derivational convenience, the following terms are defined:

\[
d_a(z, \bar{z}^*) = \int_{\Omega^S} \rho \bar{z}^* \bar{z}^* d\Omega^S
\]

(7)

\[
c_a(z, \bar{z}^*) = \int_{\Omega^S} C \bar{z}^* \bar{z}^* d\Omega^S
\]

(8)

\[
a_a(z, \bar{z}^*) = \int_{\Omega^S} \sigma(z)^T \sigma(e(z^*)) d\Omega^S
\]

(9)

\[
\ell_a(\bar{z}^*) = \int_{\Omega^S} f^T \bar{z}^* d\Omega^S + \int_{\partial \Omega^S} f^T \bar{z}^* d\Gamma
\]

(10)

where \( d_a(\bullet, \bullet) \) is the load form, \( c_a(\bullet, \bullet) \) is the kinetic form, \( a_a(\bullet, \bullet) \) is the structural energy form, and \( \ell_a(\bullet) \) is the load form. Subscribed \( u \) is used to denote that these forms depend on \( u \).

Since the structure-induced pressure within the acoustic domain is related to the velocity response, it is convenient to transfer displacement to velocity using the following relation:

\[v(x) = j \omega \bar{z}(x)\]

(11)

By using Eqs. (6)–(11), the variational equation of the frequency response problem can be obtained as

\[j \omega d_a(v, \bar{z}^*) + c_a(v, \bar{z}^*) + \frac{1}{j \omega a_a(v, \bar{z}^*)} = \ell_a(\bar{z}^*), \quad \forall \Omega \in Z\]

(12)

Structural damping, a variant of viscous damping, is caused either by internal material friction or by the connection between structural components. It has been experimentally observed that for each cycle of vibration the dissipated energy of the material is proportional to displacement. The variational equation with the structural damping effect is

\[j \omega d_a(v, \bar{z}^*) + \kappa a_a(v, \bar{z}^*) = \ell_a(\bar{z}^*), \quad \forall \Omega \in Z\]

(13)

where \( \kappa = (\varphi - j)/\omega \) and \( \varphi \) is the structural damping. In the case of an un-damped structure, \( \kappa \) must be understood as \(-j/\omega\).

After the structure is approximated using finite elements, and kinematic boundary conditions are applied, the following system of matrix equations is obtained:

\[j \omega M + \kappa K \{ \bar{v}(\omega) \} = \{ f(\omega) \}\]

(14)

where \( \{ M \} \) is the mass matrix and \( \{ K \} \) is the stiffness matrix.

2.2 Acoustic Boundary Element Method

From the structure’s velocity result, a boundary element method is used to evaluate pressure response in an acoustic domain. The standard wave equation is reduced to the Helmholtz equation in the harmonic response problem as

\[\nabla^2 p + k^2 p = 0\]

(15)

where \( p \) is the pressure, \( k = c/\omega \) is the wave number, \( c \) is the velocity of the wave propagation, and \( \nabla^2 \) is the Laplace operator.

In BEM, the structural behavior must first be computed, and then it can be used as a boundary condition to compute radiated noise \( p \). If an acoustic domain is considered to be in \( R^3 \), then the boundary of this domain constitutes the structure’s domain, \( \Omega^S \). By integrating over the domain and by using Green’s theorem, the Helmholtz equation (15) constitutes the boundary integral
equation \(^{11}\) as

\[
\int_{\Omega} \left( G(x, x_0) \frac{\partial p}{\partial n} - p(x) \frac{\partial G}{\partial n} \right) d\Omega' = \alpha p(x_0)
\] (16)

where \(G(x, x_0)\) is Green's function, \(x\) is the position of a reference point, \(x_0\) is the position of an observation point, \(\partial/\partial n\) is the normal component of the gradient, and \(S\) is the acoustic boundary, which is again a structural domain. In Eq. (16), constant \(\alpha\) is equal to 1 for \(x_0\) inside the acoustic volume, 0.5 for \(x_0\) on a smooth boundary surface, and 0 for \(x_0\) outside the acoustic volume.

