A CONTINUUM APPROACH FOR SECOND-ORDER SHAPE DESIGN SENSITIVITY OF ELASTIC SOLIDS WITH LOADED BOUNDARIES

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SUMMARY
A second-order shape design sensitivity analysis (DSA) method applicable to the shape change on the loaded boundaries is derived for three-dimensional linear elastic solids using a continuum method with the material derivative. The continuum method is also used to derive mixed second-order variations of stress and displacement performance measures with respect to shape design variables and distribution of non-conservative traction loads, and also with respect to shape design variables and material properties. A shape design acceleration field is defined for the second-order shape design sensitivity. Both direct differentiation and hybrid methods are presented in this paper. A numerical method, which can be implemented using established finite element analysis (FEA) codes, is developed. The feasibility and accuracy of the proposed second-order shape DSA method has been demonstrated by solving a structural example-doubly curved arch dam.

KEY WORDS: second-order design sensitivity; shape design; arch dam; acceleration field

1. INTRODUCTION
The second-order shape design sensitivity information is useful in many applications such as the robust design for improving the quality of product, and the reliability analysis for improving the accuracy of the prediction. To improve the quality of a product by robust design, the sensitivity of the product with respect to the environment variations needs to be reduced. To minimize sensitivity, it is necessary to have accurate second-order shape design sensitivity. By using the second-order sensitivity information, the accuracy of reliability analysis can be improved. Therefore, it is valuable to develop an accurate, reliable, and efficient method to evaluate the second-order shape design sensitivity.

Cea and Zolesio first introduced the continuum approach of first-order shape design sensitivity with a material derivative concept (the velocity method). Since then, the idea has been used extensively on structural sensitivity analysis. The first book that treats comprehensive theory and numerical methods of the continuum approach of first-order shape design sensitivity of linear structural systems was presented by Haug et al. Delfour pointed out that the material derivative method is a preferable method for shape DSA because it makes the continuous-discrete approach possible and considerably simplifies the computation of discrete shape gradients.

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By extending the concept of Dems and Mróz\(^8\) and Dems and Haftka,\(^5\) Dems\(^9\) derived the second-order shape design sensitivity of non-linear elastic structural systems with shape modification on the traction-free boundary using the hybrid method. Petryk and Mróz\(^10\) also derived the first and second time derivatives of functionals defined on time-dependent volumes or surface domains. However, they did not discuss numerical implementation issues. Chen and Choi\(^11\) first used a shape design acceleration\(^12\) concept to derive the second-order shape design sensitivity and implement a numerical method to solve the practical engineering problems without considering the shape change on the loaded boundaries.

This paper extends the concept of Chen and Choi\(^11\) to consider the second-order shape design sensitivity analysis with shape changes on the traction-loaded boundaries for linear 3-D solids. Also the mixed second-order design sensitivity of stress and displacement performance measures with respect to shape design variable and distribution of traction loads, and with respect to shape design variable and material property are derived. Both the direct differentiation and hybrid methods are used to derive the analytical formulations of second-order design sensitivity.

To demonstrate the feasibility and accuracy of the proposed second-order shape DSA method, a numerical method, which is implemented using established finite element codes, is developed. A 3-D doubly curved arch dam which is subject to water pressure and gravity force is chosen as numerical example to show excellent numerical results.

2. DEFINITIONS, ASSUMPTIONS, AND BASIC MATERIAL DERIVATIVE FORMULATIONS

In this section, some preliminary definitions, assumptions, and basic formulations are introduced. Thereafter, all necessary functions, functionals, and shapes are assumed smooth enough that their second derivatives exist.

In order not to lose integrity, the shape design velocity and acceleration fields, and first- and second-order material derivatives, which were defined and/or derived in Reference 11, are reprinted here with simplification. For detailed explanation, the reader can refer to the reference.

To define the shape design velocity and acceleration fields, consider a 3-D continuum domain with one parameter defining the transformation \(T\), as shown in Figure 1. The mapping \(T: \mathbf{x} \rightarrow \mathbf{x}_T(\mathbf{x}), \mathbf{x} \in \Omega\), is given by

\[
\mathbf{x}_t = T(\mathbf{x}, \tau), \quad \Omega_t = T(\Omega, \tau)
\]

In this paper, bold characters denote vectors or matrices. Based on equation (1), the shape design velocity field can be defined as

\[
V(\mathbf{x}_t, \tau) \equiv \frac{d\mathbf{x}_t}{d\tau} = \frac{dT(\mathbf{x}, \tau)}{d\tau} = \frac{\partial T(\mathbf{x}, \tau)}{\partial \tau}
\]

Figure 1. One parameter family of mappings
Thus, if a shape design velocity field $\mathbf{V}$ is given, then the mapping $T$ in equation (1) can be approximated, at $\tau = 0$, by

$$
T(x, \tau) = x_\tau(x) = x + \tau \mathbf{V}(x) + \frac{\tau^2}{2} \ddot{\mathbf{V}}(x)
$$

(3)

where

$$
\mathbf{V}(x) \equiv \mathbf{V}(x, 0) \quad \text{and} \quad \dot{\mathbf{V}}(x) \equiv \frac{d}{d\tau} \mathbf{V}(x_\tau, \tau)|_{\tau=0}
$$

To derive the first-order material derivative, let $z_t(x, \tau)$ be a solution of the variational equation on the deformed domain $\Omega$ (with boundary $\Gamma_t$),

$$
a_{\Omega_t}(z_t, \dot{z}_t) = \mathcal{L}_t \left( \dot{z}_t \right) \quad \text{for all} \quad \dot{z}_t \in Z_t
$$

(4)

where $\dot{z}_t = [z_t^1, z_t^2, z_t^3]$ and $Z_t$ is the space of kinematically admissible displacements. The pointwise material derivative of the state $z^i_t(x, \tau)$ (if it exists) at $x \in \Omega$ is defined as

$$
\dot{z}^i_t(x) = z''_t(x) + \nabla z^i T \mathbf{V}(x), \quad i = 1, 2, 3
$$

(5)

where $z''_t(x)$ is the partial derivative of $z^i_t(x)$ with respect to $\tau$ and $\nabla z^i = [z^i_1, z^i_2, z^i_3]^T$. Here the superscript $i$ denotes the $i$th component of vector $z$ and the subscript denotes the partial derivative of the $z^i$ with respect to $x$ (i.e. $z^i_j = \partial z^i_j / \partial x^j$, $i, j = 1, 2, 3$).

With the design velocity field as a function of $x$, and utilizing equation (2), we can derive the design acceleration field by taking the total derivative of $\mathbf{V}$ with respect to $\tau$. Then,

$$
\mathbf{V}'(x) = \ddot{\mathbf{V}}(x) - \nabla \mathbf{V}(x)^T \mathbf{V}(x)
$$

(6)

where $\mathbf{V}'(x)$ is defined as the design acceleration field.\textsuperscript{12}

Using the definition of the design acceleration field, the second-order material derivative of the state $z^i_t(x, \tau)$ at $x \in \Omega$ is defined as$^1$ (if it exists)

$$
\dddot{z}^i_t = z''''_t + 2(\nabla z^i)^T \mathbf{V}' + \nabla \mathbf{V}(\nabla z^i) \mathbf{V} + (\nabla z^i)^T \mathbf{V}' + (\nabla z^i)^T (\nabla \mathbf{V}) \mathbf{V}
$$

(7)

where $z''''_t$ is the second-order partial derivative of $z^i$ with respect to $\tau$, and $\nabla (\nabla z^i) = [\partial^2 z^i_j / \partial x^j \partial x^k] \mathbf{V}$ is the Hessian of $z^i$. With the smoothness assumption, the partial derivative with respect to $\tau$, is commutative with respect to $x$, because they are derivatives with respect to independent variables.

In order to derive the second-order variation of some functionals, we first derive the first- and second-order material derivatives of the Jacobian of the transformation $T$ and then, the first- and second-order material derivatives of $\mathbf{n}$, which is the outward unit normal vector on the boundary $\Gamma_t$ of deformed domain $\Omega$.

From equation (3), the Jacobian of the transformation $T$ can be defined as

$$
\mathbf{J} \equiv \frac{\partial T^i}{\partial x^j} = \mathbf{I} + \tau \frac{\partial \mathbf{V}^i}{\partial x^j} + \frac{\tau^2}{2} \left[ \frac{\partial \mathbf{V}^i}{\partial x^j} \right] = \mathbf{I} + \tau \mathbf{D} \mathbf{V}(x) + \frac{\tau^2}{2} \mathbf{D} \ddot{\mathbf{V}}(x)
$$

(8)

where $\mathbf{I}$ is the identity matrix, and $\mathbf{D} \mathbf{V}(x)$ and $\mathbf{D} \ddot{\mathbf{V}}(x)$ are Jacobians of $\mathbf{V}(x)$ and $\ddot{\mathbf{V}}(x)$, respectively. Let $|\mathbf{J}|$ be the determinant of $\mathbf{J}$. The first material derivatives of $\mathbf{J}$ and $|\mathbf{J}|$ can be...
obtained as
\[
\frac{dJ}{dt} \bigg|_{t=0} = DV(x) \tag{9}
\]
\[
\frac{dJ^{-1}}{dt} \bigg|_{t=0} = -DV(x) \tag{10}
\]
\[
\frac{dJ^T}{dt} \bigg|_{t=0} = DV(x)^T \tag{11}
\]
\[
\frac{dJ^{-T}}{dt} \bigg|_{t=0} = -DV(x)^T \tag{12}
\]
\[
\frac{d|J|}{dt} \bigg|_{t=0} = \text{div } V(x) \tag{13}
\]
Using equations (8)-(13), and
\[
\frac{d^2(J^T J^{-T})}{dt^2} = 0
\]
the second-order material derivatives of $J$ can be obtained as
\[
\frac{d^2J}{dt^2} \bigg|_{t=0} = D\dot{V}(x) \tag{14}
\]
\[
\frac{d^2J^T}{dt^2} \bigg|_{t=0} = D\dot{V}(x)^T \tag{15}
\]
\[
\frac{d^2J^{-T}}{dt^2} \bigg|_{t=0} = 2DV(x)^TDV(x)^T - \dot{DV}(x)^T \tag{16}
\]
The second-order material derivatives of $|J|$ can be obtained as
\[
\frac{d^2|J|}{dt^2} \bigg|_{t=0} = 2 \sum_{i=1}^{3} M_{ii} + \sum_{i=1}^{3} \frac{\partial(V^i)}{\partial x^j} \tag{17}
\]
where $M_{ii}$ is the $i$th principal minor of the matrix $DV(x)$.

