APPLICATIONS OF VARIATIONAL METHODS

In order to illustrate the variational theory developed in Chapter 10, a number of applications are studied in this chapter. The examples, which are of interest in themselves, illustrate the complexity encountered in general problems that precludes closed form solutions. Concrete applications of physical significance also show that the power of variational methods rests on careful verification of hypotheses that underlie the methods.

11.1 BOUNDARY-VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

Many problems in mechanics reduce to finding the solution of an ordinary differential equation that satisfies homogeneous boundary conditions. The Ritz method of Chapter 10 can be applied to such problems, if it can be established that the associated operator is symmetric and positive definite, or even better positive bounded below.

Theoretical Results

Consider first the ordinary differential equation

\[ Au = - \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x) u = f(x) \]  \hspace{1cm} (11.1.1)

A solution is sought in \( C^1(a, b) \) that satisfies the boundary conditions

\[ \alpha \ u'(a) - \beta \ u(a) = 0 \]
\[ \gamma \ u'(b) + \delta \ u(b) = 0 \]

(11.1.2)

where \( \alpha, \beta, \gamma, \) and \( \delta \) are constants. The following assumptions are made regarding the functions \( p(x) \) and \( q(x) \) and constants \( \alpha, \beta, \gamma, \) and \( \delta: \)

(a) \( p(x), \ p'(x), \) and \( q(x) \) are continuous in \( a \leq x \leq b. \)
(b) \( p(x) \geq 0, \ q(x) \geq 0. \)
(c) the function \( p(x) \) can vanish at a number of points in \( a \leq x \leq b, \) but the integral
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\[ F = \int_a^b \frac{dx}{p(x)} \]

has finite value. In particular, this requirement is satisfied if \( p(x) \geq k > 0 \) in \( a \leq x \leq b \).

(d) the constants \( \alpha, \beta, \gamma, \) and \( \delta \) are non-negative and neither of the pairs \( (\alpha, \beta) \) or \( (\gamma, \delta) \) vanishes.

The domain \( D_A \) of the operator \( A \) defined in Eq. 11.1.1 is the set of functions in \( C^2(a, b) \) that satisfy boundary conditions of Eqs. 11.1.2; i.e.,

\[
D_A = \{ u(x) \in C^2(a, b): \alpha u'(a) - \beta u(a) = 0, \gamma u'(b) + \delta u(b) = 0 \}
\]

(11.1.3)

To prove that the operator \( A \) is symmetric, form the scalar product \( (Au, v) \), for functions \( u \) and \( v \) in \( D_A \),

\[
(Au, v) = -\int_a^b v(x) \frac{d}{dx} \left( p(x) \frac{d}{dx} u(x) \right) dx + \int_a^b q(x) u(x) v(x) dx
\]

Integrating the first integral by parts and using Eqs. 11.1.2,

\[
(Au, v) = \int_a^b \left( p(x) \frac{d}{dx} u(x) \right) \frac{d}{dx} v(x) + q(x) u(x) v(x) dx
\]

\[
+ \frac{\beta}{\alpha} p(a) u(a) v(a) + \frac{\delta}{\gamma} p(b) u(b) v(b)
\]

(11.1.4)

if \( \alpha \neq 0 \) and \( \gamma \neq 0 \). If \( \alpha = 0 \) or \( \gamma = 0 \), then \( u(a) = 0 \) or \( u(b) = 0 \) and the corresponding terms on the right of Eq. 11.1.4 can be omitted. The expression on the right of Eq. 11.1.4 is symmetric in \( u(x) \) and \( v(x) \), hence \( (Au, v) = (u, Av) \) and the operator \( A \) is symmetric.

Putting \( v(x) = u(x) \) in Eq. 11.1.4,

\[
(Au, u) = \int_a^b \left[ p(x) \left( \frac{d}{dx} u(x) \right)^2 + q(x) u^2(x) \right] dx
\]

\[
+ \frac{\beta}{\alpha} p(a) u^2(a) + \frac{\delta}{\gamma} p(b) u^2(b)
\]

(11.1.5)

**Theorem 11.1.1.** If \( p(a) \) and \( p(b) \) are positive, then the operator \( A \) of Eq. 11.1.1 with domain \( D_A \) of Eq. 11.1.3 is positive bounded below.

To prove this result, suppose first that \( \beta > 0 \) and \( \alpha \neq 0 \neq \gamma \). On the right of Eq. 11.1.5, the second term under the integral and the last term may be deleted, to obtain the inequality
\[
(Au, u) \geq \int_a^b p \left( \frac{du}{dx} \right)^2 dx + \frac{\beta}{\alpha} p(a) u^2(a)
\]

\[
\geq B \left[ \int_a^b pu^2 \, dx + u^2(a) \right]
\]

(11.1.6)

where B is the lesser of \((\beta/\alpha) p(a)\) and 1. Since

\[
u(x) = u(a) + \int_a^x u'(t) \, dt
\]

and \((a + b)^2 \leq (a + b)^2 + (a - b)^2 = 2(a^2 + b^2),

\[
u^2(x) \leq 2u^2(a) + 2 \left[ \int_a^x u'(t) \, dt \right]^2
\]

Further,

\[
\left( \int_a^x u'(t) \, dt \right)^2 = \left( \int_a^x \frac{1}{\sqrt{p(t)}} \sqrt{p(t)} u'(t) \, dt \right)^2
\]

and by the Schwartz inequality,

\[
\left( \int_a^x u'(t) \, dt \right)^2 \leq \int_a^x \frac{dt}{p(t)} \int_a^x p(t) \, u'^2(t) \, dt
\]

Increasing the upper limit of integration on the right only strengthens the inequality, so

\[
\left( \int_a^x u'(t) \, dt \right)^2 \leq \int_a^b \frac{dt}{p(t)} \int_a^b p(t) \, u'^2(t) \, dt = F \int_a^b p(t) \, u'^2(t) \, dt
\]

and consequently

\[
u^2(x) \leq 2F \int_a^b p(t) \, u'^2(t) \, dt + 2u^2(a)
\]

Integrating this equation with respect to \(x\) in the range from \(a\) to \(b\),

\[
\|u\|^2 \leq C \left[ \int_a^b p(t) \, u'^2(t) \, dt + u^2(a) \right]
\]

(11.1.7)

where \(C\) is the larger of \(2F(b - a)\) and \(2(b - a)\). The inequalities of Eqs. 11.1.6 and 11.1.7 yield
i.e., the operator $A$ is positive bounded below.

It was assumed that $\alpha \neq 0$ and $\gamma \neq 0$. If either $\alpha$ or $\gamma$ vanishes, then the statement about the operator $A$ being positive bounded below is still true, the reasoning becoming even simpler. For example, let $\alpha = 0$. The first of the boundary conditions of Eq. 11.1.2 assumes the form $u(a) = 0$. In place of Eqs. 11.1.6 and 11.1.7, the simple inequalities

$$ (Au, u) \geq \frac{B}{C} \int_a^b p(x) u'^2(x) \, dx $$

and

$$ \|u\|^2 \leq 2F(b-a) \int_a^b p(x) u'^2(x) \, dx $$

are obtained. Equation 11.1.8 follows, if $B/C$ is replaced by $1/[2F(b-a)]$. 

By virtue of the minimum functional theorem, the problem of solving Eq. 11.1.1 subject to the boundary conditions of Eqs. 11.1.2 reduces, if $\alpha \neq 0$ and $\gamma \neq 0$, to finding the function $u(x)$ that minimizes the functional

$$ F(u) = (Au, u) - 2(f, u) $$

$$ = \frac{B}{\alpha} p(a) u^2(a) + \frac{\delta}{\gamma} p(b) u^2(b) $$

$$ + \int_a^b [p(x) u'(x) + q(x) u^2(x) - 2f(x) u(x)] \, dx \quad (11.1.9) $$

If $\alpha \neq 0$ or $\gamma \neq 0$, then the corresponding boundary conditions of Eq. 11.1.2 are natural. If $\alpha = 0$ or $\gamma = 0$, then the corresponding boundary conditions are principal.

Consider next an ordinary differential equation of arbitrary even order, of the form

$$ Au = \sum_{k=0}^m (-1)^k \frac{d^k}{dx^k} \left[ p_k(x) \frac{d^k u}{dx^k} \right] = f(x) \quad (11.1.10) $$

and confine attention to the simplest of boundary conditions, namely

$$ u(a) = u'(a) = \ldots = u^{(m-1)}(a) = u(b) = u'(b) = \ldots = u^{(m-1)}(b) = 0 \quad (11.1.11) $$

It is assumed that the coefficients $p_k(x)$, $k = 0, 1, 2, \ldots, m - 1$, are non-negative and that $p_m(x)$ is strictly positive. Integrating by parts and using Eq. 11.1.11,
\[(Au, u) = \sum_{k=0}^{m} \int_{a}^{b} p_k(x) \left( \frac{d^k u}{dx^k} \right)^2 \ dx \]

\[\geq \int_{a}^{b} p_m(x) \left( \frac{d^m u}{dx^m} \right)^2 \ dx \geq p_0 \|u^{(m)}\|^2 \quad (11.1.12)\]

where

\[p_0 \equiv \min_{x \in [a, b]} p_m(x) > 0\]

Since \(u(a) = 0\), \(u(x) = \int_{a}^{x} u'(\xi) \ d\xi\). The Schwartz inequality yields

\[u^2(x) = \left( \int_{a}^{x} 1 \times u'(\xi) \ d\xi \right)^2 \leq (x - a) \int_{a}^{x} u^2(\xi) \ d\xi \]

\[\leq (x - a) \int_{a}^{b} u^2(\xi) \ d\xi = (x - a) \|u'\|^2\]

Integrating both sides of this inequality over \((a, b)\),

\[\|u\|^2 \leq \frac{(b - a)^2}{2} \|u'\|^2\]

or

\[\|u\| \leq \frac{b - a}{\sqrt{2}} \|u'\|\]

\[\|u'\| \leq \frac{b - a}{\sqrt{2}} \|u''\|\]

\[\vdots\]

\[\|u^{(m-1)}\| \leq \frac{b - a}{\sqrt{2}} \|u^{(m)}\|\]

Thus, \(\|u^{(m)}\| \geq \left( \frac{\sqrt{2}}{b - a} \right)^m \|u\|\). Substituting this result into Eq. 11.1.12,

\[(Au, u) \geq \gamma^2 \|u\|^2\]

where \(\gamma = \sqrt{p_0} \left( \frac{\sqrt{2}}{b - a} \right)^m\). This proves the following result.
Theorem 11.1.2. Under the conditions of Eq. 11.1.11, the operator of Eq. 11.1.10 is positive bounded below.

Bending of a Beam of Variable Cross Section on an Elastic Foundation

The equation for bending of a beam on an elastic foundation [25] has the form

\[ Au = \frac{d^2}{dx^2} \left[ EI(x) \frac{d^2u}{dx^2} \right] + Ku = f(x) \]  

(11.1.13)

where \( u(x) \) is the deflection of the beam and \( I(x) \) is the moment of inertia of the cross section, which may vary with \( x \). For simplicity, assume that nowhere does the beam cross-sectional area degenerate to zero, so that the moment of inertia \( I(x) \) never vanishes. In Eq. 11.1.13, \( E \) is Young's modulus of the beam material, \( K \) is the distributed spring constant for the elastic foundation, and \( f(x) \) is the applied normal loading. The length of the beam is \( \ell \). If the ends of the beam are clamped, then the following boundary conditions must be satisfied:

\[ u(0) = u(\ell) = 0 \]
\[ u'(0) = u' (\ell) = 0 \]  

(11.1.14)

Equations 11.1.13 and 11.1.14 are special cases of Eqs. 11.1.10 and 11.1.11. It is concluded immediately that the operator defined in Eq. 11.1.13 and the boundary conditions of Eq. 11.1.14 is positive bounded below and that the problem of bending of a beam with clamped ends that rests on an elastic foundation is equivalent to the problem of minimizing the functional

\[ F(u) = (Au, u) - 2(u, f) \]

over all functions \( u(x) \) that satisfy the boundary conditions of Eq. 11.1.14. Integrating by parts,

\[ F(u) = \int_a^b \left[ EIu''^2 + Ku^2 - 2fu \right] dx \]  

(11.1.15)

The solution of the problem of finding the minimum of this functional can be obtained by the Ritz method, for which it is necessary to choose a sequence of coordinate functions \( \{ \phi_n(x) \} \) that is complete in energy. Since the boundary conditions of Eq. 11.1.14 are principal, the coordinate functions must satisfy them; e.g.,

\[ \phi_k(x) = (\ell - x)^2 x^{k+1}, \quad k = 1, 2, \ldots \]  

(11.1.16)

Theorem 10.7.4 can be used to show that these coordinate functions are complete in energy (see Exercise 10.7.1).
An approximate solution of this problem is thus obtained by putting

\[ u_n(x) = \sum_{k=1}^{n} a_k \phi_k(x) = (\mathcal{L} - x)^2 \sum_{k=1}^{n} a_k x^{k+1} \]  \hspace{1cm} (11.1.17)

and determining the coefficients from the Ritz equations of Eq. 10.8.8, which may be written as

\[ \sum_{k=1}^{n} a_k A_{ik} = b_i, \quad i = 1, 2, \ldots, n \]  \hspace{1cm} (11.1.18)

where

\[ b_k = (f, \phi_k) = \int_{0}^{\mathcal{L}} f(x) (\mathcal{L} - x)^2 x^{k+1} \, dx \]

The coefficients \( A_{ik} \) can be represented as

\[ A_{ik} = (A \phi_i, \phi_k) \]

\[ = \int_{0}^{\mathcal{L}} \phi_k A \phi_i \, dx = \int_{0}^{\mathcal{L}} \phi_k \left[ \frac{d^2}{dx^2} \left( EI \frac{d^2 \phi_i}{dx^2} \right) + K \phi_i \right] \, dx \]

or,

\[ A_{ik} = [\phi_i, \phi_k]_A = \int_{0}^{\mathcal{L}} \left[ EI \frac{d^2 \phi_i}{dx^2} \frac{d^2 \phi_k}{dx^2} + K \phi_i \phi_k \right] \, dx \]

**Eigenvalues of a Second-Order Ordinary Differential Equation**

To illustrate use of the Ritz method for an eigenvalue problem, consider the equation

\[ Au \equiv -\frac{d}{dx} \left[ \sqrt{1+x} \frac{du}{dx} \right] = \lambda u \]  \hspace{1cm} (11.1.19)

with the boundary conditions

\[ u(0) = u(1) = 0 \]  \hspace{1cm} (11.1.20)

In seeking a solution by the Ritz method, coordinate functions may be chosen as

\[ \phi_k(x) = (1 - x) x^k, \quad k = 1, 2, \ldots \]

which satisfy the principal boundary conditions of Eq. 11.1.20. Consider the first three coordinate functions,

\[ \phi_1(x) = (1 - x) x, \quad \phi_2(x) = (1 - x) x^2, \quad \phi_3(x) = (1 - x) x^3 \]
Applying the method of Section 10.9, Eq. 10.9.7 yields the characteristic equation [21]

\[
\begin{vmatrix}
0.4048 - \frac{\lambda}{30} & 0.2161 - \frac{\lambda}{60} & 0.1350 - \frac{\lambda}{105} \\
0.2161 - \frac{\lambda}{60} & 0.5108 - \frac{\lambda}{105} & 0.6754 - \frac{\lambda}{168} \\
0.1350 - \frac{\lambda}{105} & 0.6754 - \frac{\lambda}{168} & 1.0238 - \frac{\lambda}{262}
\end{vmatrix} = 0
\]

The crudest one term approximation to the first eigenvalue is obtained by equating the first diagonal minor to zero; i.e., \(0.4048 - \lambda/30 = 0\). This corresponds to using only one coordinate function \(\phi_1(x)\) in the Ritz method, yielding the approximation \(\lambda_1^{(1)} = 12.14\). The superscript denotes the number of terms used in the approximation and the subscript denotes the number of the eigenvalue.