On the surface of the acoustic boundary, the following relation between the pressure and structure's velocity is given:

\[
\nabla p = -j \rho \omega v
\] (17)

where \(\rho\) is the structural density and \(v\) is the acoustic velocity, which was computed from the frequency response in Eq.(13). If \(x_s\) is a point on the acoustic boundary surface, then the boundary integral equation (16) becomes

\[
\int_{\Omega} \left[ -j \rho \omega G(x_s, x_0) v_s(x_0) - \frac{\partial G}{\partial n} p(x_s) \right] d\Omega' = \alpha p(x_0)
\] (18)

where \(v_s\) is the normal component of surface velocity \(v\). For derivational convenience, Eq. (18) can be rewritten as follows:

\[
b(x_0, v) + e(x_0, p_3) = \alpha p(x_0)
\] (19)

where \(b(x_0, \cdot)\) and \(e(x_0, \cdot)\) are linear integral forms that correspond to the left side of Eq. (18). Note that these integral forms are independent of the sizing design variable, thus no subscript is used in their definitions.

The boundary element method has two steps: first evaluating the pressure variable on the acoustic boundary using the structural velocity, and then calculating the pressure variable within the acoustic domain using boundary pressure information. Let the acoustic boundary \(S\) be approximated by \(N\) number of nodes. If observation point \(x_0\) is positioned at every node, then the following linear system of equations is obtained:

\[
[A] [p]\ = \ [B] [v]
\] (20)

where \([p]\) is the nodal pressure vector, \([v]\) is the 3N×1 velocity vector, \([A]\) is the \(N\times N\) coefficient matrix, and \([B]\) is the \(N\times 3N\) coefficient matrix. Note that these vectors and matrices are all complex variables. The process of computing the boundary pressure \([p]\) assumes domain discretization, and the condition in Eq. (19) is imposed in every node. However, for the purposes of DSA, let us consider a continuous counterpart to Eq. (20), defined as

\[
A(p_3) = B(v)
\] (21)

where the integral forms \(A(\cdot)\) and \(B(\cdot)\) correspond to the matrices \([A]\) and \([B]\) in Eq.(20), respectively. The boundary pressure can then be calculated from \(p_3 = A^{-1}B(v)\).

Once \([p]\) has been computed, Eq. (18) can be used to compute the acoustic pressure at any point \(x_0\) within the acoustic domain in the form of a vector equation as

\[
p(x_0) = \{b(x_0)\}^\dagger \{v\} + \{e(x_0)\}^\dagger \{p_3\}
\] (22)

where \([b(x_0)]\) and \([e(x_0)]\) are the column vectors that correspond to the left-hand side of the boundary integral Eq. (18).

In a sizing design problem, in which panel thickness is a design variable, integral forms \(b(x_s;\cdot)\) and \(e(x_0;\cdot)\) in Eq. (19) are independent of the design variable. Only implicit dependence on the design exists through the state variable \(v\) and \(p\), which will be developed in the following section.

3. DESIGN SENSITIVITY ANALYSIS

The purpose of design sensitivity analysis (DSA) is to compute the dependence of performance measures on the design. In this study, only a sizing design variable is considered, such as the thickness of a plate and the cross-sectional dimension of a beam. The simplicity of a sizing design variable is that the integration domain remains fixed for different designs.

3.1 Design Sensitivity Formulas

Let us begin by defining a variation that will be frequently used in the following derivations. Let \(\psi\) be a function that depends on current design \(u\) and assume that \(\psi(u)\) is continuous with respect to design \(u\). If the current design is perturbed in the direction of \(\delta u\) (arbitrary), and \(\tau\) is a parameter that controls the perturbation size, then the variation of \(\psi(u)\) in the direction of \(\delta u\) is defined as

\[
\psi' = \frac{d}{d\tau} \psi(u + \tau \delta u) \bigg|_{\tau=0} = \frac{\partial \psi}{\partial u} \delta u
\] (23)

Throughout this paper, prime “’” plays precisely the same role as the first variation in the calculus of variations. For convenience, subscripted \(\delta u\) will often be ignored. The term “derivative” or “differentiation” will often be used to denote the variation in Eq. (23). If the variation of a function is continuous and linear with respect to \(\delta u\), the function is differentiable (even more precisely, it is Fréchet differentiable). For complicated problems, it is difficult to prove the differentiability of a general function with respect to the design. Without proof of differentiability, design sensitivity formulas will be developed as follows.