Let $\mathbf{n}$ be the outward unit normal vector on the boundary $\Gamma$ of domain $\Omega$, and $\mathbf{n}_r$ be the outward unit normal vector on the boundary $\Gamma_r$ of deformed domain $\Omega_r$. The relation between $\mathbf{n}$ and $\mathbf{n}_r$ is
\[
\mathbf{n}_r = \frac{J^{-T}(x_r)\mathbf{n}(x)}{\|J^{-T}(x_r)\mathbf{n}(x)\|} \tag{18}
\]
where $\|\mathbf{a}\| = (\mathbf{a}^T \mathbf{a})^{1/2}$ is the Euclidean norm.

Before deriving the first- and second-order material derivative of $\mathbf{n}_r$, the first- and second-order material derivative of $\|J^{-T}(x_r)\mathbf{n}(x)\|$ is derived. Using equations (12) and (16), the first-order material derivative of $\|J^{-T}(x_r)\mathbf{n}(x)\|$ is
\[
\frac{d}{dt} \|J^{-T}(x_r)\mathbf{n}(x)\| \bigg|_{t=0} = \frac{d}{dt} (J^{-T}\mathbf{n}, J^{-T}\mathbf{n})^{1/2} \bigg|_{t=0} = -DV(\mathbf{n}, \mathbf{n}) \tag{19}
\]
and the second-order material derivative of \( \| J^{-T}(x,\tau) n(x) \| \) is

\[
\frac{d}{d\tau} \left| \frac{d}{d\tau} J^{-T}(x,\tau) n(x) \right| = \left| \frac{d^2}{d\tau^2} (J^{-T} n, J^{-T} n)^{1/2} \right| = (DV^T n, DV^T n) + 2(DV n, DV^T n) - (DV n, n)^2 - (D\dot{V}^T n, n) \tag{20}
\]

Using equations (18)-(20), the first-order material derivative of \( n \) is obtained as

\[
\dot{n} = \frac{d}{d\tau} \left| \frac{d}{d\tau} J^{-T}(x,\tau) n(x) \right| - J^{-T}(x,\tau) n(x) \frac{d}{d\tau} \left( \frac{d}{d\tau} J^{-T}(x,\tau) n(x) \right) \left| \frac{d}{d\tau} J^{-T}(x,\tau) n(x) \right|^2 \tag{21}
\]

and the second-order material derivative of \( n \) is obtained as

\[
\ddot{n} = \frac{d^2}{d\tau^2} \left| \frac{d}{d\tau} J^{-T}(x,\tau) n(x) \right| - \frac{d}{d\tau} \left( J^{-T}(x,\tau) n(x) \frac{d}{d\tau} \left( \frac{d}{d\tau} J^{-T}(x,\tau) n(x) \right) \right) \left| \frac{d}{d\tau} J^{-T}(x,\tau) n(x) \right|^2 \tag{22}
\]

where \( (a, b) = a^T b \). Shape design sensitivity analysis can be carried out using these formulas. However, in the following derivation, we will use special types of design velocity field with the property \( \dot{V}(x) = 0 \) (see equation (86)). These types of design velocity fields are very much attractive from practical point of view, since the transformation mapping in (3) can be simplified to \( T(x, \tau) = x, (x) = x + \tau V(x) \). Thus, we can drop terms that involve \( \dot{V} \) in (3) and (8), and subsequent equations that are derived from these two.

Using equations (3), (5)-(7), (13) and (17), we can derive the second-order material derivative of the domain functional \( \psi \), which is defined as an integral over the domain \( \Omega \),

\[
\psi = \iiint_{\Omega} f(x) \, d\Omega \tag{23}
\]

where \( f \) is a regular function defined on the domain \( \Omega \). If the domain \( \Omega \) is \( C^k \) regular with \( k \) sufficiently large, the second-order variation of \( \psi \) at \( \Omega \) is obtained as

\[
\psi'' = \iiint_{\Omega} \left[ f''(x) + 2(\nabla f'(x))^T V + V^T \nabla \nabla f(x) V + (\nabla f(x))^T \nabla V \right.
\]

\[
\left. + (\nabla f(x))^T \nabla V + 2(f'(x) + (\nabla f(x))^T V) \text{div}(V) + 2f(x) \sum_{i=1}^3 (M_{ii}) \right] \, d\Omega \tag{24}
\]

Next, consider a functional defined as an integration over \( \Gamma \), which is the boundary of domain \( \Omega \):

\[
\psi_b = \int_{\Gamma} g(x) \, d\Gamma = \int_{\Gamma} g(x + \tau V) |J| \, d\Gamma \tag{25}
\]
where the transformation $d\Gamma_t = |J(x_1)| J^{-T}(x_2) \ n(x) \ d\Gamma$ is used. Then, using equations (5), (7), (13), (17), (19) and (20), the second-order variation of $\psi_b$ at $\Omega$ can be derived as

$$\psi_b^2 = \frac{d^2}{d\tau^2} \left[ \int_{\Gamma} g_c(x + \tau V) |J| J^{-T} n \ d\Gamma \right]_{\tau=0}$$

$$= \int_{\Gamma} \left[ \tilde{g} + 2\tilde{g} (\text{div} V) - 2\tilde{g} (DV_n, n) - 2g (\text{div} V)(DV_n, n) + 2g \sum_{i=1}^{3} M_{ii} \right.$$ 

$$+ g(DV^T n, DV^T n) + 2g(DV_n, DV^T n) - g(DV_n, n)^2 \right] d\Gamma$$

Equations (9), (7), (13), (17), (19) and (20) are used.

Another functional, which is also defined as an integration over $\Gamma$, is

$$\psi_a = \int_{\Gamma} h_s(x_1, x_2) n_1 d\Gamma, = \int_{\Gamma} \left[ \sum_{i=1}^{3} h_s(x + \tau V) n_i \right] |J| J^{-T} n \ d\Gamma$$

Its second-order variation is derived, by using equations (5), (7), (13), (17)–(22), as

$$\psi_a^2 = \frac{d^2}{d\tau^2} \left[ \int_{\Gamma} h_s(x + \tau V) n_1 |J| J^{-T} n \ d\Gamma \right]_{\tau=0}$$

$$= \int_{\Gamma} \sum_{i=1}^{3} \left\{ h_i n_i + h_i n_i + 2h_i n_i \sum_{j=1}^{3} M_{jj} + h_i n_i \left[ (DV^T n, DV^T n) + 2g(DV_n, DV^T n) \right.$$

$$- (DV_n, n)^2 \right] + 2h_i n_i + 2h_i n_i \left[ (\text{div} V) - (DV_n, n) \right] + 2h_i n_i \left[ (\text{div} V)$$

$$- (DV_n, n) \right] - 2h_i n_i (\text{div} V)(DV_n, n) \right\} d\Gamma$$

In the next two sections, equations (23)–(28) will be used to derive the second-order variation of energy bilinear and load linear forms of a 3-D elastic solid with respect to shape design changes, and second-order shape design sensitivity of stress and displacement performance measures.
3. DIRECT DIFFERENTIATION METHOD

The direct differentiation method directly utilizes the second-order variation of the energy bilinear and load linear forms to obtain the second-order sensitivity information.

3.1. Second-order shape DSA

The variational identity for the 3-D linear elastic solid can be written as

\[
\int_\Omega \int_{\Omega} \left[ \sum_{i,j=1}^{3} \sigma^{ij}(z) \varepsilon^{ij}(\bar{z}) \right] \, d\Omega - \int_\Omega \int_{\Omega} \left[ \sum_{i=1}^{3} f^i \bar{z}^i \right] \, d\Omega = \int_{\Gamma} \left[ \sum_{i,j=1}^{3} \sigma^{ij}(z) n^j \bar{z}^i \right] \, d\Gamma \quad \text{for all } \bar{z} \in [H^1(\Omega)]^3
\]

where \( n^j \) is the \( j \)-th component of the outward unit normal of the boundary and \( H^1(\Omega) \) is the Sobolev space of order one. If the boundary conditions are imposed on the boundary \( \Gamma \), of which \( \Gamma_0 \) is fixed, \( \Gamma_1 \) is traction free, and \( \Gamma_2 \) has traction loading, then

\[
z^i = 0, \quad i = 1, 2, 3, \quad x \in \Gamma_0
\]

\[
T^m(z) = \sum_{j=1}^{3} \sigma^{ij}(z) n^j = T^i, \quad i = 1, 2, 3, \quad x \in \Gamma_2
\]

Using the boundary conditions, equation (29) can be rewritten as

\[
a_\Omega(z, \bar{z}) = \sum_{i,j=1}^{3} \sigma^{ij}(z) \varepsilon_{ij}(\bar{z}) \, d\Omega = \sum_{i=1}^{3} f^i \bar{z}^i \, d\Omega + \int_{\Gamma} \left[ \sum_{i=1}^{3} T^i \bar{z}^i \right] \, d\Gamma \equiv \mathcal{E}_\Omega(\bar{z}), \quad \text{for all } \bar{z} \in Z
\]

where \( Z \) is the space of kinematically admissible displacements,

\[
Z = \{ z \in [H^1(\Omega)]^3 : z^i = 0, i = 1, 2, 3, x \in \Gamma_0 \}
\]