The two term approximation of the first eigenvalue is obtained by equating the first \(2 \times 2\) minor to zero; i.e.,

\[
\begin{vmatrix}
0.4048 - \frac{\lambda}{30} & 0.2161 - \frac{\lambda}{60} \\
0.2161 - \frac{\lambda}{60} & 0.5108 - \frac{\lambda}{105}
\end{vmatrix} = 0
\]

The smallest root of this equation is \(\lambda_1^{(2)} = 12.12\), which gives a more precise approximation to the smallest eigenvalue.

Solving the three term approximation by Newton's method yields

\[\lambda_1^{(3)} = 12.12\]

which should be quite accurate.

**Stability of a Column**

Consider a column of variable cross section that is compressed by longitudinal forces of magnitude \(P\) applied to its ends. The differential equation describing the deformed axis of the column is [25]

\[
E \frac{d^2}{dx^2} \left[ I(x) \frac{d^2u}{dx^2} \right] = P \left( - \frac{d^2u}{dx^2} \right) \quad (11.1.21)
\]

where \(E\) is Young's modulus for the material and \(I(x)\) is the centroidal moment of inertia of the cross section. In addition, boundary conditions that define the nature of the supports at the ends of the column must be specified. Consideration here is limited to the following two types of supports:

1. Both ends of the column are clamped. Assuming that the column in an unbent state occupies the segment \([0, \ell]\) of the \(x\)-axis, this is

\[u(0) = u(\ell) = 0, \quad u'(0) = u'(\ell) = 0 \quad (11.1.22)\]
Both ends of the column are pinned. In this case,

\[ u(0) = u(l) = 0, \quad u''(0) = u''(l) = 0 \]  \hspace{1cm} (11.1.23)

The investigation of other types of boundary conditions does not present any essential difficulties.

Equations 11.1.21 and 10.9.20 are of the same type, \( P \) playing the part of the parameter \( \lambda \). In this case,

\[ Au = E \frac{d^2}{dx^2} \left[ I(x) \frac{d^2 u}{dx^2} \right] \]

\[ Bu = - \frac{d^2 u}{dx^2} \]

both of which are positive definite operators, with the boundary conditions of Eqs. 11.1.22 or 11.1.23.

It is easy to show by integration by parts, with the boundary conditions of Eqs. 11.1.22 or 11.1.23, that

\[
\| u \|_A^2 = (Au, u) = E \int_0^L I(x) \left[ \frac{d^2 u(x)}{dx^2} \right]^2 dx \\
\| u \|_B^2 = (Bu, u) = \int_0^L \left[ \frac{d u(x)}{dx} \right]^2 dx
\]

The problem of stability of a column reduces to the problem of finding the eigenvalues of Eq. 11.1.21, under boundary conditions of Eqs. 11.1.22 or 11.1.23. For the clamped case of Eq. 11.1.22, the lowest critical load is given by

\[
P_1 = \min_u \frac{\| u \|_A^2}{\| u \|_B^2} = E \min_u \frac{\int_0^L I(x) u''^2(x) dx}{\int_0^L u''^2(x) dx} \hspace{1cm} (11.1.24)
\]

the minimum being sought in the class of functions that satisfy the boundary conditions of Eq. 11.1.22.

The case of pinned ends of Eq. 11.1.23 can also be investigated by the Ritz method. The lowest critical load is determined by Eq. 11.1.24, but the minimum can be sought in a wider class of functions, namely those satisfying only the principal boundary conditions \( u(0) = u(l) = 0 \), the conditions \( u''(0) = u''(l) = 0 \) of Eq. 11.1.23 being natural. In general, widening of the class of functions decreases the minimum. Hence it follows that, for pinned ends, the smallest critical load is lower than for clamped ends.
Vibration of a Beam of Variable Cross Section

The equation for free vibration of a beam of variable cross section is [25]

\[
E \frac{d^2}{dx^2} \left[ I(x) \frac{d^2u}{dx^2} \right] + \rho \frac{d^2u}{dt^2} = 0 \tag{11.1.25}
\]

where the x-axis is directed along the axis of the beam, \( u(x, t) \) is its transverse displacement, \( I(x) \) and \( S(x) \) are the geometrical moment of inertia and the area of the cross section, and \( E \) and \( \rho \) are Young's modulus and the density of the beam material. Suppose that one end of the beam is clamped and the other end is free; i.e., the beam is cantilevered. Denote the length of the beam by \( \lambda \) and place the coordinate origin at its clamped left end. The boundary conditions can then be written as

\[
u \bigg|_{x=0} = 0, \quad \frac{d\nu}{dx} \bigg|_{x=0} = 0 \tag{11.1.26}
\]

\[
\frac{d^2\nu}{dx^2} \bigg|_{x=\lambda} = 0, \quad \frac{d^3\nu}{dx^3} \bigg|_{x=\lambda} = 0 \tag{11.1.27}
\]

For harmonic motion; i.e., free vibration, a solution is sought in the form

\[
u(x, t) = \nu(x) \sin \left( \sqrt{\lambda} t + \alpha \right)
\]

Equations 11.1.25, 11.1.26, and 11.1.27 transform to the following eigenvalue problem:

\[
Au = E \frac{d^2}{dx^2} \left[ I(x) \frac{d^2\nu}{dx^2} \right] = \rho \lambda S(x) \nu \equiv \lambda Bu \tag{11.1.28}
\]

\[
\nu(0) = 0, \quad \nu'(0) = 0 \tag{11.1.29}
\]

\[
\nu''(\lambda) = 0, \quad \nu'''(\lambda) = 0 \tag{11.1.30}
\]

The smallest eigenvalue \( \lambda_1 \) of this problem equals the minimum of the integral

\[
\int_0^\lambda E \nu(x) \frac{d^2}{dx^2} \left[ I(x) \frac{d^2\nu(x)}{dx^2} \right] dx = E \int_0^\lambda I(x) \left( \frac{d^2\nu(x)}{dx^2} \right)^2 dx \tag{11.1.31}
\]

over the set of functions \( \nu(x) \) that satisfy boundary conditions of Eqs. 11.1.29 and 11.1.30 and the supplementary condition

\[
\int_0^\lambda \rho S(x) \nu^2(x) dx = 1 \tag{11.1.32}
\]
In finding the minimum of the integral of Eq. 11.1.31 by the Ritz method, coordinate functions are selected as the eigenfunctions of the operator \( \frac{d^4v}{dx^4} \), under boundary conditions of Eqs. 11.1.29 and 11.1.30. These functions form a complete orthonormal system and have the form

\[
\phi_k(\xi) = \frac{\sin \alpha_k \xi}{\sin \frac{\alpha_k}{2}} + \frac{\cosh \alpha_k \xi}{\cosh \frac{\alpha_k}{2}}, \quad k = 2m - 1
\]

\[
\phi_k(\xi) = \frac{\cos \alpha_k \xi}{\cos \frac{\alpha_k}{2}} + \frac{\sinh \alpha_k \xi}{\sinh \frac{\alpha_k}{2}}, \quad k = 2m
\]

where the \( \alpha_k \) are the roots of the equation \( \cos \alpha \cosh \alpha = 1 \) and \( \xi = (2x - \ell)/2\ell \). Putting

\[
v(x) = \sum_{k=1}^{n} a_k \psi_k(x)
\]

where

\[
\psi_k(x) = \phi_k \left( \frac{2x - \ell}{2\ell} \right)
\]

the characteristic equation is

\[
\begin{vmatrix}
A_{11} - \lambda B_{11} & A_{12} - \lambda B_{12} & \cdots & A_{1n} - \lambda B_{1n} \\
A_{21} - \lambda B_{21} & A_{22} - \lambda B_{22} & \cdots & A_{2n} - \lambda B_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} - \lambda B_{n1} & A_{n2} - \lambda B_{n2} & \cdots & A_{nn} - \lambda B_{nn}
\end{vmatrix} = 0 \quad (11.1.33)
\]

where

\[
A_{ik} = E \int_{0}^{\ell} I(x) \psi_i''(x) \psi_k''(x) \, dx
\]

\[
B_{ik} = \int_{0}^{\ell} \rho \ S(x) \psi_i(x) \psi_k(x) \, dx \quad (11.1.34)
\]

As a specific example, consider a tube with linearly varying diameter [21]. An axial section of this tube is shown in Fig. 11.1.1. In this case,

\[
I(x) = \frac{\pi}{4} \left( y^4 - a^4 \right)
\]

\[
S(x) = \pi \left( y^2 - a^2 \right)
\]
where

\[ y = R - \frac{R - r}{\lambda} x = R - \left( \xi + \frac{1}{2} \right) (R - r) \]

The coefficients \( A_{ik} \) and \( B_{ik} \) of Eq. 11.1.34 are thus

\[ A_{ik} = \frac{\pi E}{4 \lambda^3} \left\{ \left[ \frac{1}{16} (R + r)^4 - a^4 \right] I_{ik}^{(0)} - \frac{1}{2} (R + r) (R - r) I_{ik}^{(1)} \right. \]
\[ + \frac{3}{2} (R + r)^2 (R - r)^2 I_{ik}^{(2)} - 2 (R + r) (R - r)^3 I_{ik}^{(3)} \]
\[ + (R - r)^4 I_{ik}^{(4)} \right\} \]
\[ B_{ik} = \pi \rho \lambda \left\{ \left[ \frac{(R + r)^2}{4} - a^2 \right] \tilde{I}_{ik}^{(0)} \right. \]
\[ - \frac{1}{8} (R^2 - r^2) \tilde{I}_{ik}^{(1)} + (R - r)^2 \tilde{I}_{ik}^{(2)} \right\} \]

where \( I_{ik}^{(n)} \) and \( \tilde{I}_{ik}^{(n)} \) are the integrals

\[ I_{ik}^{(n)} = \int_{-1/2}^{1/2} \xi^n \phi''(\xi) \phi_k''(\xi) \, d\xi, \quad n = 0, 1, 2, 3, 4 \]
\[ \tilde{I}_{ik}^{(n)} = \int_{-1/2}^{1/2} \xi^n \phi(\xi) \phi_k(\xi) \, d\xi, \quad n = 0, 1, 2 \]

**Figure 11.1.1 Tapered Beam**

For numerical calculations, the following dimensions are adopted for the tube:

\( \lambda = 4000 \) mm, \( R = 400 \) mm, \( r = 200 \) mm, \( a = 100 \) mm.
Putting $n = 1, 2, \text{ and } 3$ in

$$v = \sum_{k=1}^{n} a_k \phi_k$$

and evaluating terms in Eq. 11.1.33, detailed calculations [21] yield the following equations for $\lambda$:

(1) \[ 0.8526 - 0.1917 \times 10^9 \lambda \frac{p}{E} = 0 \]

(2) \[
\begin{vmatrix}
0.8526 - 0.1917 \times 10^9 \lambda \frac{p}{E} & -1.5341 + 0.0632 \times 10^9 \lambda \frac{p}{E} \\
-1.5341 + 0.0632 \times 10^9 \lambda \frac{p}{E} & 39.4262 + 0.3526 \times 10^9 \lambda \frac{p}{E}
\end{vmatrix} = 0
\]

(3) \[
\begin{vmatrix}
0.8526 - 0.1917 \times 10^9 \lambda \frac{p}{E} & -1.5341 + 0.0632 \times 10^9 \lambda \frac{p}{E} & 1.1077 + 0.0037 \times 10^9 \lambda \frac{p}{E} \\
-1.5341 - 0.0632 \times 10^9 \lambda \frac{p}{E} & 39.4262 + 0.3526 \times 10^9 \lambda \frac{p}{E} & -26.614 + 0.0865 \times 10^9 \lambda \frac{p}{E} \\
1.1077 - 0.0037 \times 10^9 \lambda \frac{p}{E} & -26.614 + 0.0865 \times 10^9 \lambda \frac{p}{E} & 156.59 + 0.3168 \times 10^9 \lambda \frac{p}{E}
\end{vmatrix} = 0
\]

the smallest roots of which are

$$\lambda_1^{(1)} = 0.4446 \times 10^{-8} \frac{E}{\rho}$$

$$\lambda_1^{(2)} = 0.4226 \times 10^{-8} \frac{E}{\rho}$$

$$\lambda_1^{(3)} = 0.4171 \times 10^{-8} \frac{E}{\rho}$$

Note that the successive values of $\lambda_1$ converge rapidly. The relative error in passing from $\lambda_1^{(1)}$ to $\lambda_1^{(3)}$ is about 5% and in passing from $\lambda_1^{(2)}$ to $\lambda_1^{(3)}$, it is about 1.3%. For the second eigenvalues, the approximate values are

$$\lambda_2^{(2)} = 11.628 \times 10^{-8} \frac{E}{\rho}$$

$$\lambda_2^{(3)} = 11.074 \times 10^{-8} \frac{E}{\rho}$$

and the relative error in passing from $\lambda_2^{(2)}$ to $\lambda_2^{(3)}$ is approximately 5%.
11.2 SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Boundary Value Problems for Poisson and Laplace Equations

Consider the Poisson equation

$$-\nabla^2 u = f(x)$$  \hspace{1cm} (11.2.1)

where $x$ is a point in a bounded $m$-dimensional region $\Omega$, $m = 2$ or $3$. Consider first the Dirichlet problem; i.e., find the solution of Eq. 11.2.1 that is continuous in the closed physical domain $\overline{\Omega} = \Omega + \Gamma$ and satisfies the boundary condition

$$u|_{\Gamma} = 0$$  \hspace{1cm} (11.2.2)

As the domain of the Laplace operator $Au = -\nabla^2 u$, take the linear space of functions $D_A$ that satisfy the following conditions:

1. They are continuous, together with their first and second derivatives in the closed domain $\overline{\Omega} = \Omega + \Gamma$.
2. They vanish on $\Gamma$.