It is also assumed that the solution to the frequency response problem in Eq. (13) and the solution to the boundary integral equation (19) are differentiable with respect to the design. That is, the following forms of variation exist:

\[
v' = \frac{d}{d\tau} [v(x, u + \tau \delta u)] \bigg|_{\tau=0} = \frac{\partial v}{\partial u} \delta u
\] (24)

\[
p' = \frac{d}{d\tau} [p(x, u + \tau \delta u)] \bigg|_{\tau=0} = \frac{\partial p}{\partial u} \delta u
\] (25)

3.2 Direct Differentiation Method

A direct differentiation method computes the variation of state variables in Eqs. (24) and (25) by differentiating the state equations (13) and (19) with respect to the design. Let us first consider the structural part, i.e., the frequency response analysis in Eq. (13). The forms that appear in Eq. (13) explicitly depend on the design, and their variation can be defined as
\[ d'_{\text{sa}}(v, \tilde{v}) = \frac{d}{d\tau} \left( d_{\text{sa}}(\tilde{v}, \tilde{v}) \right) \bigg|_{v=0} \tag{26} \]
\[ a'_{\text{sa}}(v, \tilde{v}) = \frac{d}{d\tau} \left( a_{\text{sa}}(\tilde{v}, \tilde{v}) \right) \bigg|_{v=0} \tag{27} \]
\[ \ell'_{\text{sa}}(\tilde{v}) = \frac{d}{d\tau} \left( \ell_{\text{sa}}(\tilde{v}) \right) \bigg|_{v=0} \tag{28} \]

where \( \tilde{v} \) denotes state variable \( v \) with the dependence on \( \tau \) being suppressed, and \( \tilde{v} \) and its complex conjugate are independent of the design. The detailed expressions of \( d'_{\text{sa}}(\cdot, \cdot) \), \( a'_{\text{sa}}(\cdot, \cdot) \), and \( \ell'_{\text{sa}}(\cdot) \) will be developed in the analytical example section.

Thus, by taking a variation of both sides of Eq. (13) with respect to the design, and by moving terms explicitly dependent on the design to the right-hand side, the following sensitivity equation can be obtained:
\[ j\omega d_i(v', \tilde{v}^*) + \kappa a_i(v', \tilde{v}^*) = \ell'_{\text{sa}}(\tilde{v}^*) - j\omega d_{\text{sa}}(v, \tilde{v}^*) - \kappa a_{\text{sa}}(v, \tilde{v}^*) \quad \forall \tilde{v} \in Z \tag{29} \]

Presuming that velocity \( v \) is given as a solution to Eq. (13), Eq. (29) is a variational equation, with the same energy bilinear forms for displacement variation \( v' \). Note that the system matrices corresponding to Eqs. (13) and (29) are the same, and that the right side of Eq. (29) can be considered a fictitious load term. If design perturbation \( \delta u \) is defined, and if the right-hand side of Eq. (29) is evaluated with the solution to Eq. (13), then Eq. (29) can be numerically solved to obtain \( v' \) using the finite element method. By interpreting the right-hand side of Eq. (29) as another load, Eq. (29) can be solved by using the same solution process as the frequency response problem in Eq. (13).

Now the acoustic aspect will be considered, which is represented by the boundary integral equation (19). A direct differentiation of Eq. (19) yields the following sensitivity equation:
\[ b(x_0; v') + e(x_0; p'_x) = \alpha p'(x_0) \tag{30} \]

Since integral forms \( b(x_0; \cdot) \) and \( e(x_0; \cdot) \) are independent of the design, the above equation has exactly the same form as Eq. (19). Thus, using the solution (\( v' \)) to the structural sensitivity equation (29), Eq. (30) can be solved by following the same solution process as BEM, to yield the pressure sensitivity result. No additional numerical implementation is required in the DSA process. Thus, the following matrix equation has to be solved in the discrete system:
\[ \{ A \} \{ p'_x \} = \{ B \} \{ v' \} \tag{31} \]

Then, pressure sensitivity at point \( x_0 \) can be obtained from
\[ p'(x_0) = \{ b(x_0) \}^T \{ v' \} + \{ e(x_0) \}^T \{ p'_x \} \tag{32} \]

This sensitivity calculation process is exactly same as the BEM solution process described from Eq. (20) to Eq. (22).

**Structural Performance Measure**

A general performance measure that represents a variety of structural responses can be written in integral form as
\[ \psi_l = \int_{\Omega^s} g(v, u) \ d\Omega^s \quad \tag{33} \]
where function \( g(v, u) \) is assumed to be continuously differentiable with respect to its arguments. The reason for introducing the integral form of a performance measure is that in FEM the point-wise definition of a function is meaningless, since the variational formulation enforces the definition of a function value in the sense of a Sobolev norm.\(^{12}\) This will be compared to BEM, in which a function can be defined at a point. Note that \( \psi_l \) is a complex functional in the frequency response analysis.