Taking the first-order variation of equation (30),

\[
\frac{d}{d\tau} [a_\Omega(z, \bar{z}, \tau)] |_{\tau=0} = \frac{d}{d\tau} [\mathcal{E}_\Omega(\bar{z}, \tau)] |_{\tau=0}
\]

(32)

By assuming \( f' = 0 \), equation (32) yields

\[
a_\Omega(z, \bar{z}) = - a'_{\mathcal{E}}(z, \bar{z}) + \mathcal{E}'(\bar{z})
\]

(33)

where

\[
a_\Omega(z, \bar{z}) = \sum_{i,j=1}^{3} \sigma^{ij}(z) \varepsilon^{ij}(\bar{z}) \, d\Omega
\]

(34)

\[
a'_{\mathcal{E}}(z, \bar{z}) = \sum_{i,j=1}^{3} \left\{ - \sigma^{ij}(z) (\nabla \bar{z}^j)^T V_j + \sigma^{ij}(z) (\nabla \bar{z}^j)^T V_j \right\} + \sigma^{ij}(z) \varepsilon^{ij}(\bar{z}) \, d\Omega
\]

(35)

\[
\mathcal{E}'(\bar{z}) = \sum_{i,j=1}^{3} \left[ \bar{z}^i \nabla \bar{z}^j \right] d\Omega + \sum_{i=1}^{3} \left[ T^i \bar{z}^i + \bar{z}^i \nabla T^i \right] d\Omega
\]

(36)
To consider a specific non-conservative loading, if the traction load $T^i(x)$ is non-conservative pressure loading which depends not only on the position but also on the shape of the boundary, it can be expressed as

$$T^i(x) = -p(x)n^i(x), \quad x \in \Gamma_2$$

(37)

where $n^i(x)$ is the outward unit vector on boundary $\Gamma_2$ and $p(x)$ is the pressure distribution on the boundary. By assuming $p' = 0$, equation (36) can be rewritten as

$$\ell_{\nu}'(\vec{z}) \equiv \int_\Omega \left\{ \sum_{i,j=1}^{3} \left[ \delta^i_j \delta^j_i T \frac{\partial V}{\partial \nu} + f^i V T \frac{\partial T}{\partial \nu} \right] d\Omega + \int_{\Gamma_2} p [z^T D V^T n^T - z^T n (\nabla T)] d\Gamma \right\}$$

(38)

Taking the second-order variation of $a_{\Omega}(z, \vec{z})$ of equation (30), the second-order variation of $a_{\Omega}(z, \vec{z})$ can be obtained as

$$\frac{d^2}{dz^2} [a_{\Omega}(z, \vec{z})]_{t=0} = a_{\Omega}(z, \vec{z}) + 2a_{\nu}'(\vec{z}, \vec{z}) + a_{\nu}^2(\vec{z}, \vec{z}) + a_{\nu}^2(\vec{z}, \vec{z})$$

(39)

where

$$a_{\Omega}(z, \vec{z}) = \int_\Omega \left[ \sum_{i,j=1}^{3} \sigma^{ij}(\vec{z}) \varepsilon^{ij}(\vec{z}) \right] d\Omega$$

(40)

$$a_{\nu}'(\vec{z}, \vec{z}) = \int_\Omega \left\{ \sum_{i,j=1}^{3} \left[ \delta^i_j \delta^j_i T \frac{\partial V}{\partial \nu} + f^i V T \frac{\partial T}{\partial \nu} \right] d\Omega \right\}$$

(41)

$$a_{\nu}^2(\vec{z}, \vec{z}) = \int_\Omega \left\{ \sum_{i,j=1}^{3} \left[ \delta^i_j \delta^j_i T \frac{\partial V}{\partial \nu} + f^i V T \frac{\partial T}{\partial \nu} \right] d\Omega \right\}$$

(42)

and

$$a_{\nu}^2(\vec{z}, \vec{z}) = \int_\Omega \left\{ \sum_{i,j=1}^{3} \left[ \delta^i_j \delta^j_i T \frac{\partial V}{\partial \nu} + f^i V T \frac{\partial T}{\partial \nu} \right] d\Omega \right\}$$

(43)

The traction-loaded boundaries can be treated by taking the second-order variation of the right-hand side of equation (30), using equations (23)-(26), and assuming that $f$ is independent of shape design change (i.e. $f'' = f''' = 0$, $i = 1, 2, 3$):

$$\frac{d^2}{dz^2} [\ell_{\nu}(\vec{z})]_{t=0} = \int_\Omega \left\{ \sum_{i=1}^{3} \left[ V^T (\nabla (\nabla f^i)) V \varepsilon^i + \varepsilon^i (\nabla f^i)^T \frac{\partial V}{\partial \nu} V' + \varepsilon^i (\nabla f^i)^T (\nabla V) V \right] d\Omega \right\}$$

(44)
where

\[
\ell''_\nu(z) = \int_{\Omega} \sum_{i=1}^{3} \left[ V^T \nabla (\nabla f_i) \nabla \tilde{z}^i + \tilde{z}^i (\nabla f_i)^T (V V) + 2\tilde{z}^i (\nabla f_i)^T V (\text{div} V) \right] \, \text{d}\Omega - \int_{\Gamma_2} \sum_{i=1}^{3} \tilde{z}^i \left[ T''^i + 2(T T'^i) V + V^T (V T'^i) V + (V T'^i)^T (V V) V \right] \, \text{d}\Gamma \\
+ 2(T'' + (V T'^i)V)(\text{div} V) - 2(T'' + (V T'^i) V)(D \nu, n) - 2T'(\text{div} V)(D \nu, n) \\
+ 2T'(\sum_{r=1}^{3} M_{rr} + T'(D V^T n, D V^T n) + 2T'(D V^T n, D V^T n) - T'(D V^T n, n)^2) \, \text{d}\Gamma 
\]

and

\[
\ell''_\nu(z) = \int_{\Omega} \sum_{i=1}^{3} \left[ \tilde{z}^i (\nabla f_i)^T V' \right] \, \text{d}\Omega + \int_{\Gamma_2} \sum_{i=1}^{3} \left[ \tilde{z}^i (\nabla T'^i) V' \right] \, \text{d}\Gamma 
\]

In equations (44)-(46), those terms that are surface integration over boundary \( \Gamma_2 \) are the additional terms due to consideration of the shape changes of the traction-loaded boundaries.

In equations (39) and (44), those terms that contain \( \dot{\tilde{z}} \) and \( \ddot{\tilde{z}} \) have been neglected because they can be cancelled out by using the variational identity and the first-order variational equation of 3-D elastic solids.

For non-conservative pressure loading, \( T' \) can be defined as equation (37). By using equations (27), (28), and (37) and \( p'' = \nu' = 0 \), equations (45) and (46) can be rewritten as

\[
\ell''_\nu(z) = \int_{\Omega} \sum_{i=1}^{3} \left[ V^T \nabla (\nabla f_i) \nabla \tilde{z}^i + \tilde{z}^i (\nabla f_i)^T (V V) + 2\tilde{z}^i (\nabla f_i)^T V (\text{div} V) \right] \, \text{d}\Omega - 2 \int_{\Gamma_2} \left[ D V^T D V^T n + \sum_{r=1}^{3} M_{rr} n - D V^T n (\text{div} V) \right] \, \text{d}\Gamma \\
- 2 \int_{\Gamma_2} \left[ (V T' \nabla p) V + \nabla p^T (\nabla V) V \right] \tilde{z}^i n + 2 \nabla p^T V (\text{div} V) \tilde{z}^i n - 2 \nabla p^T V \tilde{z}^i D V^T n \, \text{d}\Gamma 
\]

and

\[
\ell''_\nu(z) = \int_{\Omega} \sum_{i=1}^{3} \left[ \tilde{z}^i (\nabla f_i)^T V' \right] \, \text{d}\Omega + \int_{\Gamma_2} (\nabla p)^T V' \, \text{d}\Gamma 
\]

From equations (39) and (44), the second-variation of the variational equation (30) can be written as

\[
a_\Omega(\dot{z}, \ddot{z}) = -2a_\nu'(z, \ddot{z}) - a_\nu'(z, \ddot{z}) - a_\nu'(z, \ddot{z}) + \ell''_\nu(\ddot{z}) + \ell''_\nu(\ddot{z}) \quad \text{for all } \ddot{z} \in \mathcal{Z} 
\]

where \( \ddot{z} \) is the material derivative of the displacement and can be obtained from equation (33). Note that (42) and (43) require that the design velocity field \( V(x) \) to be \( C^1 \) function and its second-order derivatives to be integrable. The second-order shape design sensitivity of the displacement \( \ddot{z} \) can then be obtained using finite element reanalysis with the fictitious load on the right-hand side of (49).
To obtain the second-order variation of a domain stress performance measure, let the general mean stress performance measure over a fixed test volume $\Omega_p$ be defined as
\[
\psi_s = \iiint_{\Omega} g(\sigma(z))m_p \, d\Omega = \frac{\iiint_{\Omega_p} g(\sigma(z)) \, d\Omega}{\iiint_{\Omega_p} \, d\Omega}
\] (50)
where $m_p$ is a characteristic function that has the constant value of $\bar{m}_p = (\iiint_{\Omega_p} \, d\Omega)^{-1}$ on volume $\Omega_p$ and zero outside volume $\Omega_p$, and whose integral is one. If the volume $\Omega_p$ becomes a point, then $m_p$ becomes the dirac delta measure and the performance measure $\psi_s$ in (50) is the stress at the point. Note that $g(\sigma(z))$ might be principal stresses, von Mises stress, or some other material failure criteria.