For the linear space $D_A$, it was shown in Section 9.2 that the operator $-\nabla^2$ is positive definite.

It follows that the Dirichlet problem of Eqs. 11.2.1 and 11.2.2 is equivalent to the problem of minimizing the functional

$$F(u) = ( -\nabla^2 u, u ) - 2 ( u, f )$$

Integrating by parts and using the boundary condition of Eq. 11.2.2, this is

$$F(u) = \int_{\Omega} \{ (\nabla u)^2 - 2uf \} \, d\Omega$$  \hspace{1cm} (11.2.3)

As shown in Section 6.3, the problem of the deflection of a membrane that is fixed at its edge, under the influence of a normal load, reduces to Eqs. 11.2.1 and 11.2.2, where $f(x)$ is proportional to the loading. The functional of Eq. 11.2.3 is proportional to the potential energy of the deformed membrane. Thus, the Minimum Functional Theorem is just the principle of minimum total potential energy.

A similar result is obtained if, in place of Eq. 11.2.2, the boundary condition for the mixed problem,

$$\left[ \frac{\partial u}{\partial n} + \sigma(x) u \right]_{\Gamma} = 0$$  \hspace{1cm} (11.2.4)

is applied, where $\sigma(x)$ is a non-negative continuous function that is not identically zero. For this problem, the domain of the Laplace operator is denoted as the linear space $M_\sigma$ of
functions that satisfy the boundary equation of Eq. 11.2.4 and the same conditions of continuity and differentiability as functions in the linear space \( D_A \). To show that, with domain \( M_\sigma \), the operator \(- \nabla^2 u\) is also positive definite, note that

\[
(- \nabla^2 u, u) = - \int_{\Gamma} u \frac{\partial u}{\partial n} \, dS + \int_{\Omega} (\nabla u)^2 \, d\Omega
= \int_{\Gamma} \sigma u^2 \, dS + \int_{\Omega} (\nabla u)^2 \, d\Omega \geq 0
\]

If \((- \nabla^2 u, u) = 0\), then it is necessary that

\[
\int_{\Gamma} \sigma u^2 \, dS = \int_{\Omega} (\nabla u)^2 \, d\Omega = 0
\]

From the second equation, it follows that \( u = c \), a constant. Substituting this in the first equation,

\[c^2 \int_{\Gamma} \sigma \, dS = 0\]

Since the function \( \sigma(x) \) is continuous, non-negative, and not identically zero, it follows that \( \int_{\Gamma} \sigma \, dS > 0 \), so \( c = 0 \). Thus, \( u(x) = 0 \), for all \( x \) in \( \Omega \), so the operator \( Au = - \nabla^2 u \) with domain \( M_\sigma \) is positive definite.

The problem of integrating the Poisson equation of Eq. 11.2.1, subject to the boundary condition of Eq. 11.2.4, can now be replaced by the problem of minimizing the functional

\[
F(u) = (- \nabla^2 u, u) - 2 (u, f)
= \int_{\Omega} \{ (\nabla u)^2 - 2uf \} \, d\Omega + \int_{\Gamma} \sigma u^2 \, dS
\]

over the linear space \( M_\sigma \). Actually, the boundary condition of Eq. 11.2.4 is natural, so it can be ignored in minimization of the functional of Eq. 11.2.5.

Finally, a solution of the Neumann problem for Eq. 11.2.1 is sought that is continuous and continuously differentiable in \( \Omega \) and that satisfies the free edge boundary condition

\[
\frac{\partial u}{\partial n} \bigg|_{\Gamma} = 0
\]

Define a linear space \( M_0 \) that consists of functions that satisfy Eq. 11.2.6 and the same conditions of continuity and differentiability as the functions of the linear space \( D_A \). How-
ever, the operator $-\nabla^2 u$ will not be positive definite on this linear space. In fact,

$$( - \nabla^2 u, u ) = \int_{\Omega} ( \nabla u )^2 \, d\Omega \geq 0 \quad (11.2.7)$$

From the equality $( - \nabla^2 u, u ) = 0$, it does not follow that $u = 0$. Indeed the function $u = 1$ clearly belongs to the linear space $M_0$ and $( - \nabla^2 u, u ) = 0$.

To resolve this problem with positive definiteness of the Laplace operator on $M_0$, the Theorem of the Alternative [18, 22] is used (see discussion in Section 3.1 for matrices and discussion following Theorem 10.6.2). The Laplace operator on $M_0$ is symmetric, so it is required that

$$\int_{\Omega} \overline{f} \, d\Omega = 0 \quad (11.2.8)$$

for all solutions $\overline{u}$ of

$$- \nabla^2 \overline{u} = 0, \quad \text{in } \Omega$$

$$\frac{\partial \overline{u}}{\partial n} = 0, \quad \text{on } \Gamma$$

Note that

$$\overline{u}(x) = c \neq 0$$

is a solution of this problem. Substituting this result into Eq. 11.2.8, it follows that

$$\int_{\Omega} f \, d\Omega = 0 \quad (11.2.9)$$

That is, the Neumann problem of Eqs. 11.2.1 and 11.2.6 is solvable only if the applied force $f$ satisfies Eq. 11.2.9. Further, if this problem is solvable, it has an infinite number of solutions, all differing from one another by a constant. To obtain a unique solution $u(x)$, it is required that

$$\int_{\Omega} u(x) \, d\Omega = 0 \quad (11.2.10)$$

Considering the Neumann problem for Eq. 11.2.1, the domain of the Laplace operator is defined as the linear space $M_0^*$ of functions that

(1) are continuous, together with their first and second derivatives in $\tilde{\Omega}$.
(2) satisfy Eq. 11.2.6.
(3) satisfy Eq. 11.2.10.
It will next be proved that the operator \(- \nabla^2 u\) is positive definite for all functions in the linear space \(M_0^*\). For any \(u\) in \(M_0^*\), Eq. 11.2.7 holds. If \((- \nabla^2 u, u) = 0\), then from Eq. 11.2.7, \(u = c\). But Eq. 11.2.10 implies \(c = u = 0\). Thus, the operator is positive definite.

This Neumann problem can thus be replaced by the variational problem of finding the function in \(M_0^*\) that minimizes

\[
F(u) = (- \nabla^2 u, u) - 2(u, f)
\]

or, by virtue of the boundary condition of Eq. 11.2.6,

\[
F(u) = \int_{\Omega} \{ (\nabla u)^2 - 2fu \} \, d\Omega \tag{11.2.11}
\]

The boundary condition of Eq. 11.2.6 is natural, so the functional of Eq. 11.2.11 can be minimized without regard for boundary conditions.

It is often required to solve the Laplace equation with non-homogeneous boundary conditions. It was shown in Section 10.5 that integration of the Laplace equation in the domain \(\Omega\), under the boundary condition of the Dirichlet problem

\[
u \mid_{\Gamma} = g(x) \tag{11.2.12}
\]

reduces to finding the minimum of the functional

\[
\int_{\Omega} (\nabla u)^2 \, d\Omega \tag{11.2.13}
\]

over the set of functions that satisfy Eq. 11.2.12.

If the boundary condition has the form of the Neumann problem

\[
\frac{\partial u}{\partial n} \mid_{\Gamma} = h(x) \tag{11.2.14}
\]

then the problem reduces to finding the minimum of the functional

\[
\int_{\Omega} (\nabla u)^2 \, d\Omega - 2\int_{\Gamma} uh \, dS \tag{11.2.15}
\]

without regard to boundary conditions.

Note that in the case of boundary conditions of the mixed problem; i.e.,

\[
\left[ \frac{\partial u}{\partial n} + \sigma(x) u \right] \mid_{\Gamma} = h_1(x) \tag{11.2.16}
\]

the energy functional has the form

\[
F(u) = \int_{\Omega} (\nabla u)^2 \, d\Omega + \int_{\Gamma} (\sigma u^2 - 2uh_1) \, dS \tag{11.2.17}
\]
The positive definite character of the operator $-\nabla^2 u$ has been established here for each of the domains $D_A$, $M_0$, $M_0^*$, and $M_0^*$. However, in order to be able to use the Ritz method with full confidence, it is important to establish that the operators are positive bounded below. It was shown in Section 9.2 that the Laplace operator is positive bounded below over the set $D_A$. It is shown in Ref. 21 that the Laplace operator is also positive bounded below over the domains $M_0$ and $M_0^*$.

**Torsion of a Rod of Rectangular Cross Section**

The problem of torsion of a rectangular rod is considered in some detail, because a relatively simple exact solution is known [26], to which approximations can be compared. This problem reduces [23, 26] to integration of the Poisson equation

$$-\nabla^2 u = 1 \quad (11.2.18)$$

in the rectangle $-a \leq x_1 \leq a$, $-b \leq x_2 \leq b$, under the boundary conditions

$$u(\pm a, x_2) = u(x_1, \pm b) = 0 \quad (11.2.19)$$

From symmetry, it is clear that the function $u$ is even in both $x_1$ and $x_2$. A sequence of polynomials that possess this property and vanish on the edges of the rectangle; i.e., on the straight lines $x_1 = \pm a$, $x_2 = \pm b$, has the form

$$(x_1^2 - a^2)(x_2^2 - b^2)(a_1 + a_2x_1^2 + a_3x_2^2 + \ldots) \quad (11.2.20)$$

Restricting attention to three terms, the approximate solution will be of the form

$$u \approx u_3 = (x_1^2 - a^2)(x_2^2 - b^2)(a_1 + a_2x_1^2 + a_3x_2^2)$$

Performing the necessary calculations [21], the Ritz equations for the unknowns $a_1$, $a_2$, and $a_3$ are

$$\begin{align*}
\frac{128}{45} a^3 b^3 \left( a^2 + b^2 \right) a_1 + \frac{128}{45} a^5 b^3 \left[ \frac{a^2}{7} + \frac{b^2}{5} \right] a_2 \\
+ \frac{128}{45} a^3 b^5 \left[ \frac{a^2}{5} + \frac{b^2}{7} \right] a_3 &= \frac{16a^3 b^3}{9} \\
\frac{128}{45} a^5 b^3 \left[ \frac{a^2}{7} + \frac{b^2}{5} \right] a_1 + \frac{128}{45 \times 7} a^5 b^5 \left[ \frac{11}{5} b^2 + \frac{1}{3} a^2 \right] a_2 \\
+ \frac{128}{45 \times 35} a^5 b^5 \left( a^2 + b^2 \right) a_3 &= \frac{16a^5 b^3}{45}
\end{align*}$$
\[
\frac{128}{45} a^3 b^3 \left[ \frac{a^2}{5} + \frac{b^2}{7} \right] a_1 + \frac{128}{45 \times 35} a^5 b^5 (a^2 + b^2) a_2 + \frac{128}{45 \times 7} \left[ \frac{11}{5} a^2 + \frac{1}{3} b^2 \right] a_3 = \frac{16 a^3 b^5}{45}
\]

The solution of these equations is

\[
a_1 = \frac{35 \left( 9a^4 + 130a^2 b^2 + 9b^4 \right)}{16 \left( 45a^6 + 509a^4 b^2 + 509a^2 b^4 + 45b^6 \right)}
\]

\[
a_2 = \frac{105 \left( 9a^2 + b^2 \right)}{16 \left( 45a^6 + 509a^4 b^2 + 509a^2 b^4 + 45b^6 \right)} \tag{11.2.21}
\]

\[
a_3 = \frac{105 \left( a^2 + 9b^2 \right)}{16 \left( 45a^6 + 509a^4 b^2 + 509a^2 b^4 + 45b^6 \right)}
\]

If only one coefficient \( a_1 \) were used; i.e., if

\[
u \approx u_1 = a_1 \left( x_1^2 - a^2 \right) \left( x_2^2 - b^2 \right)
\]

then

\[
a_1 = \frac{5}{8 \left( a^2 + b^2 \right)}
\]

A similar calculation can be performed to obtain a two term approximation.

