Chaos Theory In Practice

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Agenda

- Determination if Time-Series is Chaotic
- Calculation of Time Lag
- Calculation of Embedding Dimension
- Calculation of Maximal Lyapunov Exponent
- Calculation of System Entropy
- Non-Linear Noise Reduction
- Non-Linear Prediction
Determining Chaos

• Most books on chaos talk about infinite time scales

• We measure finite time scales, therefore we need to determine if a time-series dataset is a chaotic finite dataset.

• If the system is sensitively dependent upon time then it is most likely chaotic.

Chaotic Time Series?

• Perform any preprocessing as usual
  • Replacing missing values not recommended
  • Averaging the data is not recommended

• Create an ensemble of surrogate datasets from the original dataset.
  • Randomly arrange the time instances to ignore time.
  • Select surrogates from a similar distribution

• Measure each ensemble and the original dataset against some invariant measurement.
Chaotic Times Series?

- If the invariant measurement is the same (or close within 5%) for each of the surrogates and the original dataset then the system is not chaotic.

- If the invariant measurement is the same (or close within 5%) for each surrogate but different than the original dataset then the dataset exhibits deterministic chaos at the 95% confidence level.

- This is due to the dependence on the previous states and there for the system is deterministically chaotic.

Case Study: Santa Fe Laser

- Taken from the Santa Fe Time Series Competition

- Dataset generated from a chaotic laser

- The measurements were made on an 81.5-micron $^{14}$NH$_3$ cw (FIR) laser, pumped optically by the P(13) line of an N$_2$O laser via the vibrational aQ(8,7) NH$_3$ transition.

- 1000 data points

- Single variable (laser intensity)
Case Study: Santa Fe Laser

- Ensemble of 10 surrogate datasets created by random shuffling of the original data.

- The higher-order autocorrelation function used as the statistic to determine rejection of Null Hypothesis

\[ a_c = \left( s_n s_{n+1}^2 - s_n^2 s_{n+1} \right) \]

s is the norm of the vector
\[ s_i = \sqrt{s_i^2 + s_{i+1}^2 + \ldots + s_n^2} \]
Case Study: Santa Fe Laser

- Difference between the values of the higher-order autocorrelation function for the surrogates and the original data must be greater than 5% of the autocorrelation of the original data.

- Autocorrelation of original data = 1143.94

- 5% required to reject null hypothesis = 57.197

<table>
<thead>
<tr>
<th>Ensemble Number</th>
<th>Difference from Original</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>193.88</td>
</tr>
<tr>
<td>2</td>
<td>297.16</td>
</tr>
<tr>
<td>3</td>
<td>1168.13</td>
</tr>
<tr>
<td>4</td>
<td>1513.47</td>
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<tr>
<td>5</td>
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<td>6</td>
<td>1144.40</td>
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<td>7</td>
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<tr>
<td>9</td>
<td>1180.30</td>
</tr>
<tr>
<td>10</td>
<td>1174.89</td>
</tr>
</tbody>
</table>

Null Hypothesis reject. Data is from a deterministically chaotic time-series
Phase Space Determination

- Phase space is the mathematical space in which all possible states of a system are represented.

- Every degree of freedom, or parameter of the system, are represented as an axis of the multidimensional space.

- Phase space of deterministic chaotic systems are determined using time delay embedding.

- Delay reconstruction of the phase space for a system is in $m$ dimensional space is formed using $m$ dimensional vectors which are delayed between each other by the time lag $\tau$.

Time Lag Determination

- The autocorrelation function is used to determine the proper time lag for phase space reconstruction.

- The autocorrelation for time lag, $\tau$, is given by

$$c_\tau = \frac{\langle s_n s_{n-\tau} \rangle - \langle s \rangle^2}{\sigma^2}$$

- The appropriate time lag is the value of $\tau$ which results in the first time the autocorrelation function falls below $1/e$. 
Case Study: Santa Fe Laser

<table>
<thead>
<tr>
<th>Tau</th>
<th>Autocorrelation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>-0.1442</td>
</tr>
<tr>
<td>3</td>
<td>-0.0856</td>
</tr>
<tr>
<td>4</td>
<td>-0.0993</td>
</tr>
<tr>
<td>5</td>
<td>-0.1049</td>
</tr>
</tbody>
</table>

Autocorrelation Lag Factor = 1
Case Study: Electrical Demand

- Single variable dataset with 15198 instances

- 15 minutes averaged values of power demand in the full year 1997 for New York City
Case Study: Electrical Demand

<table>
<thead>
<tr>
<th>Tau</th>
<th>Autocorrelation</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.404</td>
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<tr>
<td>2</td>
<td>0.396</td>
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<tr>
<td>3</td>
<td>0.386</td>
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<tr>
<td>4</td>
<td>0.373</td>
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<tr>
<td>5</td>
<td>0.358</td>
</tr>
</tbody>
</table>

Autocorrelation Lag Factor = 5

Case Study: Electrical Demand

![Simple Delay Plot](image)
Determination of Embedding Dimension

- Most dynamical nonlinear systems have fractal dimensions (Lorenz Attractor Dimension = 2.06)

- Delay embedded reconstruction uses integer dimensions

- Calculation of the dimension is performed through the False-Nearest Neighbor technique

False-Nearest Neighbor

- Search for points which are neighbors in embedding space but which would not be neighbors since their future temporal evolution is too different.