Taking the second-order variation of (50), the second-order shape design sensitivity of the stress performance measure can be derived as:
\[
\psi''_s = \iiint_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{\sigma \sigma}^i \left[ (\sigma^{ij}(\hat{z}))_i \sigma^{kl}(\hat{z}) - 2 \sigma^{ij}(\hat{z}) \right] \sum_{s,t=1}^{3} C^{ijkl}(\nabla z^s)^T V_t \\
+ \sum_{s,t=1}^{3} \sum_{u,v=1}^{3} C^{ijkl} C^{kluv} (\nabla z^s)^T V_t (\nabla z^u)^T V_v \right] m_p \, d\Omega
\]
\[
+ \iiint_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{\sigma}^i \left[ (\sigma^{ij}(\hat{z}))_i - 2 \sum_{k,l=1}^{3} C^{ijkl} \sum_{s,t=1}^{3} z_s^k V_t^i - \sum_{k,l=1}^{3} C^{ijkl}(\nabla z^k)^T V_i \\
+ \sum_{s,t=1}^{3} \sum_{u,v=1}^{3} C^{ijkl} (z_s^i V_t^k V_t^k - z_s^i V_t^k V_i^k) + 2(\sigma^{ij}(\hat{z}) - \sum_{k,l=1}^{3} C^{ijkl}(\nabla z^k)^T V_i (\nabla z^l)) \right] m_p \, d\Omega
\]
\[
+ 2 \iiint_{\Omega} g \left( \sum_{r=1}^{3} M_{rr} \right) m_p \, d\Omega - 2 \iiint_{\Omega} g(\nabla V) m_p \, d\Omega \iiint_{\Omega} (\nabla V) m_p \, d\Omega
\]
\[
- \iiint_{\Omega} g m_p \, d\Omega \iiint_{\Omega} 2 \left( \sum_{r=1}^{3} M_{rr} \right) m_p \, d\Omega + 2 \iiint_{\Omega} g m_p \, d\Omega \left( \iiint_{\Omega} (\nabla V) m_p \, d\Omega \right)^2
\]
\[
- 2 \iiint_{\Omega} g_{\sigma}^i \left[ (\sigma^{ij}(\hat{z}))_i - \sum_{k,l=1}^{3} C^{ijkl}(\nabla z^k)^T V_i \right] m_p \iiint_{\Omega} (\nabla V) m_p \, d\Omega
\] (51)

The second-order shape design sensitivity of the stress performance measure can be obtained by substituting $\hat{z}$ (obtained from (49)) and $\hat{\sigma}$ (obtained from (33)) into (51).

### 3.2. Mixed second-order design sensitivity with respect to shape and non-shape design variables

As mentioned earlier, in studying robust design, it is important to know how sensitive the design is with respect to environment variations such as material and loading variations. To consider environment variations in the shape design process, the mixed second-order design sensitivity of performance measures with respect to the shape and non-shape design variables must be derived. In this paper, the non-shape design variables will include the material property and distribution of pressure loading.

The mixed second-order design sensitivity of the displacement and stress performance measures with respect to the shape design variable and Young's modulus, and with respect to the
shape design variable and distribution of non-conservative pressure loading will be derived in this section.

To treat the shape and non-shape design variables, the variational identity for the 3-D linear elastic solids is rewritten as

\[ a_{\Omega, u}(z; \bar{x}) = \mathcal{L}_{\Omega, u}(\bar{z}) \quad \text{for all } \bar{x} \in Z \]  

(52)

where \( Z \) is the kinematically admissible displacements defined as (31), and \( u \in R \) is the Young's modulus and/or distribution of non-conservative pressure loading. The first-order variation of \( z^i \) with respect to the non-shape design variable \( u \) is defined as

\[ z'^i = z'^i(x; u, \delta u) = \frac{d}{dt} z'_i(x; u + \tau \delta u)|_{\tau=0} \]  

(53)

and the mixed second-order derivative of \( z'^i(x) \) with respect to both shape and non-shape design variables (if it exists) at \( x \in \Omega \) is defined as

\[ z''_i = \frac{d^2}{dt^2} z'_i(x + \tau V; u + \tau \delta u)|_{\tau=0} = z''^i(x) + \nabla z''^i V(x) \]  

(54)

where the \( z''^i \) is the first-order variation of \( z^i \) with respect to non-shape design variable \( u \), and \( z''^i \) is the mixed second-order variation of \( z^i \) with respect to the shape and non-shape design variables.

3.2.1. Material property. To derive the mixed second-order design sensitivity of the performance measures with respect to the shape design variable and Young's modulus, the mixed second-order variation of the energy bilinear and load linear forms with respect to the shape design variable and Young's modulus is derived first. Then, the mixed second-order variation of the stress performance measures with respect to the shape design variable and Young's modulus is derived.

Considering Young's modulus \( E \) as the non-shape design variable, equation (52) can be rewritten as

\[ a_{\Omega, E}(z; \bar{x}) = \mathcal{L}_{\Omega, E}(\bar{z}) \quad \text{for all } \bar{x} \in Z \]  

(55)

The first-order variation of equation (55) with respect to Young's modulus is

\[ \frac{d}{dt} [a_{\Omega,E}(z(x; E + \tau \delta E), \bar{z})]|_{\tau=0} = \frac{d}{dt} [\mathcal{L}_{\Omega,E}(\bar{z})]|_{\tau=0} \]  

(56)

For 3-D isotropic solid, equation (56) yields

\[ a_{\Omega}(z_E, \bar{x}) = - a_{\Omega}(\bar{x}, \bar{z}) \delta E/E \]  

(57)

Taking the mixed second-order variation of equation (55) with respect to both the shape design variable and Young's modulus,

\[ \frac{d^2}{dt^2} [a_{\Omega_E}(z(x + \tau V; E + \tau \delta E), \bar{z})]|_{\tau=0} = \frac{d^2}{dt^2} [\mathcal{L}_{\Omega,E}(\bar{z})]|_{\tau=0} \]  

(58)

Using the following relations:

\[ \sum_{i,j=1}^{3} \sigma^{ij}(z) \epsilon^{ij}(\nabla z^T V) = \sum_{i,j=1}^{3} \sigma^{ij}(z)(\nabla z^T)^{ij} V + \nabla z^T V \]  

(59)
and

\[
\sum_{i,j=1}^{3} \nabla (\sigma^{ij}(z)e^{ij}(z))^{T} V = \sum_{i,j=1}^{3} \sigma^{ij}(z)(\nabla z_j)^{T} V + \sigma^{ij}(\dot{z})(\nabla z_j)^{T} V
\]

Equation (58) yields

\[
a_{o}(\ddot{z}_E, \dot{z}) = \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(\ddot{z}) e^{ij}(\dot{z}) \right] d\Omega
\]

\[
= \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z) \nabla z_j^{T} V_j + \sigma^{ij}(\dot{z})(\nabla z_j)^{T} V_j - \sigma^{ij}(\ddot{z}) e^{ij}(\dot{z})(\text{div} V) \right] d\Omega
\]

\[
+ \frac{1}{E} \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ - \sigma^{ij}(\dot{z}) e^{ij}(\dot{z}) + \sigma^{ij}(z) \nabla \dot{z}_j^{T} V_j + \sigma^{ij}(\ddot{z}) \nabla z_j^{T} V_j \right] \delta E d\Omega
\]

\[
- \frac{1}{E} \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z) e^{ij}(\dot{z})(\text{div} V) \right] \delta E d\Omega
\]

where \( \dot{z} \) is the first-order material derivative of \( z \) that can be obtained from (33), and \( \ddot{z}_E \) is the first-order variation of \( z \) with respect to Young’s modulus that can be obtained from (57). The mixed second-order sensitivity of the displacement \( \dot{z}_E \) can then be obtained by using finite element reanalysis with the fictitious load on the right-hand side of (61).

Taking the mixed second-order variation of (50) and using (54), the mixed second-order variation of the stress performance measure with respect to the shape design variable and Young’s modulus is derived as

\[
\psi_{o}'' = \iiint_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{o}^{ij} \left[ \sigma^{ij}(\ddot{z}_E) \sigma^{kl}(\dot{z}) - \sigma^{ij}(\ddot{z}_E) \sum_{s,t=1}^{3} C^{kls}(\nabla z_s)^{T} V_t \right]
\]

\[
+ \frac{\delta E}{E} \sigma^{ij}(z) \sigma^{kl}(\ddot{z}) - \frac{\delta E}{E} \sigma^{ij}(z) \sum_{k,l=1}^{3} C^{kls}(\nabla z_s)^{T} V_t \right] m_p d\Omega
\]

\[
+ \iiint_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{o}^{ij} \left[ \sigma^{ij}(\ddot{z}_E) - \sum_{k,l=1}^{3} C^{ijkl}(\nabla z_k)^{T} V_l + \frac{\delta E}{E} \sigma^{ij}(\dot{z}) \right]
\]

\[
- \frac{\delta E}{E} \sum_{k,l=1}^{3} C^{ijkl}(\nabla z_k)^{T} V_l \right] m_p d\Omega + \iiint_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{o}^{ij} \left[ \sigma^{ij}(z') + \frac{\delta E}{E} \sigma^{ij}(z) \right](\text{div} V)m_p d\Omega
\]

\[
- \iiint_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{o}^{ij} \left[ \sigma^{ij}(z') + \frac{\delta E}{E} \sigma^{ij}(z) \right] m_p d\Omega \int_{\Omega} (\text{div} V)m_p d\Omega
\]

In that \( z'_E = - z/E \) and \( \dot{z}_E = - \dot{z}/E \), it can be seen that (62) is zero. This means the stress performance measure is insensitive to the variation of Young’s modulus.