A second sequence of coordinate functions that are even and satisfy the boundary conditions is

\[
\phi_{ij}(x_1, x_2) = \cos \frac{i \pi x_1}{2a} \cos \frac{j \pi x_2}{2b} \tag{11.2.22}
\]

where \( i \) and \( j \) are odd integers. These functions are energy orthogonal, since

\[
- \int_{-b}^{b} \int_{-a}^{a} \phi_{mn} \nabla^2 \phi_{ls} \, dx_1 \, dx_2
\]

\[
= \frac{\pi^2}{4} \left[ \frac{r^2}{a^2} + \frac{s^2}{b^2} \right] \left[ \int_{-a}^{a} \cos \frac{m \pi x_1}{2a} \cos \frac{r \pi x_1}{2a} \, dx_1 \right]
\]

\[
\times \left[ \int_{-b}^{b} \cos \frac{n \pi x_2}{2b} \cos \frac{s \pi x_2}{2b} \, dx_2 \right]
\]

\[
= 0, \text{ if } m \neq r \text{ or } n \neq s
\]
Since the coordinate functions are orthogonal, the Ritz equations are decoupled and easily solved. Direct calculation [21] yields the following one, three, and six term approximations:

\[ u_1(x_1, x_2) = \frac{64a^2b^2}{\pi^4 (a^2 + b^2)} \cos \frac{\pi x_1}{2a} \cos \frac{\pi x_2}{2b} \]

\[ u_3(x_1, x_2) = \frac{64a^2b^2}{\pi^4} \left[ \frac{1}{a^2 + b^2} \cos \frac{\pi x_1}{2a} \cos \frac{\pi x_2}{2b} + \frac{1}{3 (9a^2 + b^2)} \cos \frac{\pi x_1}{2a} \cos \frac{3\pi x_2}{2b} + \frac{1}{3 (a^2 + 9b^2)} \cos \frac{3\pi x_1}{2a} \cos \frac{\pi x_2}{2b} \right] \]

\[ u_6(x_1, x_2) = \frac{64a^2b^2}{\pi^4} \left[ \frac{1}{a^2 + b^2} \cos \frac{\pi x_1}{2a} \cos \frac{\pi x_2}{2b} + \frac{1}{3 (9a^2 + b^2)} \cos \frac{\pi x_1}{2a} \cos \frac{3\pi x_2}{2b} + \frac{1}{3 (a^2 + 9b^2)} \cos \frac{3\pi x_1}{2a} \cos \frac{\pi x_2}{2b} \right. \\
+ \left. \frac{1}{5 (25a^2 + b^2)} \cos \frac{5\pi x_1}{2a} \cos \frac{\pi x_2}{2b} + \frac{1}{81 (a^2 + b^2)} \cos \frac{3\pi x_1}{2a} \cos \frac{3\pi x_2}{2b} \right] \frac{1}{5 (a^2 + 25b^2)} \cos \frac{5\pi x_1}{2a} \cos \frac{\pi x_2}{2b} \right] \]

(11.2.23)

With the aid of these approximate solutions, two mechanical characteristics of the torsional problem can be calculated, namely the torque and the maximum tangential stress, as functions of \( \alpha \). These values for the approximate solutions can be obtained and compared with known exact solutions. From Ref. 26, the torque required to create a twist of \( \alpha \) rad. per unit length of the rod is

\[ T = 4G \alpha \int_{-b}^{b} \int_{-a}^{a} u \, dx_1 \, dx_2 \quad (11.2.24) \]

which can be represented in the form

\[ T \equiv G \alpha (2a)^3 (2b) k_1(\gamma) \quad (11.2.25) \]

where \( \gamma = b/a \) and \( k_1(\gamma) \) is a nondimensional factor. For \( b > a \), the maximum tangential stress in the rectangular section occurs half-way along the side [26]. The greatest tangential stress \( \tau \) is [26]
\[ \tau = -2G\alpha \frac{\partial u}{\partial n} = 2G\alpha a k(\gamma) \quad (11.2.26) \]

where \( k(\gamma) \) is a nondimensional factor.

For numerical comparison with known solutions, denote the nondimensional approximations obtained above with the polynomial and Fourier coordinate functions as \((k_p^{(i)}, k_{1p}^{(i)})\) and \((k_f^{(i)}, k_{1f}^{(i)})\), respectively. For the Fourier approximation, numerical evaluation of the integral in Eq. 11.2.24, using the notation of Eq. 11.2.25, yields the approximations

\[
k_{1f}^{(1)}(\gamma) = \frac{256}{\pi^6} \left[ \frac{\gamma^2}{1 + \gamma^2} \right]
\]

\[
k_{1f}^{(3)}(\gamma) = \frac{256\gamma^2}{\pi^6} \left[ \frac{1}{1 + \gamma^2} + \frac{1}{9 (9 + \gamma^2)} + \frac{1}{9 (1 + 9\gamma^2)} \right]
\]

\[
k_{1f}^{(6)}(\gamma) = \frac{256\gamma^2}{\pi^6} \left[ \frac{1}{1 + \gamma^2} + \frac{1}{9 (9 + \gamma^2)} + \frac{1}{9 (1 + 9\gamma^2)} + \frac{1}{25 (25 + \gamma^2)} + \frac{1}{25 (1 + 25\gamma^2)} \right]
\]

Likewise, presuming \( b \geq a \), the maximum shear stress occurs at \( (a, 0) \). Evaluating Eq. 11.2.26 at this point, where \( \partial/\partial n = \partial/\partial x_1 \), yields the approximations

\[
k_f^{(1)}(\gamma) = \frac{32}{\pi^3} \left[ \frac{\gamma^2}{1 + \gamma^2} \right]
\]

\[
k_f^{(3)}(\gamma) = \frac{32\gamma^2}{\pi^3} \left[ \frac{1}{1 + \gamma^2} - \frac{1}{3 (\gamma^2 + 9)} + \frac{1}{1 + 9\gamma^2} \right]
\]

\[
k_f^{(6)}(\gamma) = \frac{32\gamma^2}{\pi^3} \left[ \frac{1}{1 + \gamma^2} - \frac{1}{3 (\gamma^2 + 9)} + \frac{1}{1 + 9\gamma^2} + \frac{1}{5 (\gamma^2 + 25)} + \frac{1}{1 + 25\gamma^2} \right]
\]
Tables 11.2.1 and 11.2.2 provide a comparison of the exact values [26] of the quantities $k_1$ and $k$ with their approximate values.

Table 11.2.1 Fourier Approximate Values of $k_1$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$k_1$</th>
<th>$k_{1r}^{(1)}$</th>
<th>$k_{1r}^{(3)}$</th>
<th>$k_{1r}^{(6)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1406</td>
<td>0.133</td>
<td>0.139</td>
<td>0.1401</td>
</tr>
<tr>
<td>2</td>
<td>0.229</td>
<td>0.213</td>
<td>0.225</td>
<td>0.228</td>
</tr>
<tr>
<td>3</td>
<td>0.263</td>
<td>0.240</td>
<td>0.258</td>
<td>0.261</td>
</tr>
<tr>
<td>4</td>
<td>0.281</td>
<td>0.251</td>
<td>0.273</td>
<td>0.278</td>
</tr>
<tr>
<td>5</td>
<td>0.291</td>
<td>0.256</td>
<td>0.281</td>
<td>0.287</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.333</td>
<td>0.266</td>
<td>0.299</td>
<td>0.311</td>
</tr>
</tbody>
</table>

Table 11.2.2 Fourier Approximate Values of $k$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$k$</th>
<th>$k_{f}^{(1)}$</th>
<th>$k_{f}^{(3)}$</th>
<th>$k_{f}^{(6)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.675</td>
<td>0.516</td>
<td>0.585</td>
<td>0.613</td>
</tr>
<tr>
<td>2</td>
<td>0.930</td>
<td>0.826</td>
<td>0.831</td>
<td>0.870</td>
</tr>
<tr>
<td>3</td>
<td>0.985</td>
<td>0.929</td>
<td>0.870</td>
<td>0.931</td>
</tr>
<tr>
<td>4</td>
<td>0.997</td>
<td>0.971</td>
<td>0.865</td>
<td>0.954</td>
</tr>
<tr>
<td>5</td>
<td>0.999</td>
<td>0.992</td>
<td>0.854</td>
<td>0.961</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1.000</td>
<td>1.032</td>
<td>0.803</td>
<td>1.012</td>
</tr>
</tbody>
</table>

Tables 11.2.1 and 11.2.2 show that the approximate solutions give fairly good values of the torque, but poorer values for the maximum tangential stress. This applies particularly to $u_3(x_1, x_2)$. This is caused by the fact that the precision of the evaluation of the torque $k_1$ depends on the rapidity of convergence of the series in the mean, whereas the precision of the evaluation of $k$ depends on the rapidity of uniform convergence of the series of derivatives. It is seen that the series converges in the mean fairly rapidly, but that the convergence of derivatives of this series is slow.

For comparison, values of $k_{1p}^{(1)}(\gamma)$, $k_{1p}^{(3)}(\gamma)$, $k_{p}^{(1)}(\gamma)$, and $k_{p}^{(3)}(\gamma)$, evaluated from the polynomial approximations, are given in Tables 11.2.3 and 11.2.4.
Comparison of Tables 11.2.1 – 11.2.4 shows that, for this application, an approximation of the same order with polynomials gives better results than with trigonometric functions for torque, and worse approximations for the maximum tangential stress. However, this circumstance may not be valid for approximations with greater numbers of terms.

11.3 HIGHER ORDER EQUATIONS AND SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

The power of variational methods becomes even more clear for problems that are described by higher order partial differential equations and systems of second order partial differential equations. The basic operator properties of equations that govern such systems are the same as for the simpler ordinary and second order partial differential operator equations studied in the preceding sections. More important, the Ritz method offers a practical method of solving such problems, with the aid of the modern high speed digital computer. The examples presented in this section are intended to illustrate the power and generality of the Ritz method.

**Bending of Thin Plates**

The equation for bending of a thin elastic plate may be written as [24]

\[
\nabla^4 u = \frac{f(x_1, x_2)}{D}
\]

(11.3.1)

or in more detailed form,
Here, \( u(x_1, x_2) \) is the lateral deflection of the plate, \( f(x_1, x_2) \) is the intensity of the normal load on the plate, and

\[
D = \frac{Eh^3}{12 (1 - \sigma^2)}
\]

where \( E \) and \( \sigma < 1 \) are Young's modulus and Poisson's ratio for the material from which the plate is made and \( h \) is its thickness. The region covered by the plate in the \((x_1, x_2)\) plane is denoted by \( \Omega \), with boundary \( \Gamma \). Depending on the manner of support of the plate edge, the following boundary conditions are most frequently encountered [24]:

(a) The edge of the plate is clamped;

\[
\left. u \right|_{\Gamma} = 0
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If the entire boundary of the plate is clamped, the functional of Eq. 11.3.4 can be represented in either of the following simpler forms:

\[ F(u) = \left( Au, u \right) - 2 \left( u, \frac{f}{D} \right) \]

\[ = \int \int_\Omega \left[ (\nabla^2 u)^2 - \frac{2f}{D} u \right] d\Omega \]

\[ = \int \int_\Omega \left[ \left( \frac{\partial^2 u}{\partial x_1^2} \right)^2 + 2 \left( \frac{\partial^2 u}{\partial x_1 \partial x_2} \right)^2 + \left( \frac{\partial^2 u}{\partial x_2^2} \right)^2 - \frac{2f}{D} u \right] d\Omega \quad (11.3.5) \]

Clearly, for the case of a clamped boundary, \( (Au, u) \geq 0 \). If \( (Au, u) = 0 \), all the second derivatives of \( u \) are zero in \( \Omega \). In particular, \( -\nabla^2 u = 0 \). Since \( u = 0 \) on \( \Gamma \), this implies \( u = 0 \) in \( \Omega \). Thus, the biharmonic operator with clamped boundary conditions is positive definite. While not shown for other boundary conditions, the same result holds. In fact, the operator is positive bounded below [18].

The foregoing results show that the problem of bending of a plate is equivalent to minimizing the energy functional over the set of functions in the class \( H_A \) that satisfy Eqs. 11.3.2 on the clamped part of the boundary and the condition \( u |_\Gamma = 0 \) on the simply supported part of the boundary. The remaining boundary condition is natural and need not be satisfied by coordinate functions.

If the plate has variable thickness \( h(x) \), then Eq. 11.3.1 is replaced by

\[ \frac{\partial^2}{\partial x_1^2} \left( h^3 \frac{\partial^2 u}{\partial x_1^2} \right) + \sigma \frac{\partial^2}{\partial x_2^2} \left( h^3 \frac{\partial^2 u}{\partial x_1^2} \right) + \sigma \frac{\partial^2}{\partial x_1^2} \left( h^3 \frac{\partial^2 u}{\partial x_2^2} \right) \]

\[ + \frac{\partial^2}{\partial x_2^2} \left( h^3 \frac{\partial^2 u}{\partial x_2^2} \right) + 2 \left( 1 - \sigma \right) \frac{\partial^2}{\partial x_1 \partial x_2} \left( h^3 \frac{\partial^2 u}{\partial x_1 \partial x_2} \right) = \frac{f}{D'} \]

\[ (11.3.6) \]

where

\[ D' = \frac{E}{12 \left( 1 - \sigma^2 \right)} \]

The boundary conditions on the clamped or simply supported segments of the boundary have the same form as for a plate of constant thickness. If the thickness \( h \) of the plate is nonzero, the operator of Eq. 11.3.6 is positive definite on the set of functions that satisfy Eqs. 11.3.2 and 11.3.3 on the clamped and simply supported parts of the boundary, respectively. The corresponding variational problem consists of minimizing the integral
subject to the conditions of Eqs. 11.3.2 on the clamped part of the boundary and the condition \( u|_{\Gamma} = 0 \) on the simply supported part of the boundary.

If the thickness of the plate is constant the frequency of vibration of a thin elastic plate is proportional to the eigenvalues of the biharmonic operator; i.e.,

\[
\mathbf{V}^4 u \equiv \frac{\partial^4 u}{\partial x_1^4} + 2 \frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4} = \lambda hu
\]

subject to the boundary conditions of Eqs. 11.3.2 or 11.3.3. For the general case, the boundary of the plate is split into segments \( \Gamma_1 \) and \( \Gamma_2 \), on which Eqs. 11.3.2 and 11.3.3, respectively, are satisfied.

The smallest eigenvalue of the biharmonic operator is the minimum value of the functional

\[
(Au, u) = \iint_{\Omega} \left\{ \left( \frac{\partial^2 u}{\partial x_1^2} \right)^2 + 2\sigma \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 u}{\partial x_2^2} + \left( \frac{\partial^2 u}{\partial x_2^2} \right)^2 \right\} d\Omega
\]

the minimum taken over the set of functions that satisfy

\[
(Bu, u) = \iint_{\Omega} hu^2 d\Omega = 1
\]

and the boundary conditions

\[
u|_{\Gamma_1 + \Gamma_2} = 0
\]

\[
\frac{\partial u}{\partial n}|_{\Gamma_1} = 0
\]

If the entire boundary of the plate is clamped, the smallest eigenvalue of the biharmonic operator is equal to the minimum of the functional
where the function \( u(x) \) satisfies Eq. 11.3.9 and the boundary conditions \( u|_\Gamma = 0 \) and \( (\partial u/\partial n)|_\Gamma = 0 \).

The \( n^{th} \) eigenvalue \( \lambda_n \) of the biharmonic operator is related to the \( n^{th} \) natural frequency \( \omega_n \) of the plate by the expression [24]

\[
\lambda_n = \frac{\gamma \omega_n^2}{D}
\]

where \( \gamma \) is the density of the plate and \( D \) is the resistance of the plate to bending.