The variation of \( \psi_l \) with respect to the design becomes
\[ \psi'_l = \frac{d}{d\tau} \left[ \int_{\Omega^s} g(v(x); u + \tau \delta u), u + \tau \delta u) \ d\Omega^s \right] \bigg|_{\tau=0} \tag{34} \]

\[ = \int_{\Omega^s} (g'_v v' + g'_u \delta u) \ d\Omega^s \]

where \( g'_v = \partial g/\partial v \) and \( g'_u = \partial g/\partial u \) are column vectors, and their expressions are known from the definition of function \( g \). The objective of DSA is to obtain an explicit expression of \( \psi'_l \) in terms of \( \delta u \). If the structural design sensitivity equation (29) is solved for the variation \( v' \), then the sensitivity of \( \psi_l \) can be calculated from Eq. (34) using the numerical integration process.

**Acoustic Performance Measure**

Consider a performance measure that is defined at point \( x_0 \) within an acoustic domain as
\[ \psi_2(x_0) = h(p(x_0), u(x_0)) \tag{35} \]
where the function \( h(p,u) \) is assumed to be continuously differentiable with respect to its arguments. Note that acoustic performance \( \psi_2 \) is not defined in the integral form, as was the case for structural performance \( \psi_l \).

The variation of the performance measure with respect to the design variable becomes
\[ \psi'_2 = \frac{d}{d\tau} \left[ h(p(x); u + \tau \delta u), u + \tau \delta u) \right] \bigg|_{\tau=0} \tag{36} \]

\[ = h'_p p' + h'_u \delta u \]

where the expression of \( h'_p = \partial h/\partial p \) and \( h'_u = \partial h/\partial u \) are known from the definition of the function \( h \). Thus, from the solution to the acoustic design sensitivity equation (30), the sensitivity of \( \psi_2 \) can readily be calculated. However, the calculation of \( p' \) also requires the solution to the structural sensitivity equation (29).

### 3.3 Adjoint Variable Method

Since the number of design variables is larger than the number of active constraints in most optimization problems, the adjoint variable method is more efficient than the direct differentiation method.\(^{3}\) However, the adjoint variable method is known to be limited in the case of a symmetric operator problem. In this section, the adjoint variable method is further extended to non-symmetric complex operator problems. Since the adjoint variable method is directly related to the performance measure type, structural and acoustic performance measures are treated
Structural Performance Measure
To obtain an explicit expression for \( \psi' \) in terms of \( \delta u \), it is necessary to rewrite the first term in Eq. (34) explicitly in terms of \( \delta u \). As with the static problem, an adjoint equation can be introduced by replacing \( v' \) in Eq. (34) with the complex virtual displacement \( \widetilde{u} \) and by equating it to the variational equation (13) with respect to adjoint variable \( \lambda' \) as

\[
\int_{\Omega} g_{\lambda}^* \lambda' d\Omega^2, \quad \forall \lambda' \in \mathbb{R}
\]

where an adjoint solution, \( \lambda' \), or equivalently its complex conjugate \( \lambda'^* \), is desired. Since Eq. (37) satisfies for all \( \lambda' \in \mathbb{R} \), and since \( \delta v' \in \mathbb{R} \), Eq. (37) may be evaluated at \( \lambda' = v' \), to obtain

\[
\int_{\Omega} g_{\lambda}^* v' d\Omega^2
\]

where the right-hand side corresponds to the adjoint load in the discrete system. Instead of computing the inverse matrix, let us consider a discrete form of the adjoint load. Equation (43) can be written in the discrete system as

\[
[[\omega + kK]\{\lambda'\} = \{b\} + [B]^T\{\eta\}
\]

where the right-hand side corresponds to the adjoint load in the discrete system. Instead of computing the inverse matrix, let us define an acoustic adjoint problem in BEM as

\[
[A]\{\eta\} = \{e\}
\]

where the acoustic adjoint solution \( \{\eta\} \) is desired. Even if the coefficient matrix \( [A] \) is not symmetric, the adjoint equation (46) can still use the factorized matrix of the boundary element equation (20). By substituting \( \{\eta\} \) into Eq. (45), we obtain the structural adjoint problem, as

\[
[[\omega + kK]\{\lambda'\} = \{b\} + [B]^T\{\eta\}
\]

\[
\{\omega + kK\} = [A]^T\{\eta\}
\]

3.4 Analytical Example
In many structural-acoustic problems, a structural part is described by using a plate/shell component, and an acoustic domain is the region the structure surrounds. A typical design problem would reduce sound pressure levels in the passenger position by changing the plate thickness. In such a problem, the design variable is the thickness of a plate/shell component, and the performance measure is the sound pressure level at a point in the acoustic domain. Moreover, in order to reduce the radiated acoustic power from the structure, the structure’s velocity is also considered as a performance measure.