3.2.2. Distribution of non-conservative pressure loading. Utilizing the mixed second-order variation of the energy bilinear and load linear forms with respect to the shape design variable and distribution of non-conservative pressure loading, the mixed second-order variation of the stress performance measures with respect to the shape design variable and distribution of nonconservative pressure loading can be derived.
Considering the distribution of non-conservative pressure loading $p$ as the non-shape design variable, Equation (52) can be rewritten as

$$a_{\Omega,p}(z, \tilde{z}) = \mathcal{E}_{\Omega,p}(\tilde{z}) \quad \text{for all } \tilde{z} \in Z$$

(63)

The first-order variation of (63) with respect to distribution of nonconservative pressure loading is

$$\frac{d}{dt} \left[ a_{\Omega,p}(z(x; p + \tau \delta p), \tilde{z}) \right]_{t=0} = \frac{d}{dt} \left[ \mathcal{E}_{\Omega,p}(\tilde{z}) \right]_{t=0}$$

(64)

Equation (64) yields

$$a_{\Omega}(z'_p, \tilde{z}) = - \int_{\Gamma_2} \left( \sum_{i=1}^{3} n' \tilde{z}^i \right) \delta p \, d\Gamma$$

(65)

Taking the mixed second-order variation of (63) with respect to both the shape design variable and distribution of non-conservative pressure loading

$$\frac{d^2}{dt^2} \left[ a_{\Omega,p}(z(x + \tau \mathbf{V}; p + \tau \delta p), \tilde{z}) \right]_{t=0} = \frac{d^2}{dt^2} \left[ \mathcal{E}_{\Omega,p}(\tilde{z}) \right]_{t=0}$$

(66)

Using equations (54), (59), and (60), equation (66) yields

$$a_{\Omega}(z'_p, \tilde{z}) \equiv \int_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z'_p) \varepsilon^{ij}(\tilde{z}) \right] \, d\Omega$$

$$= \int_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z'_p) (\nabla \tilde{z}^i)^T \mathbf{V}_j + \sigma^{ij}(\tilde{z})(\nabla z'_p)^T \mathbf{V}_j \right] \, d\Omega$$

$$- \int_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z'_p) \varepsilon^{ij}(\tilde{z}) \right] \, d\Omega + \int_{\Gamma_2} \sum_{i,j=1}^{3} \tilde{z}^i \left[ D V^{ji} n^i - n'(\text{div } \mathbf{V}) \right] \delta p \, d\Gamma$$

(67)

where $z'_p$ is the first-order variation of $z$ with respect to the distribution of non-conservative pressure loading $p$ that can be obtained from (65). The mixed second-order material derivative of displacement $z'_p$ can be obtained by using finite element reanalysis with the fictitious load on the right-hand side of (67).

Taking the second-order variation of (50) and using (54), the mixed second-order variation of the stress performance measure with respect to the shape design variable and distribution of pressure loading is derived as

$$\psi'' \equiv \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{\rho''\sigma''} \left[ \sigma^{ij}(z'_p) \sigma^{kl}(\tilde{z}) - \sigma^{ij}(z'_p) \sum_{s,t=1}^{3} C^{kis}(\nabla z^s)^T \mathbf{V}_i \right] m_p \, d\Omega$$

$$+ \int_{\Omega} \sum_{i,j=1}^{3} g_{\rho''} \left[ \sigma^{ij}(z'_p) \sigma^{kl}(\tilde{z}) - \sum_{k,l=1}^{3} C^{jkl}(\nabla z'_p)^T \mathbf{V}_i - \sigma^{ij}(z'_p)(\text{div } \mathbf{V}) \right] m_p \, d\Omega$$

$$- \int_{\Omega} \sum_{i,j=1}^{3} g_{\rho''}(z'_p) m_p \, d\Omega \int_{\Omega} (\text{div } \mathbf{V}) m_p \, d\Omega$$

(68)

For the direct differentiation method, $z'_p$ obtained from (67), $\tilde{z}$ obtained from (33), and $z'_p$ obtained from (65) can be used in (68) to obtain the design sensitivity.
3.3. Computational cost

The computational cost of the direct differentiation method is expensive if the design problem has a large number of shape and non-shape design variables. For the shape design problem with \( n \)-shape design variables and \( m \) non-shape design variables, the method requires one original finite element (FE) analysis and \((3n + n^2)/2 + m(n + 1)\) FE reanalyses which include \( n + m \) FE reanalyses (equations (33), (57), and (65)) for the first-order shape and non-shape design sensitivity, and \( n(n + 1)/2 + mn \) FE reanalyses (equations (49), (61), and (67)) for the second-order shape design sensitivity and mixed design sensitivity.

4. HYBRID METHOD

The hybrid method uses the direct differentiation method to compute the first-order sensitivity, and the adjoint variable method to evaluate the second-order sensitivity.

4.1. Second-order shape DSA

The displacement performance measure can be defined as

\[
\psi_d = \iiint_{\Omega} e^T z \delta(x - \hat{x}) \, d\Omega, \quad i = 1, 2, \text{or } 3 \tag{69}
\]

where \( e^1 = [1, 0, 0]^T \), \( e^2 = [0, 1, 0]^T \), and \( e^3 = [0, 0, 1]^T \) are are three constant vectors. The second-order variation of \( \psi_d \) is

\[
\psi_d'' = \iiint_{\Omega} e^T \ddot{z} \delta(x - \hat{x}) \, d\Omega \tag{70}
\]

The adjoint equation for the displacement performance can be defined by replacing \( \ddot{z} \) in (69) by a virtual displacement \( \ddot{\lambda} \) and equating the result to the energy bilinear form

\[
a_{\Omega}(\lambda, \ddot{\lambda}) = \iiint_{\Omega} e^T \ddot{\lambda} \delta(x - \hat{x}) \, d\Omega \quad \text{for all } \ddot{\lambda} \in Z \tag{71}
\]

where \( Z \) is the space of kinematically admissible displacements. The adjoint structural system can be interpreted as the original structure with a positive unit load applied at location \( \hat{x} \).

Let \( \ddot{\lambda} = \ddot{z} \in Z \) in (71) and \( \ddot{\lambda} = \ddot{\lambda} \in Z \) in (49). Then, (71) becomes

\[
a_{\Omega}(\lambda, \ddot{z}) = \iiint_{\Omega} e^T \ddot{z} \delta(x - \hat{x}) \, d\Omega \tag{72}
\]

and (49) becomes

\[
a_{\Omega}(\ddot{z}, \lambda) = -2a_{\psi}(\ddot{z}, \lambda) - a_{\psi}(\ddot{r}, \lambda) + a_{\psi}(z, \lambda) + \ell_{\psi}(\lambda) + \ell'_{\psi}(\lambda) \tag{73}
\]

Since the energy bilinear form \( a_{\Omega}(\cdot, \cdot, \cdot) \) is symmetric in its arguments, by equating the left-hand sides of (72) and (73), the second-order sensitivity of the displacement performance measure can be written as

\[
\psi_d'' \equiv \iiint_{\Omega} e^T \ddot{z} \delta(x - \hat{x}) \, d\Omega = -2a_{\psi}(\ddot{z}, \lambda) - a_{\psi}(\ddot{r}, \lambda) + \ell_{\psi}(\lambda) + \ell'_{\psi}(\lambda) \tag{74}
\]

where \( \lambda \) is the adjoint response of (71). To evaluate (74), \( \ddot{z} \) can be obtained from (33).
Let $\psi$ be the stress performance measure defined by (50). The adjoint equation for the stress performance measure can be defined by replacing the term that contains $\bar{z}$ in (51) by a virtual displacement $\bar{\lambda}$ and equating the result to the energy bilinear form

$$a_\Omega(\lambda, \bar{\lambda}) = \iint_{\Omega} \left[ \sum_{i,j=1}^{3} g_{\sigma^{ij}} \sigma^{ij}(\bar{\lambda}) \right] m_p \, d\Omega \quad \text{for all } \bar{\lambda} \in Z. \quad (75)$$

By setting $\bar{\lambda} = \bar{z} \in Z$ in (75) and $\bar{z} = \lambda \in Z$ in (49) and using the symmetricity of the energy bilinear form $a_\Omega(\cdot, \cdot)$, the following equation can be obtained:

$$\iint_{\Omega} \left[ \sum_{i,j=1}^{3} g_{\sigma^{ij}} \sigma^{ij}(\bar{z}) \right] m_p \, d\Omega = -2a_\gamma(\bar{z}, \lambda) - a_\gamma(z, \lambda) - a_\gamma(z, \lambda) + \ell_\gamma(t) + \ell_\gamma(t) \quad (76)$$

where $\lambda$ is the adjoint response of equation (75).

Substituting equation (76) for the corresponding term in equation (51), the second-order shape design sensitivity of the stress performance measure can be written as

$$\psi_s'' = \iint_{\Omega} \left[ \sum_{i,j=1}^{3} g_{\sigma^{ij}} \sigma^{ij}(\bar{z}) - 2a_\gamma(\bar{z}, \lambda) \right] \sqrt{s_i} \sqrt{t_j} \, d\Omega$$

where $\bar{z}$ can be obtained from equation (33)

4.2. Mixed second-order design sensitivity with respect to shape and non-shape design variables

In this section, the hybrid method is used to derive the mixed second-order design sensitivity of performance measures with respect to the shape and nonshape design variables.
4.2.1. Material property. The mixed second-order variation of the displacement performance measure with respect to the shape design variable and Young's modulus is

\[
\psi_{a}^{\bullet} = \iiint_{\Omega} e^{T} \hat{z}_{k} \delta(x - \xi) \, d\Omega
\]  

(78)

Replacing \( \hat{z}_{k} \) in equation (78) by a virtual displacement \( \tilde{\lambda} \) and equating the result to the energy bilinear form, the same adjoint equation as equation (71) can be obtained.

By setting \( \tilde{\lambda} = \hat{z}_{k} \in Z \) in equation (71) and \( \tilde{\lambda} = \lambda \in Z \) in equation (61) and using the symmetry of the energy bilinear form \( a_{\Omega}(\cdot, \cdot) \), the mixed second-order variation of the displacement performance measure with respect to the shape design variable and Young's modulus is

\[
\psi_{a}^{\bullet} = \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z_{k}) (\nabla \lambda_{i})^{T} V_{j} + \sigma^{ij}(\lambda) (\nabla z_{k})^{T} V_{j} \right] \, d\Omega
\]

\[
- \frac{\delta E}{E} \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(\tilde{\lambda}) \epsilon^{ij}(\lambda) \right] \, d\Omega
\]

\[
+ \frac{\delta E}{E} \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \epsilon^{ij}(z_{k}) \nabla \lambda_{i}^{T} V_{j} + \sigma^{ij}(\lambda) \nabla z_{k}^{T} V_{j} - \epsilon^{ij}(\tilde{\lambda}) \epsilon^{ij}(\lambda) (\text{div } V) \right] \, d\Omega
\]  

(79)

where \( \lambda \) is the adjoint response of equation (71) and \( z_{k} \) and \( \tilde{\lambda} \) can be obtained from equations (57) and (33), respectively.

