7-3 LAGRANGE'S EQUATIONS FOR CONSERVATIVE SYSTEMS

- For conservative systems $\delta W = -\delta V$, thus Hamilton’s principle can be written as
  \[ \int_{t_0}^{t_1} (\delta T - \delta V)dt = 0; \quad \delta \int_{t_0}^{t_1} (T - V)dt = 0 \]
  \[ \delta A = 0; \quad A = \int_{t_0}^{t_1} L dt; \quad L = T - V \]

- $L = L(x, \dot{x}, t)$ is called the Lagrangian function; $A$ is called the Action Integral.

- Among all motions that will carry a conservative system from a given initial configuration $X_0$ to a given final configuration $X_1$ in a given time interval $(t_0, t_1)$, that which actually occurs provides a stationary value to the integral $A$.

- For a nonholonomic system, the variation of virtual displacement $S$ must be consistent with the nonholonomic constraints.

- For holonomic system, the variation of virtual displacement $S$ is arbitrary.

- For holonomic system with a finite number of generalized coordinates $x_i$, Calculus of Variations procedure is used to impose the condition $\delta A = 0$. This gives the differential equation of motion as
  \[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \]

These are called Lagrange's equation or Euler-Lagrange equations.
Example: Double pendulum

7-4 TIME-DEPENDENT CONSTRAINTS

- Systems subjected to time-dependent constraints are usually nonconservative in the sense that the total mechanical energy varies with time, since the constraining forces perform work on the system; i.e., $T + V \neq \text{constant}$.

- Hamilton's principle is valid for systems with time-dependent constraints, provided that the terminal configurations $(X_0, X_1)$ and the variations $S$ satisfy the constraints.

- If constraints are held constant instantaneously, the work $V$ that we perform against noninertial forces in giving the system any kinematically admissible virtual displacement may be independent of path. Then $\delta W = -\delta V$ and $V$ may be regarded as a time-dependent potential energy function. However, in this case, we cannot use $T + V = \text{constant}$.

- Example: Mechanism with time-dependent constraints.

7-5 THE HAMILTONIAN FUNCTION

- Let $(x_1, x_2, x_3)$ be the coordinates of a particle that moves in a conservative force field $(F_1, F_2, F_3)$. Then the Lagrangian function is given as

$$L(x, \dot{x}, t) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V$$
The Lagrange's equations for the system are given as
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0; \quad \frac{\partial L}{\partial \dot{x}_i} = m\ddot{x}_i; \quad \frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i
\]

Let \( p_i \) be the components of the momentum. The above equation gives
\[
\frac{\partial L}{\partial \dot{x}_i} = p_i
\]

The Lagrange's equation can be written as
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0; \quad \frac{\partial L}{\partial \dot{x}_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \dot{p}_i
\]

Thus, Lagrange's equation yields Newton's second law.

For a conservative system with enumerable degrees of freedom, the quantities \( p_i \) are called components of generalized momentum. An alternate form of the Lagrange's equation is given as
\[
\frac{\partial L}{\partial \dot{x}_i} = \dot{p}_i
\]

Hamilton's function: Let a function be defined as
\[
H = \sum_{i=1}^{n} p_i \dot{x}_i - L(x_i, \dot{x}_i, t)
\]

Differentiating \( H \) with respect to \( t \):
\[
\frac{dH}{dt} = \sum \dot{p}_i \dot{x}_i + \sum p_i \ddot{x}_i - \sum \frac{\partial L}{\partial x_i} \dot{x}_i - \sum \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i - \frac{\partial L}{\partial t}
\]
\[
\frac{dH}{dt} = -\frac{\partial L}{\partial t}
\]
If \( L \) is not time dependent which is usually the case, then the previous equation gives \( H = \text{constant} \). Also \( H = 2T - L = T + V = \text{constant} \) which expresses the conservation of energy principle.