The structure’s variational equation of harmonic motion is given by Eq. (13). The objective is to derive explicit forms of \( a_0(a, \cdot, \cdot), a_0, L_0(a, \cdot) \) for a plate/shell component. In general, a shear-deformable plate/shell has three translation degrees-of-freedom and two rotational degrees-of-freedom. Thus, the structural state variable \( z \) is defined by

\[
\int_{\Omega} g_{\lambda}^* \lambda' d\Omega^2, \quad \forall \lambda' \in Z
\]

where an adjoint solution \( \lambda' \) is desired. By following the same process, described from Eqs. (37) to (41), the sensitivity of \( \psi_2 \) can be obtained as

\[
\psi' = h^* \delta u + \lambda' \lambda - [a_0 \lambda' \lambda - \kappa a_0 \lambda' \lambda]
\]

It is interesting to note that even if \( \psi_2 \) is a function of pressure \( p \), its sensitivity expression in Eq. (44) does not require the value of \( p \); only the structural solution \( \lambda' \) and the adjoint solution \( \lambda' \) are required in the calculation of \( \psi' \).

Even if Eq. (44) looks similar to the structural performance measure in Eq. (41), a fundamental difference exists in the calculation of the adjoint load in Eq. (43). To illustrate, let us consider a discrete form of the adjoint load. Equation (43) can be written in the discrete system as

\[
[[\omega + kK]\{\lambda'\} = \{b\} + [B]^T\{\eta\}
\]

Note that the acoustic adjoint solution \( \{\eta\} \), which is obtained from BEM, is required to compute the structural adjoint load, and the frequency response re-analysis then provides the structural adjoint solution \( \{\lambda'\} \). Thus, two different adjoint problems are defined: the first is similar to BEM, and is used to compute the adjoint load, while the second is similar to the structural frequency-response problem.
\( \mathbf{z} = [z_1, z_2, z_3, \theta_1, \theta_2]^T \) 

Strain is decomposed into membrane, bending, and transverse shear parts as

\[
\mathbf{e}^m = \begin{bmatrix} z_{1,1} \\ z_{2,2} \\ z_{1,2} + z_{2,1} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \theta_{1,1} \\ \theta_{2,2} \\ \theta_{1,2} + \theta_{2,1} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} z_{1,2} - \theta_2 \\ z_{3,1} - \theta_1 \end{bmatrix}
\]

(49)

Note that the strain resultants given in Eq. (49) have the following properties: \( \theta_{1,1} \) is the curvature in the \( x_1 \) direction, \( \theta_{2,2} \) is the curvature in the \( x_2 \) direction, and \( (\theta_{1,2} + \theta_{2,1}) \) is the twisting curvature. In Eq. (49), \( z_{3,2} - \theta_2 \) and \( z_{3,1} - \theta_1 \) are the shear rotation in the 2–3 and 1–3 plane, respectively. By using the above definitions, the structural energy form is defined as

\[
a_{s}(z, \varepsilon^m) = \int_T \left[ \varepsilon^m (\varepsilon^m)^T + \frac{E}{2(1+v)} \right] \varepsilon^m d\Omega^t
\]

(50)

where

\[
C = \frac{E}{1-v^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix}, \quad D = \frac{E\zeta}{2(1+v)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

and \( \zeta \) is the shear correction factor. The three terms within the integral of Eq. (50) represent the membrane, bending, and transverse shear contribution. Note that the design variable \( h \) is explicitly denoted for DSA convenience. Since the calculation of \( a_{s}(z, \varepsilon^m) \) is the same as differentiating \( a_{s}(\varepsilon^m) \) with respect to \( h \), but without considering the effect of the state variable, the following formula can be obtained:

\[
a_{s}(z, \varepsilon^m) = \int_T \left[ \varepsilon^m (\varepsilon^m)^T C \varepsilon^m + \frac{E}{4} \kappa (\varepsilon^m)^T C \kappa (\varepsilon^m) + \frac{E}{2} \zeta (\varepsilon^m)^T D \zeta (\varepsilon^m) \right] d\Omega^t
\]

(52)

From the known structural solution \( z \) (or \( v \)), the structural variation of Eq. (52) can be calculated using the numerical integration process.