4.2.2. Distribution of non-conservative pressure loading. The mixed second-order variation of the displacement performance measure with respect to the shape design variable and distribution of non-conservative pressure loading is

\[
\psi_{a}^{\bullet} = \iiint_{\Omega} e^{T} \hat{z}_{p} \delta(x - \xi) \, d\Omega
\]  

(80)

Replacing \( \hat{z}_{p} \) equation (80) by a virtual displacement \( \tilde{\lambda} \) and equating the result to the energy bilinear form, the same adjoint equation as equation (71) can be obtained.

By setting \( \tilde{\lambda} = \hat{z}_{p} \in Z \) in equation (71) and \( \tilde{\lambda} = \lambda \in Z \) in equation (67) and using the symmetry of the energy bilinear form \( a_{\Omega}(\cdot, \cdot) \), form the mixed second-order variation of the displacement performance measure with respect to the shape design variable and distribution of nonconservative pressure loading is

\[
\psi_{a}^{\bullet} = \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z_{p}) (\nabla \lambda_{i})^{T} V_{j} + \sigma^{ij}(\lambda) (\nabla z_{p})^{T} V_{j} \right] \, d\Omega
\]

\[
- \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z_{p}) \epsilon^{ij}(\lambda) \right] \, d\Omega
\]

\[
+ \iiint_{\Gamma_{3}} \sum_{i,j=1}^{3} \hat{\lambda}^{i} [DV^{ji} n^{i} - n^{i}(\text{div } V)] \, d\Gamma
\]  

(81)

where \( \lambda \) is the adjoint response of equation (71), and \( z_{p} \) can be obtained from equation (65).

By setting \( \tilde{\lambda} = \hat{z}_{p} \in Z \) in equation (75) and \( \tilde{\lambda} = \lambda \in Z \) in equation (67) and using the symmetry of the energy bilinear form \( a_{\Omega}(\cdot, \cdot) \), the mixed second-order variation of the stress performance
measures with respect to shape design variable and distribution of non-conservative pressure loading can be written as
\[ \psi_i^{\mu} = \iiint_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{\alpha_i^j} \delta_{\alpha_j^l} \left[ \sigma^{ij}(z'_p) \sigma^{kl}(\tilde{z}) - \sigma^{ij}(z'_p) \sum_{s,t=1}^{3} C^{kst} (\nabla z^s)^T \nabla V_t \right] m_p d\Omega \\
+ \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z'_p) (\nabla z^i)^T \nabla V_j + \sigma^{ij}(\tilde{z}) (\nabla z^i)^T \nabla V_j - \sigma^{ij}(z'_p) \delta^{ij}(\lambda)(\text{div } V) \right] d\Omega \\
+ \int_{\Gamma_2} \sum_{i,j=1}^{3} \lambda^j [D V^j_n] - n^i(\text{div } V)] \delta p d\Gamma \\
- \iiint_{\Omega} \sum_{i,j=1}^{3} \delta^{ij}(z'_p) \left( \sum_{k,l=1}^{3} C^{kjl} (\nabla z^k)^T \nabla V_l \right) m_p d\Omega \\
- \iiint_{\Omega} \sum_{i,j=1}^{3} g_{\alpha_i^j} \delta^{ij}(z'_p) (\text{div } V) m_p d\Omega \\
- \iiint_{\Omega} \sum_{i,j=1}^{3} g_{\alpha_i^j} \delta^{ij}(z'_p) m_p d\Omega \iiint_{\Omega} (\text{div } V) m_p d\Omega \] (82)

where \( \lambda \) is the adjoint response of equation (75), and \( z'_p \) and \( z \) can be obtained from equation (65) and (33), respectively.

4.3. Computational cost

If the design problem has \( n \) shape design variables and \( m \) non-shape design variables and \( s \) performance measures, to obtain the second-order shape design sensitivity and mixed second-order design sensitivity using the hybrid method, one original FE analysis and \( n + m + s \) FE reanalyses are required. This includes \( n + m + s \) FE reanalyses (Equations (33), (57), and (65)) for the first-order shape and non-shape design sensitivity and \( s \) FE reanalyses (Equations (71) and/or (75)) for the adjoint structures. The numerical computation of the method is more efficient than the direct differentiation method if the total number of performance measures is less than \( (n + n^2)/2 + mn \).

5. NUMERICAL METHODS OF SECOND-ORDER SHAPE DESIGN SENSITIVITY

In this section, a numerical method based on FEA results for computation of the second-order shape design sensitivity and mixed second-order design sensitivity is presented. It is similar to the method presented by Choi et al.\textsuperscript{15} although they used the method to solve sizing design sensitivity problems. Both the direct differentiation and hybrid methods are implemented. A 3-D doubly curved arch dam that is subject to water pressure and gravity force is chosen as an example. For this example, the design boundaries are defined on both traction-free and traction-loaded boundaries. The example shows excellent numerical results which verify the proposed second-order shape DSA method.

A shape design parametrization method is used to parametrize the geometric model. With the parametric representation of the boundary, an isoparametric mapping method is used to compute the boundary and domain velocity fields. There are two advantages in using the isoparametric mapping method to obtain the domain velocity. First, it can guarantee the regularity requirement for the domain velocity fields as mentioned in Section 3.1. Second, it takes less computational cost.
to obtain the domain velocity fields compared to boundary displacement method\textsuperscript{14} which use FEA method to obtain the domain velocity.

5.1. Numerical methods for shape DSA

The numerical computation of second-order shape DSA is based on the FEA results, so it can be implemented outside of the FEA code used. The numerical algorithm for the second-order shape DSA is shown in Figure 2. For the mixed second-order design sensitivity, the procedure is the same.

For the direct differentiation method, the first-order shape design sensitivity of the displacement is evaluated using equation (33). Then, the second-order shape design sensitivity of the displacement performance measure is computed from equation (49) by using the finite element reanalysis with the fictitious load on the right-hand side of equation (49). The second-order shape design sensitivity of the stress performance measure can be computed from equation (51).

For the hybrid method, the first-order shape design sensitivity of displacement is evaluated first using equation (33). After that, based on the performance type, adjoint equations (71) and (75) are used to solve adjoint responses. Finally, the second-order shape design sensitivity for the displacement and stress performance measures can be obtained from equations (74) and (77), respectively.

In the finite element reanalysis, both fictitious loads and adjoint loads can be treated as additional load cases applied to the original structure.

5.2. Shape design parametrization

Shape design parameterization must be executed carefully, because an inappropriate parametrization method will yield unacceptable shape optimal design. The shape design parameterization

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{algorithm.png}
\caption{Numerical algorithm for second-order shape DSA}
\end{figure}
method developed by Chang and Choi is used in this research because it can be applied to a broad class of shape design problems.

A 3-D geometric surface is used to illustrate the shape design parametrization method. The geometric surface has one important characteristic that the \( C^1 \)-continuity can be preserved at the connecting boundary of two consecutive surfaces. This continuity property is important in order to keep the smoothness of the design boundary since equations (42), (43), (74), and (77) require that the second-order partial derivatives of the design velocity field to be integrable in order to compute the second-order shape design sensitivity.

A 3-D geometric surface can be mathematically represented by a geometric coefficient matrix \( B \). The geometric coefficients are the positions, tangent vectors, and twist vectors at four corner points of the geometric surface. The matrix \( B \) can be written as

\[
B = \begin{bmatrix}
p_{00} & p_{01} & p_{0w} & p_{0t} \\
p_{10} & p_{11} & p_{1w} & p_{1t} \\
p_{w0} & p_{w1} & p_{ww} & p_{wt} \\
p_{t0} & p_{t1} & p_{tw} & p_{tt}
\end{bmatrix}_{4 \times 4 \times 3}
\]  

where \( u \in [0, 1] \) and \( w \in [0, 1] \) are local co-ordinates on the patch. The 48 geometric coefficients of the matrix \( B \) can be defined as the shape design parameters.

### 5.3. The numerical computation of design velocity and acceleration fields

The numerical computation methods for velocity and acceleration field are presented in this section. Using the geometric coefficients as the shape design parameters, the isoparametric mapping method is used to compute the boundary and domain velocity fields. The domain acceleration field can be computed from the domain velocity field with little additional computational cost.

To compute the boundary velocity field, the geometric coefficient matrix \( B \) should be transformed to the algebraic coefficient matrix \( A \) by premultiplying a constant universal matrix \( M \). From the algebraic coefficient matrix \( A \), the boundary velocity field can be computed as

\[
V_b = U\delta A W^T \tag{84}
\]

where vectors \( U \) and \( W \) is the location of the nodes in the parametric direction of the geometric surface. The reason that the geometric surface must be transformed to the algebraic surface is that the algebraic surface format can be used to compute the boundary velocity field directly. For the geometric surface, \( A = MBM^T \) and (85)

\[
V_b = U\delta BM^T W^T \tag{85}
\]

The domain velocity field can be computed in a similar way using the isoparametric mapping method. By parametrically mapping the variations of the design boundaries into the solid geometry, the movements of the finite element nodes within the structure can be obtained.

Finally, from equation (85), note that the design velocity field depends on the position \( x \), linearly depends on the perturbations of the shape design parameters \( \delta b \), and does not depend on design parameters \( b \) (see also equations (2) and (6)). Thus, \( V = 0 \), and the design acceleration field can be computed from

\[
\dot{V} = V' + (\nabla V)^T V = 0 \Rightarrow V' = - (\nabla V)^T V \tag{86}
\]
where $V'$ is the design acceleration field. Note that only little computational effort is required to obtain the domain acceleration field, once the domain velocity field is obtained.