If the thickness \( h \) of the plate varies, the equation for natural vibration of the plate takes the form

\[
\frac{\partial^2}{\partial x_1^2} \left( h^3 \frac{\partial^2 u}{\partial x_1^2} \right) + \sigma \frac{\partial^2}{\partial x_2^2} \left( h^3 \frac{\partial^2 u}{\partial x_2^2} \right) + 2 (1 - \sigma) \frac{\partial^2}{\partial x_1 \partial x_2} \left( h^3 \frac{\partial^2 u}{\partial x_1 \partial x_2} \right) = \lambda h u
\]

(11.3.11)

where the parameter \( \lambda \) is related to the natural frequency \( \omega \) by the expression [24]

\[
\lambda = \frac{12 (1 - \sigma^2)}{E} \gamma \omega^2
\]

The smallest eigenvalue \( \lambda_1 \) is the minimum of the functional

\[
\iint_\Omega h^3 \left[ \left( \frac{\partial^2 u}{\partial x_1^2} \right)^2 + 2\sigma \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_2^2} + \left( \frac{\partial^2 u}{\partial x_2^2} \right)^2 \right] + 2 (1 - \sigma) \left( \frac{\partial^2 u}{\partial x_1 \partial x_2} \right)^2 \, d\Omega
\]

(11.3.12)

subject to the conditions of Eqs. 11.3.9 and 11.3.10.

**Bending of a Clamped Rectangular Plate**

Lateral displacement \( u \) of a plate satisfies the biharmonic equation

\[
\nabla^4 u = \frac{f}{D} = p
\]

(11.3.13)
where \( f \) is the load intensity and \( D \) is the rigidity of the plate, subject to the boundary conditions

\[
\begin{align*}
  u \bigg|_{\Gamma} & = 0 \\
  \frac{\partial u}{\partial n} \bigg|_{\Gamma} & = 0
\end{align*}
\]  

(11.3.14)

The lengths of the sides of the plate are 2a and 2b, where the coordinate axes are parallel to the sides of the plate and the origin of coordinates is at the center of the plate.

As was shown in Eq. 11.3.5, this problem reduces to finding the minimum of the functional

\[
F(u) = \int_{-a}^{a} \int_{-b}^{b} \{ (\nabla^2 u)^2 - 2pu \} \, dx_2 \, dx_1
\]

(11.3.15)

over the space of functions that satisfy Eqs. 11.3.14. This problem can be solved by the Ritz method. In order to simplify calculations, suppose that the load is distributed uniformly; i.e., \( p = \text{constant} \).

As coordinate functions, consider polynomials of the form

\[
( x_1^2 - a^2 )^2 ( x_2^2 - b^2 )^2 ( a_1 + a_2 x_1^2 + a_3 x_2^2 + \ldots )
\]

(11.3.16)

Odd powers of \( x_1 \) and \( x_2 \) are omitted, because the solution \( u(x_1, x_2) \) must be symmetric about the coordinate axes.

Restricting attention to three terms in Eq. 11.3.16,

\[
\phi_1 = ( x_1^2 - a^2 )^2 ( x_2^2 - b^2 )^2, \quad \phi_2 = x_1^2 \phi_1, \quad \phi_3 = x_2^2 \phi_1
\]

the three term Ritz approximation is

\[
u = u_3 = a_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3
\]

The Ritz equations in this case have the form

\[
\sum_{k=1}^{3} ( \nabla^2 \phi_k, \nabla^2 \phi_m ) a_k = p ( 1, \phi_m ), \quad m = 1, 2, 3
\]

(11.3.17)

Performing the necessary calculations, Eqs. 11.3.17 are [21]
\[
\left( \gamma^2 + \frac{1}{\gamma^2} + 4 \right) a_1 + \left( \frac{1}{7} + \frac{1}{11 \gamma^4} \right) b^2 a_2 + \left( \frac{1}{7} + \frac{\gamma^4}{11} \right) a^2 a_3 = \frac{7}{128 a^2 b^2} p
\]

\[
\left( \frac{\gamma^2}{7} + \frac{1}{11 \gamma^2} \right) a_1 + \left( \frac{3}{7} + \frac{4}{143 \gamma^4} + \frac{4}{77 \gamma^2} \right) b^2 a_2 + \frac{1}{77} \left( 1 + \gamma^4 \right) a^2 a_3 = \frac{1}{128 a^2 b^2} p
\]

\[
\left( \frac{1}{7 \gamma^2} + \frac{\gamma^2}{11} \right) a_1 + \frac{1}{77} \left( 1 + \frac{1}{\gamma^4} \right) b^2 a_2 + \left( \frac{3}{7} + \frac{3 \gamma^4}{143} + \frac{4 \gamma^2}{77} \right) a^2 a_3 = \frac{1}{128 a^2 b^2} p
\]

where \( \gamma = b/a \). If the plate is square, then \( \gamma = 1 \) and these equations assume the simpler form

\[
\frac{18}{7} a_1 + \frac{18}{77} a^2 a_2 + \frac{18}{77} a^2 a_3 = \frac{7}{128 a^4} p
\]

\[
\frac{18}{77} a_1 + \frac{502}{1001} a^2 a_2 + \frac{2}{77} a^2 a_3 = \frac{1}{128 a^4} p
\]

\[
\frac{18}{77} a_1 + \frac{2}{77} a^2 a_2 + \frac{502}{1001} a^2 a_3 = \frac{1}{128 a^4} p
\]

Solving this system,

\[
u = u_3 = \frac{p}{a^4} \left( x_1^2 - a^2 \right)^2 \left( x_2^2 - a^2 \right)^2 \left( 0.02067 + 0.0038 \frac{x_1^2 + x_2^2}{a^2} \right)
\]

The approximate deflection at the center of the plate is thus

\[
u_3 \bigg|_{x_1=0, x_2=0} = 0.02067pa^4
\]

If the approximation for \( u \) were restricted to one term; i.e.,

\[
u = u_1 = a_1 \phi_1 = a_1 \left( x_1^2 - a^2 \right)^2 \left( x_2^2 - b^2 \right)^2
\]

then the Ritz equations reduce to the single equation
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\[
\left( \gamma^2 + \frac{1}{\gamma^2} + \frac{4}{7} \right) a_1 = \frac{7}{128a^2b^2} \rho
\]

so

\[
a_1 = \frac{49}{128 \left( 7a^4 + 7b^4 + 4a^2b^2 \right)} \rho
\]

which gives the value

\[
\left. u_1 \right|_{x_1=0,x_2=0} = \frac{49a^4b^4}{128 \left( 7a^4 + 7b^4 + 4a^2b^2 \right)}
\]

for the deflection at the center of the plate. In the case of a square plate \((a = b)\),

\[
\left. u_1 \right|_{x_1=0,x_2=0} = 0.02127pa^4
\]

Note that \(u_1\) and \(u_3\) give similar values at the center of the plate, suggesting that \(u_3\) is a good approximation of the solution.

**Computer Solution of a Clamped Plate Problem**

To illustrate the effect of the number of coordinate functions used in solving problems with the Ritz technique, the clamped plate with both constant thickness and variable thickness is solved for displacement due to a uniformly distributed lateral load \(f\) and for the fundamental natural frequency and the associated eigenfunction. In these problems, static deflection is governed by the differential equation of Eq. 11.3.6 and the boundary conditions of Eqs. 11.3.10. For vibration, the deflection satisfies Eq. 11.3.11 and the same boundary conditions. Since all boundary conditions in Eqs. 11.3.10 are principal, the coordinate functions must satisfy them.

Since the problem is symmetric, the following even trigonometric coordinate functions are selected:

\[
u_M = \sum_{m=1}^{M} \sum_{n=1}^{M} a_{mn} \cos \frac{(2m-1)\pi x_1}{a} \cos \frac{\pi x_1}{a} \cos \frac{(2n-1)\pi x_2}{b} \cos \frac{\pi x_2}{b}
\]

\[
= \frac{1}{4} \sum_{m=1}^{M} \sum_{n=1}^{M} a_{mn} \left( \cos \frac{2m\pi x_1}{a} + \cos \frac{(2m-2)\pi x_1}{a} \right)
\]

\[
\times \left( \cos \frac{2n\pi x_2}{b} + \cos \frac{(2n-2)\pi x_2}{b} \right)
\]

\[(11.3.18)\]
The Ritz equations for this formulation are

\[
\frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} a_{mn} \int_{-b}^{b} \int_{-a}^{a} \{ \left[ \left( \frac{2m\pi}{a} \right)^2 \cos \frac{2m\pi x_1}{a} + \left( \frac{(2m - 2)\pi}{a} \right)^2 \cos \frac{(2m - 2)\pi x_1}{a} \right] \times \left[ \cos \frac{2m\pi x_2}{b} + \cos \frac{(2n - 2)\pi x_2}{b} \right] + \nu \left[ \cos \frac{2m\pi x_1}{a} + \cos \frac{(2m - 2)\pi x_1}{a} \right] \times \left[ \cos \frac{2m\pi x_2}{b} + \cos \frac{(2n - 2)\pi x_2}{b} \right] \}
\]

\[
\times \left[ \left( \frac{2m\pi}{b} \right)^2 \cos \frac{2m\pi x_2}{b} + \left( \frac{(2m - 2)\pi}{b} \right)^2 \cos \frac{(2m - 2)\pi x_2}{b} \right] \times \left[ \left( \frac{2n\pi}{b} \right)^2 \cos \frac{2n\pi x_2}{b} + \left( \frac{(2n - 2)\pi}{b} \right)^2 \cos \frac{(2n - 2)\pi x_2}{b} \right]
\]

\[
+ \nu \left[ \cos \frac{2m\pi x_1}{a} + \cos \frac{(2m - 2)\pi x_1}{a} \right] \times \left[ \cos \frac{2m\pi x_2}{b} + \cos \frac{(2n - 2)\pi x_2}{b} \right] \}
\]

\[
\times \left[ \left( \frac{2n\pi}{b} \right)^2 \cos \frac{2n\pi x_2}{b} + \left( \frac{(2n - 2)\pi}{b} \right)^2 \cos \frac{(2n - 2)\pi x_2}{b} \right] \times \left[ \left( \frac{2m\pi}{b} \right)^2 \cos \frac{2m\pi x_2}{b} + \left( \frac{(2m - 2)\pi}{b} \right)^2 \cos \frac{(2m - 2)\pi x_2}{b} \right]
\]

\[
+ 2(1 - \nu) \left[ \left( \frac{2m\pi}{a} \right)^2 \sin \frac{2m\pi x_1}{a} + \left( \frac{(2m - 2)\pi}{a} \right)^2 \sin \frac{(2m - 2)\pi x_1}{a} \right]
\]

\[
\times \left[ \left( \frac{2n\pi}{b} \right)^2 \sin \frac{2n\pi x_2}{b} + \left( \frac{(2n - 2)\pi}{b} \right)^2 \sin \frac{(2n - 2)\pi x_2}{b} \right]
\]

\[
\times \left[ \left( \frac{2m\pi}{b} \right)^2 \sin \frac{2m\pi x_2}{b} + \left( \frac{(2m - 2)\pi}{b} \right)^2 \sin \frac{(2m - 2)\pi x_2}{b} \right]
\]

\[
\times \left[ \left( \frac{2n\pi}{a} \right)^2 \sin \frac{2n\pi x_2}{a} + \left( \frac{(2n - 2)\pi}{a} \right)^2 \sin \frac{(2n - 2)\pi x_2}{a} \right] \}\ dx_1 \ dx_2
\]

\[
= \frac{1}{2} \int_{-b}^{b} \int_{-a}^{a} f \left( \cos \frac{2m\pi x_1}{a} + \cos \frac{(2m - 2)\pi x_1}{a} \right) \left( \cos \frac{2m\pi x_2}{b} + \cos \frac{(2m - 2)\pi x_2}{b} \right) \ dx_1 \ dx_2,
\]

\[\kappa = 1, 2, \ldots, M, \quad \ell = 1, 2, \ldots, M \quad (11.3.19)\]

These equations were programmed and solved for a square plate \((b = a)\), with 9 and 16 coordinate functions; i.e., \(M = 3\) and \(M = 4\), respectively. Results for deflection and bending moment \(M_{x_1}\) at the points of interest shown in Fig. 11.3.1 and for the smallest eigenvalue for a square plate of constant thickness are given in Table 11.3.1. The Ritz eigenvalue equations were also solved for the eigenvalue \(\zeta = \rho \omega^2\), where \(\rho\) is the mass.

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density of material and $\omega$ is the natural frequency, in rad/sec. Results are presented in Table 11.3.1. Note that the polynomial three term approximation for deflection was $0.0206 \left( \frac{f{\alpha}^4}{D} \right)$, whereas the 16 term approximation here gives $0.0201 \left( \frac{f{\alpha}^4}{D} \right)$. Note also that while accurate displacement and eigenvalues are obtained, since bending moment involves two derivatives of displacement [26] (see also Eq. 11.4.11), it is less accurate.

Table 11.3.1 Results for Clamped Plate (Constant Thickness)

<table>
<thead>
<tr>
<th></th>
<th>$M_{\kappa_1} @ A$</th>
<th>$M_{\kappa_1} @ B$</th>
<th>$u @ A$</th>
<th>$\zeta = \rho \omega^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\times 4f{\alpha}^2$</td>
<td>$\times 16f{\alpha}^4$</td>
<td>$\times \frac{1}{4a^2} \sqrt{\frac{D}{\rho h}}$</td>
<td></td>
</tr>
<tr>
<td>$M = 3$</td>
<td>0.025092</td>
<td>-0.038922</td>
<td>0.0012626</td>
<td>36.106</td>
</tr>
<tr>
<td>$M = 4$</td>
<td>0.021518</td>
<td>-0.041473</td>
<td>0.0012586</td>
<td>36.044</td>
</tr>
<tr>
<td>Solution [26]</td>
<td>0.0231</td>
<td>-0.0513</td>
<td>0.00126</td>
<td>35.98</td>
</tr>
</tbody>
</table>

Figure 11.3.1 Points of Interest on Plate

The same problem was solved for a clamped plate of variable thickness, with

$$h(x) = 2 \left( 1 - \left| \frac{x_1}{a} \right| \right) \left( 1 - \left| \frac{x_2}{a} \right| \right) h_0$$

The same coordinate functions, with 4, 9, and 16 terms (corresponding to $M = 2$, 3, and 4, respectively) were used to solve this problem. Numerical results are given in Table 11.3.2.
Table 11.3.2 Results for Clamped Plate (Varying Thickness)

<table>
<thead>
<tr>
<th></th>
<th>$M_{x_1} \times 4fa^2$</th>
<th>$M_{x_1} \times 4fa^2$</th>
<th>$u @ A$</th>
<th>$\zeta = \rho \omega^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 2$</td>
<td>0.029754</td>
<td>-0.0062549</td>
<td>0.00060060</td>
<td>40.706</td>
</tr>
<tr>
<td>$M = 3$</td>
<td>0.063371</td>
<td>-0.0085592</td>
<td>0.00065468</td>
<td>39.376</td>
</tr>
<tr>
<td>$M = 4$</td>
<td>0.040864</td>
<td>-0.010112</td>
<td>0.00067205</td>
<td>38.774</td>
</tr>
</tbody>
</table>

* $D = \frac{Eh_0^3}{12 (1 - \nu)}$

11.4 SOLUTION OF BEAM AND PLATE PROBLEMS BY THE GALERKIN METHOD

To illustrate use of the Galerkin method, problems of the type treated in Section 11.3 for beams and plates are solved numerically using the Galerkin method.