Usually, it is easier to define kinetic energy form \( d_s(\varepsilon^m) \) than form \( a_{s}(\varepsilon^m) \). For the plate/shell component, the kinetic energy form and its variation can be defined as

\[
d_s(\varepsilon^m) = \int_T \rho \frac{\partial^2 \varepsilon^m}{\partial t^2} \varepsilon^m d\Omega^t
\]

(53)

\[
d_s'(\varepsilon^m) = \int_T \rho \frac{\partial^2 \varepsilon^m}{\partial t^2} \delta h \varepsilon^m d\Omega^t
\]

(54)

If the applied load consists of externally applied pressure \( F(x) \) and the self-weight given by

\[
f(x) = F(x) + \beta gh(x)
\]

where \( \beta \) is the weight density of the plate, and \( g \) is a unit vector in the direction of gravity, then the load linear form \( f_s(\varepsilon^m) \) and its variation can be defined as

\[
f_s(\varepsilon^m) = \int_T \left[ F + \beta gh \right] \varepsilon^m d\Omega^t
\]

(56)

\[
f_s'(\varepsilon^m) = \int_T \beta g^T \varepsilon^m \delta h d\Omega^t
\]

(57)

Consider an acoustic cavity with a flexible panel, as illustrated in Figure 1. The cavity is surrounded on all but one side by rigid walls, and the open side is closed by a clamped panel of linear elastic material with the structural damping coefficient \( \varphi \). The panel’s uniform thickness, \( h \), is selected as the design variable, i.e., \( u(\mathbf{x}) = \{h\} \). Let us consider such performance measures as the acoustic pressure \( p(x^a) \) at point \( x^a \) in the acoustic cavity, and the \( x^3 \)-directional velocity \( v_3(x^a) \) at point \( x^a \) on the structural panel. A harmonic force \( f(x^f, t) \) with frequency \( \omega \) is applied to the plate. Here, \( f(x, t) \) is assumed to be independent of the design variable \( u(x) \).

| Figure 1 Acoustic Cavity with Flexible Wall |

**Structural Performance Measure**

The performance measure in this example is the vertical velocity at point \( x^a \), whose mathematical expression is

\[
\psi_1 = v_3(x^a) = \int_{\Omega^t} \delta(x-x^a) v_3 d\Omega^t
\]

(58)

where \( \delta(\cdot) \) is the Dirac-delta measure at zero. Equation (58) is a simple form of Eq. (33), which is the general form for a structural performance measure. The variation of \( \psi_1 \) is

\[
\psi_1' = v_3'(x^a) = \int_{\Omega^t} \delta(x-x^a) v_3' d\Omega^t
\]

(59)

Working from Eq. (37), the corresponding adjoint equation is obtained as

\[
j_0 d_s'(\varepsilon^m, \lambda^c) + \kappa a_{s}(\varepsilon^m, \lambda^c) = \int_{\Omega^t} \delta(x-x^a) \lambda^c d\Omega^t, \quad \forall \lambda^c \in Z
\]

(60)

In Eq. (60), the term on the right side is the adjoint load for the structural velocity. The physical meaning of the adjoint load, which corresponds to the harmonic velocity at a point, is a unit harmonic force applied at point \( x^a \). The design sensitivity of \( \psi_1 \) is obtained from Eq. (41) as

\[
\psi_1' = c_{s}(\lambda^c) - j_0 d_s'(\varepsilon^m, \lambda^c) - \kappa a_{s}(\varepsilon^m, \lambda^c)
\]

(61)

Substituting the variations of those forms in Eqs. (52), (54), and (57), the design sensitivity expression becomes
\[ \psi'(x) = \int_{\Omega} \gamma^T \lambda' \delta h d\Omega^S - jo \int_{\Omega} (\rho v^T \lambda') \delta h d\Omega^S \]

\[-\kappa \int_{\Omega} \left( C e''(\lambda')^T C e''(v) + \frac{h^2}{4} \kappa(\lambda')^T \kappa(v) + \gamma(\lambda')^T D\gamma(v) \right) \delta h d\Omega^S \]  

(62)

Thus, \( \psi' \) is expressed in terms of \( \delta h \).

**Acoustic Performance Measure**

Consider a pressure performance measure at point \( x_0 \), given as

\[ \psi_2 = p(x_0) \]  

(63)

Equation (63) is a simple form of Eq. (35), a general form of the acoustic performance measure. The variation of the performance measure, corresponding to Eq. (63), is

\[ \psi'_2 = p'(x_0) = b(x_0; v') + e(x_0, A^{-1} \circ B(v')) \]  

(64)

The adjoint equation for \( \psi'_2 \) is formed from Eq. (43) as

\[ jod_a(\lambda^*, \lambda') + \kappa a \delta(\lambda^*, \lambda') \]

\[ = b(x_0, \lambda') + e(x_0, A^{-1} \circ B(\lambda)), \quad \forall \lambda \in Z \]  