5.4. A numerical example

A 3-D doubly curved arch dam based on the optimum design of Wassermann\textsuperscript{17} shown in Figure 3(a) and (b), is chosen for a numerical example to demonstrate accuracy and feasibility of the second-order shape design sensitivity with shape changes on loaded boundaries. The arch dam is subject to water pressure and gravity force. The foundation of the dam is assumed to be

![Figure 3. Arch dam (a) Two design boundary patches (b) Finite element mesh](image-url)
rigid. The material of the dam is assumed to be homogeneous and to behave elastically. The temperature effect is neglected. The physical properties include water weight density (10.0 kN/m$^3$), the gravity acceleration (10.0 m/s$^2$), the concrete weight density (25.0 kN/m$^3$), Young's modulus of concrete (21.0 GPa), and Poisson's ratio of concrete (0.2).

It is assumed that the arch dam is constructed on an idealized valley. The normal cross-section of an idealized valley is shown in Figure 4. It is assumed that the structural and loading are symmetric with respect to the crown cross-section, so only half of the span of the arch dam is analysed. Two patches, which are used to simulated water face and air face of the arch dam, are chosen as design boundaries.

The patches for water and air faces are parametrized using geometric surfaces. To parametrize a geometric surface, any of the entities in the geometric coefficient matrix $B$ can be treated as independent design parameters. In this model, the $x^2$-co-ordinates of the grids 21, 24, 33, 36, 41, 44, 53, and 56 are chosen as independent design parameters, so there are eight design parameters in this model and four of them are defined on the loaded boundary (water face). The definitions and values of each design parameter are given in Table I.

In the finite element model, the water pressure and gravity force are considered. They will contribute to the second-order shape DSA through equations (47) and (48). The water pressure is assumed linearly varied from zero at the top of the dam to 1200 kN/m$^2$ at the bottom of the dam. However, because the ANSYS FEA code is used to analyse the structural response, and it only allows step pressure to be applied on the loading surface, a step pressure profile is used in the sensitivity analyses. All nodes that connect the dam and foundation are fixed. The $x^1$-positions of

---

<table>
<thead>
<tr>
<th>Table I. Definitions of design parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design variable ID</td>
</tr>
<tr>
<td>-------------------</td>
</tr>
<tr>
<td>Grid ID</td>
</tr>
<tr>
<td>Design parameter values</td>
</tr>
</tbody>
</table>

---

![Figure 4. Normal cross-section of an idealized valley](image-url)
the nodes on the centre cross-section are fixed. The finite element model of the arch dam includes 315 nodes, 36 20-node isoparametric elements, and 726 active degrees of freedom.

Eight shape design velocity fields are generated corresponding to the eight independent design parameters. Sixty-four shape design acceleration fields are generated from the eight shape design velocity fields. The direct differentiation method and hybrid method are used to compute the second-order shape design sensitivity and mixed second-order design sensitivity. The domain velocity is computed by the isoparametric mapping method, so the $C^1$-continuity of the velocity fields can be satisfied.

In this example, the first principal stresses at Gauss points of finite elements are chosen as the stress performance measures. For a numerical test, the first eight elements with high tensile and compressible first principal stress are selected. In order to verify the mixed second-order design sensitivity with respect to the shape design variable and material property, eight displacement performance measures are defined at nodes that have maximum displacements in $x^2$-direction. The second-order sensitivity results are verified by using the finite difference method.

Table II shows the values of the selected stress performance measures and the first-order sensitivity of these performance measures with respect to the variations of water pressure. Obviously, the stress performance measures are not sensitive to the variation of the water pressure. Tables III–IV present the verification results of the first-order sensitivity. Tables III and IV are first-order shape design sensitivity with respect to shape design variables $b_6$ and $b_8$.

<table>
<thead>
<tr>
<th>Element ID</th>
<th>Gauss point</th>
<th>First principal stress</th>
<th>1st-order DSA w.r.t. $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.217672707561858E + 04</td>
<td>0.691413456D + 01</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.213157640998988E + 04</td>
<td>0.673086159D + 01</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.200545302385018E + 04</td>
<td>0.625868795D + 01</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.178274336797057E + 04</td>
<td>0.551374682D + 01</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>-0.101712111123081E + 04</td>
<td>-0.921305376D + 00</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>-0.97375302365233E + 03</td>
<td>-0.840378822D + 00</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>-0.883229150143620E + 03</td>
<td>0.857373634D + 00</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>-0.753388300835157E + 03</td>
<td>-0.260082701D + 00</td>
</tr>
</tbody>
</table>

Table III. Verification of first-order shape design sensitivity of first principal stress with respect to shape design variable $b_6$.

<table>
<thead>
<tr>
<th>Element ID</th>
<th>Gauss point</th>
<th>$\delta\psi(b)$</th>
<th>$\psi'(b)$</th>
<th>$\psi'(b)/\delta\psi(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-0.643537179D + 02</td>
<td>-0.642763379D + 02</td>
<td>99.88</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-0.619405836D + 02</td>
<td>-0.618627572D + 02</td>
<td>99.87</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-0.560541237D + 02</td>
<td>-0.559828536D + 02</td>
<td>99.87</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-0.465250312D + 02</td>
<td>-0.474645314D + 02</td>
<td>99.87</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>-0.631979095D + 01</td>
<td>-0.623057290D + 01</td>
<td>98.90</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>-0.129379632D + 02</td>
<td>-0.128201061D + 02</td>
<td>99.09</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>-0.320606220D + 02</td>
<td>-0.329942234D + 02</td>
<td>99.79</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>-0.125831626D + 02</td>
<td>-0.125410448D + 02</td>
<td>99.67</td>
</tr>
</tbody>
</table>

$b_6 = 129$, $\delta b_6 = 0.645$
Table IV. Verification of first-order shape design sensitivity of first principal stress with respect to shape design variable $b_1'$

<table>
<thead>
<tr>
<th>Element ID</th>
<th>Gauss point</th>
<th>$\delta \psi(b)$</th>
<th>$\psi'(b)$</th>
<th>$\psi'(b)/\delta \psi(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-0.799369699D + 02</td>
<td>-0.799296768D + 02</td>
<td>99.99</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-0.750269271D + 02</td>
<td>-0.750288859D + 02</td>
<td>99.99</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-0.667135811D + 02</td>
<td>-0.667082826D + 02</td>
<td>99.99</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-0.575078430D + 02</td>
<td>-0.575057023D + 02</td>
<td>100.00</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>0.375322103D + 01</td>
<td>0.375644301D + 01</td>
<td>100.09</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>0.495610419D + 01</td>
<td>0.495626881D + 01</td>
<td>100.01</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>0.162137517D + 02</td>
<td>0.162195583D + 02</td>
<td>100.00</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>0.723811736D + 02</td>
<td>0.722128383D + 01</td>
<td>99.77</td>
</tr>
</tbody>
</table>

* $b_8 = 130, \delta b_8 = 0.65$

Table V. Verification of first-order design sensitivity of first principal stress with respect to pressure loading $p^d$

<table>
<thead>
<tr>
<th>Element ID</th>
<th>Gauss point</th>
<th>$\delta \psi(b)$</th>
<th>$\psi'(b)$</th>
<th>$\psi'(b)/\delta \psi(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.138277447D + 03</td>
<td>0.138282691D + 03</td>
<td>100.00</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.134612200D + 03</td>
<td>0.134627232D + 03</td>
<td>100.00</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.125169225D + 03</td>
<td>0.125273759D + 03</td>
<td>100.00</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.110271072D + 03</td>
<td>0.110274936D + 03</td>
<td>100.00</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>-0.183801236D + 02</td>
<td>-0.184261147D + 02</td>
<td>100.25</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>-0.951656869D + 01</td>
<td>-0.960757644D + 01</td>
<td>100.95</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>0.171327034D + 02</td>
<td>0.171474727D + 02</td>
<td>100.09</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>-0.512337906D + 01</td>
<td>-0.520165402D + 01</td>
<td>101.53</td>
</tr>
</tbody>
</table>

* $\delta p = 10$

Table VI. Verification of second-order shape design sensitivity of first principal stress with respect to shape design variable $b_1^d$

<table>
<thead>
<tr>
<th>Element ID</th>
<th>Gauss point</th>
<th>$\delta^2 \psi(b)$</th>
<th>$\psi''(b)$</th>
<th>$\psi''(b)/\delta^2 \psi(b)$</th>
<th>$\psi''(b)/\delta \psi(b)$</th>
<th>$\psi''(b)/\delta^2 \psi(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.130895613D + 02</td>
<td>0.130762340D + 02</td>
<td>100.10</td>
<td>100.10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.121978259D + 03</td>
<td>0.121669853D + 03</td>
<td>100.25</td>
<td>100.25</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.107518907D + 02</td>
<td>0.106854091D + 02</td>
<td>100.62</td>
<td>100.62</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.893278337D + 01</td>
<td>0.890867650D + 01</td>
<td>100.27</td>
<td>100.27</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>0.429678661D + 01</td>
<td>0.434543928D + 01</td>
<td>98.88</td>
<td>98.88</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>0.251031667D + 01</td>
<td>0.253784373D + 01</td>
<td>98.92</td>
<td>98.92</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>0.134155199D + 01</td>
<td>0.162106989D + 01</td>
<td>98.25</td>
<td>98.25</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>0.832807276D + 02</td>
<td>0.830271689D + 00</td>
<td>100.31</td>
<td>100.31</td>
<td></td>
</tr>
</tbody>
</table>

* $b_1 = 115, \delta b_1 = 2.3$ respectively, which are defined on the loaded boundary. Table V is the first-order design sensitivity with respect to the pressure loading. Tables VI–X show the verification results of the second-order sensitivity. Tables VI, VII and IX are second-order shape design sensitivity, and Table VIII is mixed second-order sensitivity with respect to both shape design variable-$b_1$ and non-shape design variable distribution of water pressure. Table X is mixed second-order sensitivity with respect to both shape design variable-$b_1$ and Young’s modulus.
Table VII. Verification of second-order shape design sensitivity of first principal stress with respect to shape design variable $b_a^*$.  