Static Analysis of Beams

The governing equations of a beam may be written in the second order form

$$ A\begin{bmatrix} u \\ M \end{bmatrix} = \begin{bmatrix} -\frac{d^2M}{dx^2} \\ -\frac{d^2u}{dx^2} - \frac{M}{EI} \end{bmatrix} \begin{bmatrix} f(x) \\ 0 \end{bmatrix} $$  \hspace{1cm} (11.4.1)

or in the more usual fourth order form

$$ \frac{d^2}{dx^2} \left( EI \frac{d^2u}{dx^2} \right) = f(x) $$  \hspace{1cm} (11.4.2)

where $u$ is deflection, $M$ is moment, $E$ is Young’s modulus, $I$ is moment of inertia of the beam cross section, and $f(x)$ is distributed load, taken as constant in the examples treated here.
The second order form of Eq. 11.4.1 has the advantage that both displacement $u$ and moment $M$ are determined with comparable accuracy. This feature allows direct computation of bending stress in the beam, without having to compute derivatives of an approximate numerical solution $u(x)$, as was done in the case of plates in Tables 11.3.1 and 11.3.2.

As has been shown, under the usual boundary conditions, the fourth order operator of Eq. 11.4.2 is symmetric and positive definite. For the second order operator of Eq. 11.4.1, under the usual boundary conditions,

$$
\int_{-l/2}^{l/2} \left[ \begin{array}{c} u' \\ \bar{M} \end{array} \right]^T A \left[ \begin{array}{c} u \\ \bar{M} \end{array} \right] \, dx = \int_{-l/2}^{l/2} \left( -\bar{u}M'' - \bar{M}u'' - \frac{\bar{M}M}{EI} \right) \, dx
$$

$$
= \int_{-l/2}^{l/2} \left( -\bar{u}M'' - \bar{M}u'' - \frac{\bar{M}M}{EI} \right) \, dx
$$

$$
= \int_{-l/2}^{l/2} \left[ u \right]^T A \left[ \begin{array}{c} \bar{u} \\ \bar{M} \end{array} \right] \, dx
$$

so the operator $A$ is symmetric. However,

$$
\int_{-l/2}^{l/2} \left[ u \right]^T A \left[ \begin{array}{c} \bar{u} \\ \bar{M} \end{array} \right] \, dx = \int_{-l/2}^{l/2} \left[ 2u'M' - \frac{M^2}{EI} \right] \, dx
$$

may be made negative by choosing admissible $u'$ and $M'$ that are orthogonal, so the first term vanishes and a negative integral results. Thus, the second order operator $A$ is not positive definite and the Ritz method cannot be used directly for this formulation.

**Simply Supported Beam**

Two simply supported beam problems are solved using the Galerkin method. One has a uniform cross section and the other has a varying cross section, with area

$$
a(x) = 2a_0 \left( 1 - \frac{|x|}{\ell} \right), \quad -\frac{\ell}{2} < x < \frac{\ell}{2}
$$

and moment of inertia $I(x) = \alpha \left[ a(x) \right]^2$, which is symmetric with respect to the center of the span. The boundary conditions are

$$
\begin{align*}
\left. u \right|_{x=\pm \ell/2} & = M \left( \pm \frac{\ell}{2} \right) = 0
\end{align*}
$$

(11.4.3)

Under symmetric loading, the solution will be symmetric, so cosine functions are
selected as coordinate functions. This leads to the approximate solutions

\[ u = \sum_{n=1}^{N} a_n \cos \left( \frac{(2n-1)\pi x}{\ell} \right) \]
\[ M = \sum_{m=1}^{M} b_m \cos \left( \frac{(2m-1)\pi x}{\ell} \right) \]
\[ (11.4.4) \]

Substituting these sums into the Galerkin equations of Eq. 10.10.13 and performing the necessary calculations,

\[ \int_{-\ell/2}^{\ell/2} \bar{u} A u M \, dx = \sum_{m=1}^{M} b_m \left( \frac{(2m-1)\pi}{\ell} \right)^2 \]
\[ \times \int_{-\ell/2}^{\ell/2} \left[ \cos \left( \frac{(2m-1)\pi x}{\ell} \right) \cos \left( \frac{(2n-1)\pi x}{\ell} \right) \right] dx = \int_{-\ell/2}^{\ell/2} f \cos \left( \frac{(2n-1)\pi x}{\ell} \right) dx = 0, \quad n = 1, 2, \ldots, N \]
\[ (11.4.5) \]

where \([\bar{u}, \bar{M}]^T = \left[ \cos \left( \frac{(2n-1)\pi x}{\ell} \right), 0 \right]^T\), and

\[ \int_{-\ell/2}^{\ell/2} \bar{u} A u M \, dx = \sum_{n=1}^{N} a_n \left( \frac{(2n-1)\pi}{\ell} \right)^2 \]
\[ \times \int_{-\ell/2}^{\ell/2} \cos \left( \frac{(2n-1)\pi x}{\ell} \right) \cos \left( \frac{(2m-1)\pi x}{\ell} \right) dx \]
\[ - \sum_{j=1}^{M} b_j \int_{-\ell/2}^{\ell/2} \left[ \frac{1}{EI} \cos \left( \frac{(2j-1)\pi x}{\ell} \right) \cos \left( \frac{(2m-1)\pi x}{\ell} \right) \right] dx = 0, \quad m = 1, 2, \ldots, M \]
\[ (11.4.6) \]
where \([ \tilde{u}, \tilde{M} ]^T = \left[ 0, \cos \frac{(2m-1)\pi x}{\ell} \right]^T \).

Numerical results for the uniform beam are presented in the first two columns of Table 11.4.1, for six and ten coordinate functions. Similar results for a beam with variable cross section are given in Table 11.4.2. As expected, results for deflections are very accurate. While there is some error in the calculation of bending moment, it is much smaller than would be experienced if numerically determined displacements were differentiated twice.

### Table 11.4.1
Results for Simply Supported Beam
(Uniform Cross Section, Second Order Method)

<table>
<thead>
<tr>
<th>( \ell/2 )</th>
<th>( -\ell/2 )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \ell/2 )</td>
<td></td>
</tr>
</tbody>
</table>

\( \text{E}I_0 = \text{bending rigidity of beam} \)
\( a_0 = \text{cross sectional area of beam} \)
\( \rho = \text{mass density per unit volume} \)

<table>
<thead>
<tr>
<th>( N ) = 3, ( M ) = 3</th>
<th>\text{Moment @ A} \times f x^2</th>
<th>\text{u @ A} \times \frac{f x^4}{\text{E}I_0}</th>
<th>\zeta = \rho \omega^2 \times \frac{1}{\ell^2} \sqrt{\frac{\text{E}I_0}{\rho a_0}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12526</td>
<td>0.013021</td>
<td>9.86960</td>
<td></td>
</tr>
<tr>
<td>( N ) = 5, ( M ) = 5</td>
<td>0.12506</td>
<td>0.013021</td>
<td>9.86960</td>
</tr>
<tr>
<td>( \text{EXACT} )</td>
<td>0.12500</td>
<td>0.013021</td>
<td>9.86960</td>
</tr>
</tbody>
</table>
Chap. 11  Applications of Variational Methods

Table 11.4.2
Results for Simply Supported Beam
(Varying Cross Section, Second Order Method)

\[
a(x) = 2a_0 \left( 1 - \frac{|x|}{\ell} \right)
\]

\[
I_0 = \alpha a_0^2
\]

<table>
<thead>
<tr>
<th></th>
<th>Moment @ A ( \times f \ell^2 )</th>
<th>( u @ A ) ( \times \frac{f \ell^4}{E I_0} )</th>
<th>( \zeta = \rho \omega^2 ) ( \times \frac{1}{\ell^2 \sqrt{\frac{E I_0}{\rho a_0}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 3 ) ( M = 3 )</td>
<td>0.12526</td>
<td>0.0046279</td>
<td>12.598</td>
</tr>
<tr>
<td>( N = 5 ) ( M = 5 )</td>
<td>0.12506</td>
<td>0.0046247</td>
<td>12.599</td>
</tr>
</tbody>
</table>

Clamped Beam

Clamped beam problems are now solved, using both the second and fourth order formulations. The boundary conditions for this case are

\[
u \left( \pm \frac{\ell}{2} \right) = \frac{du}{dx} \left( \pm \frac{\ell}{2} \right) = 0
\]

(11.4.7)

The following coordinate function approximation, which satisfies these principal boundary conditions, is selected:

\[
u = \sum_{n=1}^{N} a_n \cos \left( \frac{(2n-1) \pi x}{\ell} \right) \cos \frac{\pi x}{\ell}
\]

\[
= \sum_{n=1}^{N} a_n \left[ \cos \left( \frac{2n \pi x}{\ell} \right) + \cos \left( \frac{(2n - 2) \pi x}{\ell} \right) \right]
\]

(11.4.8)

\[
M = \sum_{m=1}^{M} b_m \cos \left( \frac{(m-1) \pi x}{\ell} \right)
\]
where \( u \) and \( M \) are represented in symmetric form.

Substituting the above into the Galerkin equations of Eq. 10.10.13 and carrying out the necessary computation yields

\[
\int_{-\ell/2}^{\ell/2} \begin{bmatrix} \tilde{u}^T \\ \tilde{M} \end{bmatrix} A \begin{bmatrix} u \\ M \end{bmatrix} \, dx = \sum_{m=0}^{M} b_m \int_{-\ell/2}^{\ell/2} \begin{bmatrix} (m - 1) \pi \\ \ell \\ \ell \end{bmatrix}^2 \cos \left( \frac{2n\pi x}{\ell} \right) \cos \left( \frac{(m - 1)\pi x}{\ell} \right) \cos \left( \frac{2n\pi x}{\ell} \right) \cos \left( \frac{(m - 1)\pi x}{\ell} \right) \, dx
\]

\[
= \int_{-\ell/2}^{\ell/2} f \left[ \cos \left( \frac{2n\pi x}{\ell} \right) + \cos \left( \frac{(2n - 2)\pi x}{\ell} \right) \right] \, dx, \quad n = 1, \ldots, N
\]

(11.4.9)

where \( [\tilde{u}, \tilde{M}]^T = [\cos \frac{2n\pi x}{\ell} + \cos \{ (2n - 2)\pi x/\ell \}, 0]^T \), and

\[
\int_{-\ell/2}^{\ell/2} \begin{bmatrix} \tilde{u}^T \\ \tilde{M} \end{bmatrix} A \begin{bmatrix} u \\ M \end{bmatrix} \, dx = \sum_{n=1}^{N} a_n \int_{-\ell/2}^{\ell/2} \begin{bmatrix} (2n\pi) \ell^2 \cos \frac{2n\pi x}{\ell} \\ \ell \end{bmatrix}^2 \cos \left( \frac{(2n - 2)\pi x}{\ell} \right) \cos \left( \frac{(m - 1)\pi x}{\ell} \right) \cos \left( \frac{(2n - 2)\pi x}{\ell} \right) \cos \left( \frac{(m - 1)\pi x}{\ell} \right) \, dx
\]

\[
- \sum_{j=1}^{M} b_j \int_{-\ell/2}^{\ell/2} \frac{2}{E\ell} \cos \left( \frac{(j - 1)\pi x}{\ell} \right) \cos \left( \frac{(m - 1)\pi x}{\ell} \right) \, dx = 0, \quad m = 1, \ldots, M
\]

(11.4.10)

where \( [\tilde{u}, \tilde{M}]^T = [0, \cos \{ (m - 1)\pi x/\ell \}]^T \).

Inserting the first of Eqs. 11.4.8 into the Galerkin equations of Eq. 10.10.13 for the fourth order operator and performing the necessary calculations yields

\[
\sum_{n=0}^{N} a_n \int_{-\ell/2}^{\ell/2} \frac{1}{2} E\ell \left\{ \left( \frac{2n\pi}{\ell} \right)^2 \cos \left( \frac{2n\pi x}{\ell} \right) + \left( \frac{(2n - 2)\pi}{\ell} \right)^2 \cos \left( \frac{2n - 2)\pi x}{\ell} \right) \right\}
\]

\[
\times \left\{ \left( \frac{2k\pi}{\ell} \right)^2 \cos \left( \frac{2k\pi x}{\ell} \right) \left( \frac{(2k - 2)\pi}{\ell} \right)^2 \cos \left( \frac{2k - 2)\pi x}{\ell} \right) \right\} \, dx
\]

\[
= \int_{-\ell/2}^{\ell/2} f \left[ \cos \left( \frac{2k\pi x}{\ell} \right) + \cos \left( \frac{(2k - 2)\pi x}{\ell} \right) \right] \, dx,
\]

\[
k = 1, \ldots, N
\]

(11.4.11)
The second order approximate solution of Eqs. 11.4.9 and 11.4.10 and the fourth order approximate solution of Eqs. 11.4.11 were solved numerically for uniform and tapered clamped beams. Results for varying numbers of coordinate functions are given in Tables 11.4.3 and 11.4.4. Note that displacement results are accurate with both second and fourth order formulations, slightly more accurate with the fourth order formulation. Bending moments, in contrast, are much more accurate with the second order formulation. This is due partly to error induced in differentiating numerically computed displacements to obtain bending moments in the fourth order formulation.