(65)

The term on the right side of this equation is referred to as the acoustic adjoint load. In actual implementation, the adjoint load is calculated from the secondary adjoint problem defined in Eq. (46). The discrete adjoint problem in Eq. (47) can then be solved for \( \lambda' \). From Eq. (41), the design sensitivity expression of the acoustic pressure becomes

\[ \psi'_2 = \psi'_2(\lambda) - jod_a(v, \lambda') - \kappa a \delta(v, \lambda') \]  

(66)

Note that the design sensitivity expression in Eqs. (61) and (66) has identical forms. The same numerical integration process can be used for both structural and the acoustic performance measures. However, in the case of an acoustic performance measure, a secondary adjoint problem must be solved in order to define the structural adjoint load.

### 4. NUMERICAL EXAMPLES

#### 4.1 Numerical Method

A structural-acoustic system is solved using both finite element and the boundary element methods. The variational equation of the harmonic motion of a continuum model, Eq. (13), can be reduced to a set of linear algebraic equations by discretizing the model into elements and by introducing shape functions and nodal variables for each element. It is assumed that the structural finite element and the acoustic boundary element use the same mesh. Acoustic pressure \( p(x) \) and structural velocity \( v(x) \) are approximated using shape functions and nodal variables for each element in the discretized model as

\[ v(x) = N_v(x) v' \]

\[ p(x) = N_p(x) p' \]  

(67)

where \( N_v(x) \) and \( N_p(x) \) are matrices of shape functions for velocity and pressure, respectively, and \( v' \) and \( p' \) are the element nodal variable vectors. Substituting Eq. (67) into Eq. (13) and carrying out integration yields the same matrix equation as Eq. (14), rewritten here

\[ [ioaM + \kappa K] \{v(\omega)\} = \{f(\omega)\} \]  

(68)

After obtaining the structural velocity, BEM is used to evaluate the pressure response on the boundary, as well as within the acoustic domain, as explain in Section 2.2.

Figure 2 shows the computational procedure for the adjoint variable method with a structural FEA and an acoustic BEA code. Even if FEM and BEM are used to evaluate the acoustic performance measure, only structural response \( v \) is required to perform the design sensitivity analysis. The adjoint load is calculated from the transposed boundary element analysis, and the adjoint equations are then numerically solved using the FEA with the same finite element model used in the initial analysis.

**Figure 2 Computational Procedure of DSA**

Numerical solutions are used to compute design sensitivity, and the integration of the design sensitivity expressions in Eq. (44) can be evaluated using a numerical integration method, such as the Gauss quadrature method\(^\text{11}\).

#### 4.2 Design Sensitivity Analysis of a Box Model

Figure 3 depicts the acoustic cavity and the panel, previously discussed in Section 3.4. The acoustic medium in the cavity is air, with a mass density \( \rho_a = 1.205 \text{ Kg/m}^3 \) and a wave propagation velocity \( c = 344 \text{ m/sec}^2 \). The panel is an aluminum plate with thickness of 0.01 m, mass density \( \rho_p = 2,700 \text{ Kg/m}^2 \), Young's modulus \( E = 7.1 \times 10^{10} \text{ Pa} \), Poisson's ratio \( \nu = 0.334 \), and structural-
damping coefficient $\phi = 0.06$. Harmonic force $f = 1.0 \text{ N}$ in the $x_3$-direction is applied at the four points on the plate. The whole structure is discretized by 864 elements and 866 nodes. In the frequency response analysis, the five sides of the structure are fixed to simulate the rigid wall; only the bottom panel is allowed to move. In the acoustic analysis, the pressure value of each node is calculated from the structural velocity data and the pressure value in the acoustic cavity is then evaluated.

![Figure 3 Box Acoustic model](image)

Panel thickness is chosen as the design variable, and only one design variable is considered in this example. The following design sensitivities are considered: the acoustic pressure at points $A_1 (0.5, 0.6, 0.0)$, the interface point at the panel center, and at $A_2 (0.5, 0.6, 1.5)$, the cavity center, and the panel velocity in the $x_3$-direction at points $A_1$. The MSC/NASTRAN program is used for the direct frequency analysis of primary and adjoint structural problems, whereas BEM is used for primary and adjoint acoustic problems. Figure 4 provides the amplitude of pressure at points $A_1$ and $A_2$ for various frequency ranges between 1–140 Hz. The pick values of the pressure appear at the frequencies corresponding to the natural frequencies of the plate.