<table>
<thead>
<tr>
<th>Element ID</th>
<th>Gauss point</th>
<th>$\delta^2 \psi(b)$</th>
<th>$\psi''(b)$</th>
<th>$\psi''(b)$</th>
<th>$\psi''(b)/\delta^2 \psi(b)$</th>
<th>$\psi''(b)/\delta^2 \psi(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.598193155D + 01</td>
<td>0.598717853D + 01</td>
<td>0.598717855D + 01</td>
<td>99.91</td>
<td>99.91</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.507540824D + 01</td>
<td>0.507540823D + 01</td>
<td>0.507540823D + 01</td>
<td>100.02</td>
<td>100.02</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.463400780D + 01</td>
<td>0.459019023D + 01</td>
<td>0.459019023D + 01</td>
<td>100.96</td>
<td>100.96</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.56896501D + 01</td>
<td>0.564474763D + 01</td>
<td>0.564474762D + 01</td>
<td>100.48</td>
<td>100.48</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>-0.16816415D + 00</td>
<td>-0.16547566D + 00</td>
<td>-0.16547566D + 00</td>
<td>102.02</td>
<td>102.02</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>0.480638420D + 01</td>
<td>0.478340279D + 01</td>
<td>0.478340279D + 01</td>
<td>100.48</td>
<td>100.48</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>0.877708602D + 01</td>
<td>0.878715176D + 01</td>
<td>0.878715176D + 01</td>
<td>99.89</td>
<td>99.89</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>0.748838016D + 01</td>
<td>0.752140476D + 01</td>
<td>0.752140476D + 01</td>
<td>99.56</td>
<td>99.56</td>
</tr>
</tbody>
</table>

For Tables III–V, the first column is the element ID, the second column is Gauss point ID, the third column is the first-order finite difference results, and the fourth column is the first-order design sensitivity results from the direct differentiation method, and the last column is the verification results. In these tables,  

$$ \psi' = 2(\partial \psi/\partial b_i)(\delta b_i) $$  \hspace{1cm} (87)  

and  

$$ \delta \psi = \psi(b_i + \delta b_i) - \psi(b_i - \delta b_i) $$ \hspace{1cm} (88)  

where $\delta b_i$ are the variations of design variables $b_i$ for $i = 1-8$.

The first column is the element ID for Tables VI–IX and the node ID for Table X, and the second column is Gauss point ID for Tables VI–IX and co-ordinate direction for Table X. In these tables, the third column is the second-order finite difference result, and the fourth and fifth columns are the second-order design sensitivity results from the direct differentiation and hybrid methods, respectively. The last two columns are verification results of the second-order design sensitivity from two methods. In these tables, $\psi''$ and $\delta^2 \psi$ are  

$$ \psi'' = (\partial^2 \psi/\partial b_i^2)(\delta b_i)^2 $$ \hspace{1cm} (89)
Table IX. Verification of second-order shape design sensitivity of first principal stress with respect to shape design variables $b_a^*$ and $b_s^b$

<table>
<thead>
<tr>
<th>Element ID</th>
<th>Gauss point</th>
<th>$\delta^2\psi(b)$ Finite difference</th>
<th>$\psi''(b)$ Direct method</th>
<th>$\psi''(b)$ Hybrid method</th>
<th>$\psi''(b)/\delta^2\psi(b)$ Direct</th>
<th>$\psi''(b)/\delta^2\psi(b)$ Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-0.872342681D - 01</td>
<td>-0.87223694D - 01</td>
<td>-0.872296794D - 01</td>
<td>99.99</td>
<td>99.99</td>
</tr>
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<td>2</td>
<td>1</td>
<td>-0.865118581D - 01</td>
<td>-0.865781485D - 01</td>
<td>-0.865071485D - 01</td>
<td>99.99</td>
<td>99.99</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-0.821275380D - 01</td>
<td>-0.821272206D - 01</td>
<td>-0.821272201D - 01</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-0.68993271D - 01</td>
<td>-0.690058524D - 01</td>
<td>-0.690058519D - 01</td>
<td>100.01</td>
<td>100.01</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>0.257293490D - 02</td>
<td>0.257156786D - 02</td>
<td>0.257156786D - 02</td>
<td>99.95</td>
<td>99.95</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>0.78607421D - 02</td>
<td>0.78586481D - 02</td>
<td>0.78586481D - 02</td>
<td>99.97</td>
<td>99.97</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>0.16764633D - 02</td>
<td>0.167643877D - 01</td>
<td>0.167643877D - 01</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>-0.61876444D - 02</td>
<td>-0.619122369D - 02</td>
<td>-0.619122358D - 02</td>
<td>100.06</td>
<td>100.06</td>
</tr>
</tbody>
</table>

*a* $b_a = 103$, $\delta b_a = 0.103$

*b* $b_s = 130$, $\delta b_s = 0.13$

Table X. Verification of second-order design sensitivity of first principal stress with respect to shape design variable $b_i^*$ and Young's modulus $E^b$

<table>
<thead>
<tr>
<th>Node ID</th>
<th>Co-ordinate direction</th>
<th>$\delta^2\psi(b)$ Finite difference</th>
<th>$\psi''(b)$ Direct method</th>
<th>$\psi''(b)$ Hybrid method</th>
<th>$\psi''(b)/\delta^2\psi(b)$ Direct</th>
<th>$\psi''(b)/\delta^2\psi(b)$ Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>87</td>
<td>2</td>
<td>0.829904564D - 05</td>
<td>0.828007744D - 05</td>
<td>0.828007735D - 05</td>
<td>99.77</td>
<td>99.77</td>
</tr>
<tr>
<td>88</td>
<td>2</td>
<td>0.84160840D - 05</td>
<td>0.832255428D - 05</td>
<td>0.832255419D - 05</td>
<td>99.77</td>
<td>99.77</td>
</tr>
<tr>
<td>89</td>
<td>2</td>
<td>0.843925020D - 05</td>
<td>0.841964878D - 05</td>
<td>0.841964848D - 05</td>
<td>99.77</td>
<td>99.77</td>
</tr>
<tr>
<td>90</td>
<td>2</td>
<td>0.859352034D - 05</td>
<td>0.857357547D - 05</td>
<td>0.857357515D - 05</td>
<td>99.77</td>
<td>99.77</td>
</tr>
<tr>
<td>300</td>
<td>2</td>
<td>0.832750985D - 05</td>
<td>0.830848288D - 05</td>
<td>0.830848279D - 05</td>
<td>99.77</td>
<td>99.77</td>
</tr>
<tr>
<td>301</td>
<td>2</td>
<td>0.879848947D - 05</td>
<td>0.877820128D - 05</td>
<td>0.877820128D - 05</td>
<td>99.77</td>
<td>99.77</td>
</tr>
<tr>
<td>302</td>
<td>2</td>
<td>0.887522474D - 05</td>
<td>0.885457271D - 05</td>
<td>0.885457253D - 05</td>
<td>99.77</td>
<td>99.77</td>
</tr>
<tr>
<td>303</td>
<td>2</td>
<td>0.85243776D - 05</td>
<td>0.850458677D - 05</td>
<td>0.850458644D - 05</td>
<td>99.77</td>
<td>99.77</td>
</tr>
</tbody>
</table>

*a* $b_i = 115$, $\delta b_i = 0.575$

*b* $E = 1.0E + 6$

and

$$\delta^2\psi = \psi(b_i + \delta b_i) - 2\psi(b_i) + \psi(b_i - \delta b_i)$$  \hspace{1cm} (90)

for second-order derivative with respect to the design variable $b_i$, and

$$\psi'' = 4(\delta^2\psi/\partial b_i\partial b_j)(\delta b_i)(\delta b_j)$$  \hspace{1cm} (91)

and

$$\delta^2\psi = \psi(b_i + \delta b_i, b_j + \delta b_j) - \psi(b_i - \delta b_i, b_j + \delta b_j) - \psi(b_i + \delta b_i, b_j - \delta b_j)$$

$$+ \psi(b_i - \delta b_i, b_j - \delta b_j)$$  \hspace{1cm} (92)

for the mixed second-order derivative, where $\delta b_i$ is the perturbation of the design variable $b_i$ for $i, j = 1-8$.

Tables III and IV give very good first-order sensitivity results for shape design variable defined on the loaded boundary. Table V demonstrates accurate first-order sensitivity with respect to the pressure loading variable. Results in Tables VI-X indicate that the accuracy of second-order shape design sensitivity is very good, even though the stress performance measures are defined on the elements with high tensile and compressible stresses, and displacement performance measures are defined on the nodes with maximum deformation. For the design variables defined on the loaded boundary, Tables VII and IX show pretty good results. These numerical results
demonstrate accuracy and feasibility of the theoretical derivations and numerical implementations of the second-order shape DSA by considering shape changes on the loaded boundaries.

6. CONCLUSIONS

A continuum design sensitivity approach with the material derivative idea is used to derive the explicit formulations of the second-order shape DSA which considers shape changes on the loaded boundaries. Also the mixed second-order design sensitivity with respect to shape and nonshape design variables are obtained which are valuable information for studying robust design. Both direct differentiation and hybrid methods are presented. In determining which of the two methods discussed in this paper to be employed, it depends on the number of shape and non-shape design variables and performance measures of shape design problems.

It is interesting to note that the computation of acceleration fields need only little additional effort if the design velocity fields are available. The $C^1$-continuity of the velocity fields can be obtained by using the isoparametric mapping method to compute the domain velocity. A 3-D doubly curved arch dam example demonstrates feasibility and accuracy of the proposed numerical method. The numerical results show that the proposed method can be used to support the design problems that have shape changes on loaded boundaries.

ACKNOWLEDGEMENT


REFERENCES