**Table 11.4.3**
Results for Clamped Beam (Uniform Cross Section)

<table>
<thead>
<tr>
<th></th>
<th>Moment @ A $\times f^2$</th>
<th>Moment @ B $\times f^2$</th>
<th>$u @ A \frac{f^4}{EI_0}$</th>
<th>$\zeta = \frac{\rho\omega^2}{\xi}$ $\frac{1}{\xi^2} \sqrt{\frac{EI_0}{\rho \omega_0}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2\textsuperscript{nd} Order</td>
<td>N = 3, M = 4</td>
<td>0.041676</td>
<td>-0.083179</td>
<td>0.0026213</td>
</tr>
<tr>
<td></td>
<td>N = 5, M = 6</td>
<td>0.041668</td>
<td>-0.083404</td>
<td>0.0026131</td>
</tr>
<tr>
<td>4\textsuperscript{th} Order</td>
<td>N = 5</td>
<td>0.042485</td>
<td>-0.074147</td>
<td>0.0026023</td>
</tr>
<tr>
<td></td>
<td>N = 10</td>
<td>0.041185</td>
<td>-0.078765</td>
<td>0.0026023</td>
</tr>
<tr>
<td>Exact</td>
<td></td>
<td>0.041667</td>
<td>-0.083333</td>
<td>0.0026042</td>
</tr>
</tbody>
</table>
### Table 11.4.4
Results for Clamped Beam (Varying Cross Section)

<table>
<thead>
<tr>
<th>Order</th>
<th>N</th>
<th>Moment @ A $\times f\ell^2$</th>
<th>Moment @ B $\times f\ell^2$</th>
<th>$u$ @ A $\times f\ell^4/EI_0$</th>
<th>$\xi = \rho \omega^2 \times \frac{1}{\ell^2} \sqrt{\frac{EI_0}{\rho a_0}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; Order</td>
<td>3</td>
<td>0.056862</td>
<td>-0.067993</td>
<td>0.0013586</td>
<td>23.410</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.056855</td>
<td>-0.068216</td>
<td>0.0013476</td>
<td>23.413</td>
</tr>
<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt; Order</td>
<td>5</td>
<td>0.061550</td>
<td>-0.053045</td>
<td>0.0013252</td>
<td>23.487</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.056509</td>
<td>-0.060293</td>
<td>0.0013320</td>
<td>23.428</td>
</tr>
</tbody>
</table>

#### Static Analysis of Plates

The governing equations for bending of a plate [24] can be written in the second order form

\[
\begin{bmatrix}
M_{x_1} \\
M_{x_2} \\
M_{x_1 x_2} \\
u
\end{bmatrix}
= \begin{bmatrix}
-\frac{\partial^2 u}{\partial x_1^2} - \frac{12}{Eh^3} M_{x_1} + \nu \frac{12}{Eh^3} M_{x_2} \\
-\frac{\partial^2 u}{\partial x_2^2} - \frac{12}{Eh^3} M_{x_2} + \nu \frac{12}{Eh^3} M_{x_1} \\
2 \frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{24 (1 + \nu)}{Eh^3} M_{x_1 x_2} \\
-\frac{\partial^2 M_{x_1}}{\partial x_1^2} - \frac{\partial^2 M_{x_2}}{\partial x_2^2} + 2 \frac{\partial^2 M_{x_1 x_2}}{\partial x_1 \partial x_2}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
f(x_1, x_2)
\end{bmatrix}
\tag{11.4.12}
\]
where $M_{x_1}$ and $M_{x_2}$ are bending moments on cross sections normal to the $x_1$ and $x_2$ axes, $M_{x_1x_2}$ is a torsional moment, $u$ is deflection, $h$ is thickness of the plate, $v$ is Poisson's ratio, $E$ is Young's modulus, $D = Eh^3/[12(1 - v)]$, and $f$ is the distributed applied load. Equations 11.4.12 apply to plates with varying thickness. The applied load $f$ is taken as uniformly distributed in the examples treated here.

**Simply Supported Plate**

Simply supported plates with uniform and varying thickness are solved using the second order equations of Eqs. 11.4.12. The boundary conditions are

$$u(± a, x_2) = u(x_1, ± b) = 0$$

(11.4.13)

The solution is approximated by the following coordinate function expansions:

$$M_{x_1} = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} \cos \left( \frac{(2m - 1)\pi x_1}{2a} \right) \cos \left( \frac{(2n - 1)\pi x_2}{2b} \right)$$

$$M_{x_2} = \sum_{m=1}^{M} \sum_{n=1}^{N} b_{mn} \cos \left( \frac{(2m - 1)\pi x_1}{2a} \right) \cos \left( \frac{(2n - 1)\pi x_2}{2b} \right)$$

(11.4.14)

$$M_{x_1x_2} = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \sin \left( \frac{m\pi x_1}{2a} \right) \sin \left( \frac{n\pi x_2}{2b} \right)$$

$$u = \sum_{m=1}^{M} \sum_{n=1}^{N} d_{mn} \cos \left( \frac{(2m - 1)\pi x_1}{2a} \right) \cos \left( \frac{(2n - 1)\pi x_2}{2b} \right)$$

where the coordinate functions are chosen so that $M_{x_1}, M_{x_2},$ and $u$ satisfy the principal boundary conditions. Note that since the moment $M_{x_1x_2}$ is antisymmetric, its coordinate functions are selected in antisymmetric form.

Substituting Eqs. 11.4.14 into the Galerkin equations of Eq. 10.10.13 and carrying out necessary computations,

$$\int_{\Omega} [\bar{M}_{x_1}, \bar{M}_{x_2}, \bar{M}_{x_1x_2}, \bar{u}] A [M_{x_1}, M_{x_2}, M_{x_1x_2}, u]^T d\Omega$$

$$= \int_{\Omega} \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ d_{mn} \left( \frac{(2m - 1)\pi}{2a} \right)^2 - \frac{12}{Eh^3} a_{mn} + v \frac{12}{Eh^3} b_{mn} \right\}$$

$$\times \cos \left( \frac{(2m - 1)\pi x_1}{2a} \right) \cos \left( \frac{(2n - 1)\pi x_2}{2b} \right) \cos \left( \frac{(2\alpha - 1)\pi x_1}{2a} \right) \cos \left( \frac{(2\beta - 1)\pi x_2}{2b} \right) \cos \left( \frac{(2\alpha - 1)\pi x_1}{2a} \right) \cos \left( \frac{(2\beta - 1)\pi x_2}{2b} \right) d\Omega = 0,$$

$$\alpha, \beta = 1, \ldots, M$$

where $[\bar{M}_{x_1}, \bar{M}_{x_2}, \bar{M}_{x_1x_2}, \bar{u}]^T = \left[ \cos \left( \frac{(2\alpha - 1)\pi x_1}{2a} \right) \cos \left( \frac{(2\beta - 1)\pi x_2}{2b} \right), 0, 0, 0 \right]^T$.
Sec. 11.4  Solution of Beam and Plate Problems by the Galerkin Method

\[ \int_\Omega [ M_{x_1}, M_{x_2}, M_{x_1 x_2}, \ddot{u}] A [ M_{x_1}, M_{x_2}, M_{x_1 x_2}, u]^T d\Omega \]

\[ = \int_\Omega \sum_{m=1}^{M} \sum_{n=1}^{M} \left\{ d_{mn} \left( \frac{(2m-1)\pi}{2a} \right)^2 - \frac{12}{Eh^3} b_{mn} + v \frac{12}{Eh^3} a_{mn} \right\} \]

\[ \times \cos \frac{(2m-1)\pi x_1}{2a} \cos \frac{(2n-1)\pi x_2}{2b} \cos \frac{(2\alpha-1)\pi x_1}{2a} \cos \frac{(2\beta-1)\pi x_2}{2b} \]

\[ d\Omega = 0, \quad \alpha, \beta = 1, \ldots, M \]

where \([ \bar{M}_{x_1}, \bar{M}_{x_2}, \bar{M}_{x_1 x_2}, \ddot{u}]^T = [0, \cos \frac{(2\alpha-1)\pi x_1}{2a}, \cos \frac{(2\beta-1)\pi x_2}{2b}, 0, 0]^T\),

\[ \int_\Omega [ \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1 x_2}, \ddot{u}] A [ M_{x_1}, M_{x_2}, M_{x_1 x_2}, u]^T d\Omega \]

\[ = \int_\Omega 2 \sum_{m=1}^{M} \sum_{n=1}^{M} \left\{ d_{mn} \left( \frac{(2m-1)\pi}{2a} \right)^2 - e_{mn} \frac{24 (1 + v)}{Eh^3} \right\} \]

\[ \times \sin \frac{m\pi x_1}{2a} \sin \frac{n\pi x_2}{2b} \sin \frac{\alpha\pi x_1}{2a} \sin \frac{\beta\pi x_2}{2b} d\Omega = 0, \quad \alpha, \beta = 1, \ldots, M \]

where \([ \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1 x_2}, \ddot{u}]^T = [0, 0, \sin \frac{\alpha\pi x_1}{2a}, \sin \frac{\beta\pi x_2}{2b}, 0]^T\), and

\[ \int_\Omega [ \bar{M}_{x_1}, \bar{M}_{x_2}, \bar{M}_{x_1 x_2}, \ddot{u}] A [ M_{x_1}, M_{x_2}, M_{x_1 x_2}, u]^T d\Omega \]

\[ = \int_\Omega \sum_{m=1}^{M} \sum_{n=1}^{M} \left\{ a_{mn} \left( \frac{(2m-1)\pi}{2a} \right)^2 + b_{mn} \left( \frac{(2n-1)\pi}{2b} \right)^2 \right\} \]

\[ \times \cos \frac{(2m-1)\pi x_1}{2a} \cos \frac{(2n-1)\pi x_2}{2b} \cos \frac{(2\alpha-1)\pi x_1}{2a} \cos \frac{(2\beta-1)\pi x_2}{2b} \]

\[ + 2c_{mn} \left( \frac{m\pi}{2a} \frac{n\pi}{2b} \right) \cos \frac{m\pi x_1}{2a} \cos \frac{n\pi x_2}{2b} \cos \frac{(2\alpha-1)\pi x_1}{2a} \cos \frac{(2\beta-1)\pi x_2}{2b} \}

\[ d\Omega = \int_\Omega f \cos \frac{(2\alpha-1)\pi x_1}{2a} \cos \frac{(2\beta-1)\pi x_2}{2b} d\Omega, \quad \alpha, \beta = 1, \ldots, M \]

(11.4.15)

where \([ \bar{M}_{x_1}, \bar{M}_{x_2}, \bar{M}_{x_1 x_2}, \ddot{u}]^T = [0, 0, 0, \cos \frac{(2\alpha-1)\pi x_1}{2a}, \cos \frac{(2\beta-1)\pi x_2}{2b}]^T\).

Numerical solutions for both uniform and variable thickness simply supported square plates are presented in Tables 11.4.5 and 11.4.6, where points of interest are defined in Fig. 11.4.1. Note that the accuracy of both displacement and bending moments is good for the uniform plate and only \( M_{x_1 x_2} \) at point C shows significant error (or slower convergence) for the variable thickness plate.
### Table 11.4.5
Results for Simply Supported Plate
(Uniform Thickness, Second Order Method)

<table>
<thead>
<tr>
<th>M</th>
<th>( M_{x_1} @ A \times 4fa^2 )</th>
<th>( M_{x_1x_2} @ C \times 4fa^2 )</th>
<th>( u @ A \times 16fa^4D )</th>
<th>( \zeta = \rho \omega^2 \times \frac{1}{4a^2 \sqrt{\frac{D}{\rho h_0}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = 2</td>
<td>0.046895</td>
<td>0.033244</td>
<td>0.0040547</td>
<td>19.739</td>
</tr>
<tr>
<td>M = 3</td>
<td>0.048235</td>
<td>0.032865</td>
<td>0.0040639</td>
<td>19.739</td>
</tr>
<tr>
<td>M = 4</td>
<td>0.047709</td>
<td>0.032271</td>
<td>0.0040619</td>
<td>19.739</td>
</tr>
<tr>
<td>EXACT [26]</td>
<td>0.0479</td>
<td>0.0325</td>
<td>0.00406</td>
<td>19.739</td>
</tr>
</tbody>
</table>

### Table 11.4.6
Results for Simply Supported Plate
(Varying Thickness, Second Order Method)

\[
h = 2h_0 \left( 1 - \frac{|x_1|}{2a} \right) \left( 1 - \frac{|x_2|}{2a} \right)
\]

<table>
<thead>
<tr>
<th>M</th>
<th>( M_{x_1} @ A \times 4fa^2 )</th>
<th>( M_{x_1x_2} @ C \times 4fa^2 )</th>
<th>( u @ A \times 16fa^4D )</th>
<th>( \zeta = \rho \omega^2 \times \frac{1}{4a^2 \sqrt{\frac{D}{\rho h_0}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = 2</td>
<td>0.075359</td>
<td>0.0054289</td>
<td>0.0019412</td>
<td>22.923</td>
</tr>
<tr>
<td>M = 3</td>
<td>0.079413</td>
<td>0.0039519</td>
<td>0.0019674</td>
<td>22.942</td>
</tr>
<tr>
<td>M = 4</td>
<td>0.080911</td>
<td>0.0040777</td>
<td>0.0019570</td>
<td>22.955</td>
</tr>
</tbody>
</table>
Sec. 11.4 Solution of Beam and Plate Problems by the Galerkin Method

Clamped Plate

Clamped plates with uniform and varying thickness are solved, using both the second and fourth order equations. The boundary conditions are

\[ u(\pm a, x_2) = u(x_1, \pm b) = 0 \]
\[ \frac{\partial u}{\partial x_1}(\pm a, x_2) = \frac{\partial u}{\partial x_2}(x_1, \pm b) = 0 \]  

(11.4.16)

The solution is approximated by coordinate functions of the form

\[ M_{x_1} = \sum_{m=0}^{M} \sum_{n=0}^{M} a_{mn} \cos \frac{2m\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \]
\[ M_{x_2} = \sum_{m=0}^{M} \sum_{n=0}^{M} b_{mn} \cos \frac{2m\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \]
\[ M_{x_1x_2} = \sum_{m=1}^{M} \sum_{n=1}^{M} c_{mn} \sin \frac{4m\pi x_1}{a} \sin \frac{4n\pi x_2}{b} \]
\[ u = \sum_{m=1}^{M} \sum_{n=1}^{M} d_{mn} \cos \frac{2(2m-1)\pi x_1}{a} \cos \frac{2\pi x_1}{a} \cos \frac{2(2n-1)\pi x_2}{b} \cos \frac{2\pi x_2}{b} \]

\[ = \frac{1}{4} \sum_{m=1}^{M} \sum_{n=1}^{M} d_{mn} \left[ \cos \frac{4m\pi x_2}{b} + \cos \frac{2(2m-2)\pi x_2}{b} \right] \]

\[ \times \left[ \cos \frac{4n\pi x_2}{b} + \cos \frac{2(2n-2)\pi x_2}{b} \right] \]  

(11.4.17)
where \( u \) in Eqs. 11.4.17 satisfies the boundary conditions. The moments \( M_{x_1} \) and \( M_{x_2} \) are symmetric with respect to the \( x_1 \) and \( x_2 \) axes, so they are approximated by cosine functions. The moment \( M_{x_1 x_2} \), on the other hand, is antisymmetric and must be zero along the boundary, so its coordinate functions are selected with these properties.