Design sensitivities are computed at 76 Hz, which is close to the resonant frequency. Design sensitivity results are shown in Table 1, in which the forward finite difference in the design sensitivity is denoted by $\Delta \psi/\Delta u$, and $\psi'$ is the predicted design sensitivity using the proposed method. Design perturbation $\Delta u = 1.0 \times 10^{-6} \text{ m}$ is used. Good agreement is obtained between $\psi'$ and $\Delta \psi/\Delta u$. Since the applied load magnitude is fixed, an increase in panel thickness reduces plate vibration and radiated pressure. Consequently, all sensitivities show negative values.

![Figure 4 Analysis Results of Acoustic Cavity with Flexible Wall](image)

<table>
<thead>
<tr>
<th>Performance</th>
<th>$\Delta \psi/\Delta u$</th>
<th>$\psi'$</th>
<th>Ratio (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement at $A_1$</td>
<td>-0.040403</td>
<td>-0.040181</td>
<td>100.55</td>
</tr>
<tr>
<td>Velocity at $A_1$</td>
<td>-19.219</td>
<td>-19.187</td>
<td>100.17</td>
</tr>
<tr>
<td>Pressure at $A_1$</td>
<td>-2191.8</td>
<td>-2223.3</td>
<td>98.58</td>
</tr>
<tr>
<td>Pressure at $A_2$</td>
<td>-1200.8</td>
<td>-1198.4</td>
<td>100.20</td>
</tr>
</tbody>
</table>

A major advantage of the adjoint variable method appears when a large number of design variables exist. In the early development stage, for example, a design engineer may want to decide on the panel thickness for each section in order to minimize acoustic noise. To this end, the element sensitivity plot (Figure 5) clearly shows the contribution of each element to the cavity pressure, and helps to determine new panel thicknesses. If the direct differentiation method is employed, then 864 design sensitivity equations must be solved in order to obtain such information, while with the adjoint variable method only one adjoint equation needs to be solved.

![Figure 5 Element Sensitivity for the Pressure at the Cavity](image)

### 4.3 Design Sensitivity Analysis of a Vehicle Model

Figure 6 shows a concept design finite element model of a hydraulic hybrid vehicle. In addition to power train vibration and wheel/terrain interaction, a hydraulic pump is also a source of vibration. Because of this additional source of excitation, vibration and noise is more significant than with a conventional vehicle. In this example, the noise level of the passenger compartment is chosen as a performance measure, and panel thicknesses are chosen as design variables. From the power train analysis and rigid-body dynamic analysis, the harmonic excitations at twelve locations are obtained. The frequency response analysis is carried out using a MSC/NASTRAN to obtain the velocity response at eight frequency values, which correspond to the structure’s natural frequencies at less than 100 Hz.

After solving the structure’s velocity response, an acoustic BEA is carried out using a cabin part boundary element model, as shown in Figure 6. Table 2 shows sound pressure levels at the position of the driver’s ear. Since the sound pressure level at 93.6 Hz is higher than other frequency values, design modification will be carried out at that frequency. Figure 7 shows the sound pressure level inside the cabin compartment. The maximum
The acoustic adjoint is calculated with respect to panel thicknesses. For ten design sensitivity results agree with the finite difference sensitivity results within a range of 10% when 0.1% of the thickness is perturbed. The numerical integration process given in Eq. (66) calculates the sensitivity of the sound pressure level at the driver’s ear between the initial and updated design. The maximum value of the sound pressure level is reduced from 77.8 dB to 75.0 dB.

As was shown in Table 3, the rail component has the highest sensitivity value for the sound pressure level. The rail is therefore increased by 1.0 mm. The whole analysis process is repeated for the modified design. Figure 9 shows sound pressure levels at the driver’s ear when the excitation frequency is 93.6 Hz at the updated design. The maximum value of the sound pressure level is reduced from 77.8 dB to 75.0 dB.

NVH performance improvement at the updated design can be investigated further by considering the pressure results around the critical frequency. Figure 10 plots the change in the level of sound pressure at the driver’s ear between the initial and updated design. Thus, sound pressure levels are effectively reduced by increasing the thickness of the rail component.

5. CONCLUSION

Based on the assumption that acoustic pressure does not influence structural behavior, design sensitivity analysis of a sequential structural-acoustic problem is presented. Using the adjoint variable method, a sequential adjoint problem is presented in which the adjoint load is calculated by solving a boundary adjoint problem, and the adjoint solution is calculated from a structural adjoint problem.

ACKNOWLEDGMENTS

We gratefully acknowledge the support of the Automotive Research Center (ARC), which is sponsored by the U.S. Army TARDEC.

REFERENCES