Substitution of these approximations into the Galerkin equations of Eq. 10.10.13 and performing necessary calculations yields the following equations:

\[
\int_\Omega \left[ \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1 x_2}, \tilde{u} \right] A [ M_{x_1}, M_{x_2}, M_{x_1 x_2}, u ]^T d\Omega \\
= \int_{-b}^{b} \int_{-a}^{a} \left( \sum_{m=1}^{M} \sum_{n=1}^{M} d_{mn} \left[ \left( \frac{2m\pi}{a} \right)^2 \cos \frac{4m\pi x_1}{a} + \left( \frac{2m - 2}{a} \right)^2 \cos \frac{2(2m - 2)\pi x_1}{a} \right] \right) \\
\times \left\{ \cos \frac{4n\pi x_2}{b} + \cos \frac{2(2n - 2)\pi x_2}{b} \right\} - \frac{12}{Eh^3} \sum_{m=0}^{M} \sum_{n=0}^{M} a_{mn} \cos \frac{2m\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \\
+ \nu \frac{12}{Eh^3} \sum_{m=0}^{M} \sum_{n=0}^{M} b_{mn} \cos \frac{2m\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \right) \right] \\
\times \cos \frac{2n\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \right) dx_1 \right) dx_2 = 0, \\
\alpha, \beta = 0, 1, \ldots, M (11.4.18)
\]

where \( [ \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1 x_2}, \tilde{u} ]^T = \left[ \begin{array}{c} \cos \frac{2\alpha\pi x_1}{a} \\
\cos \frac{2\beta n x_2}{b} \\
0 \\
0 \end{array} \right] \),

\[
\int_\Omega \left[ \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1 x_2}, \tilde{u} \right] A [ M_{x_1}, M_{x_2}, M_{x_1 x_2}, u ]^T d\Omega \\
= \int_{-b}^{b} \int_{-a}^{a} \left( \sum_{m=1}^{M} \sum_{n=1}^{M} d_{mn} \left[ \left( \frac{2n\pi}{b} \right)^2 \cos \frac{4n\pi x_2}{b} + \left( \frac{2n - 2}{b} \right)^2 \cos \frac{2(2n - 2)\pi x_2}{b} \right] \right) \\
\times \left\{ \cos \frac{4m\pi x_1}{a} + \cos \frac{2(2m - 2)\pi x_1}{a} \right\} - \frac{12}{Eh^3} \sum_{m=0}^{M} \sum_{n=0}^{M} a_{mn} \cos \frac{2m\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \\
+ \nu \frac{12}{Eh^3} \sum_{m=0}^{M} \sum_{n=0}^{M} b_{mn} \cos \frac{2m\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \right) \right] \\
\times \cos \frac{2n\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \right) dx_1 \right) dx_2 = 0, \\
\alpha, \beta = 1, 2, \ldots, M (11.4.19)
\]

where \( [ \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1 x_2}, \tilde{u} ]^T = \left[ \begin{array}{c} 0 \\
\cos \frac{2\alpha\pi x_1}{a} \\
\cos \frac{2\beta n x_2}{b} \\
0 \end{array} \right] \),
Sec. 11.4  Solution of Beam and Plate Problems by the Galerkin Method

\[ \int_{\Omega} \begin{bmatrix} \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1x_2}, \tilde{u} \end{bmatrix} A \begin{bmatrix} M_{x_1}, M_{x_2}, M_{x_1x_2}, u \end{bmatrix}^T d\Omega \]

\[ = \int_{-b}^{b} \sum_{m=1}^{M} \sum_{n=1}^{M} \left[ d_{mn} \left( \frac{2m\pi}{a} \right) \sin \frac{4m\pi x_1}{a} + \left( \frac{2(2m-2)\pi}{a} \right) \sin \frac{2(2m-2)\pi x_1}{a} \right] \]

\times \left[ \left( \frac{2n\pi}{b} \right) \sin \frac{4n\pi x_2}{b} + \left( \frac{2(2n-2)\pi}{b} \right) \sin \frac{2(2n-2)\pi x_2}{b} \right] \]

\[ - \frac{24(1+\nu)}{Eh^3} \sin \frac{4m\pi x_1}{a} \sin \frac{4n\pi x_2}{b} \left[ \frac{4\phi_1x_1}{a} \sin \frac{4\beta\pi x_2}{b} \right] d_1 d_2 \]

\( = 0, \quad \alpha, \beta = 1, 2, \ldots, M \) (11.4.20)

where \( \begin{bmatrix} M_{x_1}, M_{x_2}, M_{x_1x_2}, \tilde{u} \end{bmatrix}^T = \begin{bmatrix} 0, 0, \sin \frac{4\alpha\pi x_1}{a}, \sin \frac{4\beta\pi x_2}{b}, 0 \end{bmatrix}^T \), and

\[ \int_{\Omega} \begin{bmatrix} \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1x_2}, \tilde{u} \end{bmatrix} A \begin{bmatrix} M_{x_1}, M_{x_2}, M_{x_1x_2}, u \end{bmatrix}^T d\Omega \]

\[ = \int_{-b}^{b} \int_{-a}^{a} \left\{ \sum_{m=0}^{M} \sum_{n=0}^{M} \left[ a_{mn} \left( \frac{2m\pi}{a} \right)^2 + b_{mn} \left( \frac{2n\pi}{b} \right)^2 \right] \cos \frac{2m\pi x_1}{a} \cos \frac{2n\pi x_2}{b} \right\} \]

\[ + \sum_{m=1}^{M} \sum_{n=1}^{M} \left( \frac{4m\pi}{a} \right) \left( \frac{4n\pi}{b} \right) \cos \frac{4m\pi x_1}{a} \cos \frac{4n\pi x_2}{b} \]

\times \cos \frac{2(2\alpha-1)\pi x_1}{a} \cos \frac{2\pi x_1}{a} \cos \frac{2(2\beta-1)\pi x_2}{b} \cos \frac{2\pi x_2}{b} dx_1 dx_2 \]

\[ = \int_{-b}^{b} \int_{-a}^{a} f(x_1, x_2) \cos \frac{2(2\alpha-1)\pi x_1}{a} \cos \frac{2\pi x_1}{a} \cos \frac{2(2\beta-1)\pi x_2}{b} \cos \frac{2\pi x_2}{b} dx_1 dx_2, \]

\( \alpha, \beta = 1, 2, \ldots, M \) (11.4.21)

where

\[ \begin{bmatrix} \tilde{M}_{x_1}, \tilde{M}_{x_2}, \tilde{M}_{x_1x_2}, \tilde{u} \end{bmatrix}^T = \begin{bmatrix} 0, 0, \cos \frac{2(2\alpha-1)\pi x_1}{a}, \cos \frac{2\pi x_1}{a} \cos \frac{2(2\beta-1)\pi x_2}{b} \cos \frac{2\pi x_2}{b} \end{bmatrix}^T \]

Numerical results, obtained with both the second order and the fourth order equations for square plates, are presented in Tables 11.4.7 and 11.4.8, where points of interest are defined in Fig. 11.4.1. As in prior examples, the accuracy of displacement and bending moments is uniformly good with the second order formulation. In contrast, bending moments calculated by differentiating approximate displacements in the fourth order formulation show significant error for even the uniform thickness plate and large error for the variable thickness plate.
Table 11.4.7
Results for Clamped Plate (Uniform Thickness)

<table>
<thead>
<tr>
<th></th>
<th>Moment @ A $\times 4a^2$</th>
<th>Moment @ B $\times 4a^2$</th>
<th>$u@A\times 16fa^4D$</th>
<th>$\zeta = \rho\omega^2 \times \frac{1}{4a^2} \sqrt{\frac{D}{\rho h_0}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd Order</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M = 2</td>
<td>0.022891</td>
<td>-0.048742</td>
<td>0.0012075</td>
<td>36.047</td>
</tr>
<tr>
<td>M = 3</td>
<td>0.022788</td>
<td>-0.050480</td>
<td>0.0012958</td>
<td>35.990</td>
</tr>
<tr>
<td>M = 4</td>
<td>0.022987</td>
<td>-0.050932</td>
<td>0.0012438</td>
<td>35.974</td>
</tr>
<tr>
<td>4th Order</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M = 3</td>
<td>0.025092</td>
<td>-0.038922</td>
<td>0.0012626</td>
<td>36.106</td>
</tr>
<tr>
<td>M = 4</td>
<td>0.021518</td>
<td>-0.041473</td>
<td>0.0012586</td>
<td>36.044</td>
</tr>
<tr>
<td>EXACT [26]</td>
<td>0.0213</td>
<td>-0.0513</td>
<td>0.00126</td>
<td>35.98</td>
</tr>
</tbody>
</table>

Table 11.4.8
Results for Clamped Plate (Varying Thickness)

$$h = 2h_0 \left( 1 - \frac{|x_1|}{2a} \right) \left( 1 - \frac{|x_2|}{2a} \right)$$

<table>
<thead>
<tr>
<th></th>
<th>Moment @ A $\times 4a^2$</th>
<th>Moment @ B $\times 4a^2$</th>
<th>$u@A\times 16fa^4D$</th>
<th>$\zeta = \rho\omega^2 \times \frac{1}{4a^2} \sqrt{\frac{D}{\rho h_0}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd Order</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M = 2</td>
<td>0.040755</td>
<td>-0.011680</td>
<td>0.00061224</td>
<td>38.058</td>
</tr>
<tr>
<td>M = 3</td>
<td>0.041235</td>
<td>-0.014495</td>
<td>0.00075683</td>
<td>38.141</td>
</tr>
<tr>
<td>M = 4</td>
<td>0.043061</td>
<td>-0.014452</td>
<td>0.00065104</td>
<td>38.175</td>
</tr>
<tr>
<td>4th Order</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M = 2</td>
<td>0.029754</td>
<td>-0.0062549</td>
<td>0.00060060</td>
<td>40.706</td>
</tr>
<tr>
<td>M = 3</td>
<td>0.063371</td>
<td>-0.0085592</td>
<td>0.00065468</td>
<td>39.376</td>
</tr>
<tr>
<td>M = 4</td>
<td>0.040864</td>
<td>-0.0101120</td>
<td>0.00067205</td>
<td>38.774</td>
</tr>
</tbody>
</table>
Vibration of Beams and Plates

The beam vibration eigenvalue problem can be written in second order form as

\[ A \begin{bmatrix} u \\ \dot{M} \end{bmatrix} = \begin{bmatrix} -\frac{d^3M}{dx^2} \\ -\frac{d^2u}{dx^2} - \frac{M}{EI} \end{bmatrix} = \begin{bmatrix} \zeta u \\ 0 \end{bmatrix} \]  

(11.4.22)

where \( M \) is bending moment, or in fourth order form as

\[ \frac{d^2}{dx^2} \left[ \frac{EI}{dx^2} \frac{d^2u}{dx^2} \right] = \zeta au \]  

(11.4.23)

where \( \zeta = \rho \omega^2 \) and \( a \) is the cross-sectional area.

The equations for vibration of a plate can be written in second order form as

\[ A \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ M_{x_1x_2} \\ u \end{bmatrix} \equiv \begin{bmatrix} -\frac{\partial^2u}{\partial x_1^2} - \frac{12}{Eh^3} M_{x_1} + \nu \frac{12}{Eh^3} M_{x_2} \\ -\frac{\partial^2u}{\partial x_2^2} - \frac{12}{Eh^3} M_{x_2} + \nu \frac{12}{Eh^3} M_{x_1} \\ 2 \frac{\partial^2u}{\partial x_1 \partial x_2} - \frac{24 (1 + \nu)}{Eh^3} M_{x_1x_2} \\ -\frac{\partial^2M_{x_1}}{\partial x_1^2} - \frac{\partial^2M_{x_2}}{\partial x_2^2} + 2 \frac{\partial^2M_{x_1x_2}}{\partial x_1 \partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \zeta hu \end{bmatrix} \]  

(11.4.24)

or in fourth order form as

\[ \frac{\partial^2}{\partial x_1^2} \left\{ D \left[ \frac{\partial^2u}{\partial x_1^2} + \nu \frac{\partial^2u}{\partial x_2^2} \right] \right\} + \frac{\partial^2}{\partial x_2^2} \left\{ D \left[ \frac{\partial^2u}{\partial x_2^2} + \nu \frac{\partial^2u}{\partial x_1^2} \right] \right\} + 2 \frac{\partial^2}{\partial x_1 \partial x_2} \left\{ D (1 - \nu) \frac{\partial^2u}{\partial x_1 \partial x_2} \right\} = \zeta hu \]  

(11.4.25)

where \( \zeta = \rho \omega^2 \) and \( h \) is the plate thickness.

The Galerkin method for eigenvalue analysis was implemented for simply supported and clamped beams and plates, with both uniform and variable cross sections and thick-
nesses. The same coordinate functions employed in static analysis were used to obtain numerical results that are presented in Tables 11.4.1 through 11.4.8. It is instructive to note that when the second order equations are employed, the approximate eigenvalues do not necessarily decrease as more coordinate functions are employed; e.g., see Tables 11.4.2, 11.4.4, 11.4.6, and 11.4.8. This is due to the fact that the operators for second order systems of equations are not positive definite. Note, however, that since the fourth order operators are positive definite, the approximate eigenvalues always decrease as more coordinate functions are employed.