- For each point in the time series, take its closest neighbors in \( m \) dimensions and compute the ratio of the distances between the points in \( m+1 \) dimension and in \( m \) dimensions.

- If ratio is larger than some threshold \( r \), the neighbor was false.

- Embedding dimension is the dimension in which false nearest neighbors cease to be detected.
False-Nearest Neighbor

False-Nearest Neighbor
Case Study: Astrophysical Data

- Set of measurements of the light curve (time variation of the intensity) of the variable white dwarf star PG1159-035 during March 1989.

- Recorded by the Whole Earth Telescope

- 618 instances of the single variable
**False-Nearest Neighbor**

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Embedding Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Santa Fe Chaotic Laser</td>
<td>2</td>
</tr>
<tr>
<td>Electricity Demand</td>
<td>2</td>
</tr>
<tr>
<td>Astrophysical</td>
<td>3</td>
</tr>
</tbody>
</table>

**Lyapunov Exponents**

- Measure the rate at which trajectories within the attractor diverge or converge.
- Positive Lyapunov Exponent indicates rapid divergence
- Negative Lyapunov indicates convergence
- Lyapunov Exponent of 0.00 indicates a limit cycle
- There exists as many Lyapunov Exponents as there are dimension to the embedded space.
- The larger the Lyapunov Exponent the shorter the good prediction time
Calculation of Max Lyapunov

- Requires both the embedding dimension and the time delay to have already been calculated.
- Consider two trajectories, in the reconstructed phase space

\[ x(n_i) = \{x_0, x_0, x_0, x_0, \ldots, x_0, y_0, y_0, y_0, y_0, \ldots, y_0\} \]

\[ y(n_i) = \{y_0, y_0, y_0, y_0, \ldots, y_0, y_0, y_0, y_0, \ldots, y_0\} \]

- Let the distance between \( x(n_i) \) and \( y(n_i) \) be some small value \( \varepsilon \).

Calculation of Max Lyapunov

- Evolve the attractor for some time \( \Delta t \) and recalculate the distances between \( x(n_i) \) and \( y(n_i) \) as:

\[ \varepsilon_{\Delta t} = \|x_{n+\Delta t} - y_{n+\Delta t}\| \]

- The maximum Lyapunov Exponent, \( \lambda \), of the trajectories is then:

\[ \varepsilon_{\Delta t} \approx \varepsilon_0 e^{\lambda \Delta t} \quad \text{or} \]

\[ \lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left( \frac{\|x_{n_i+\Delta t} - y_{n_i+\Delta t}\|}{\|x_{n_i} - y_{n_i}\|} \right) \]
Lyapunov Exp.: Case Study

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Time Delay</th>
<th>Embedding Dim</th>
<th>Lyapunov</th>
</tr>
</thead>
<tbody>
<tr>
<td>Santa Fe Laser</td>
<td>1</td>
<td>2</td>
<td>0.2214</td>
</tr>
<tr>
<td>Electric Demand</td>
<td>5</td>
<td>2</td>
<td>0.0061</td>
</tr>
<tr>
<td>Astrophysical</td>
<td>1</td>
<td>3</td>
<td>5.7185</td>
</tr>
</tbody>
</table>

Indicates the complexity of the time-series data.

- Large values of entropy (greater than 1.5) increase the difficulty of prediction (known as loss of predictibility)
- Negative values do not exist for entropy
System Entropy

- Entropy should be calculated in embedded phase space for deterministically chaotic systems.

- Create phase space vectors for each point in embedding space.

- Entropy is then calculated as

\[ H = -\sum_i p(s_i) \ln(p(s_i)) \]

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Time Delay</th>
<th>Embedding Dim</th>
<th>Entropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electric Demand</td>
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<tr>
<td>Santa Fe Laser</td>
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<tr>
<td>Astrophysical</td>
<td>1</td>
<td>3</td>
<td>9.2644</td>
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</table>
Nonlinear Noise Reduction

- Must be performed in reconstructed phase space

- Relies heavily on Lyapunov exponents since rapid divergence increases the difficulty of noise reduction

- Uses neighborhoods of embedded points to attempt to reduce noise and determine true dynamics of the attractor

\[ S^*_{m/2} - m/2 = \frac{1}{U_{\varepsilon}(S_{n_0})} \sum_{s_n(U_{\varepsilon}(S_{n_0})))} S_{n-m/2} \]

\[ U_{\varepsilon} \text{ is a small neighborhood about the point } S_{n_0} \]
Nonlinear Noise Reduction

• Locally Projective Noise Reduction
  • Uses Principal Component Analysis to determine actual data versus noise in tangent space (a sub embedding of phase space)
  • First few principle components correspond to actual data while the remaining correspond purely to noise
  • As in the simple algorithm calculate a neighborhood $U_\varepsilon$ around each vector.

Nonlinear Noise Reduction cont.

• Locally Projective Noise Reduction cont.
  • PCA is performed by first computing a mean

$$s^{(n)} = \frac{1}{|U_n|} \sum_{k \in U_n} \tilde{s}_k$$

• Next calculate the covariance matrix

$$C_{ij} = \sum_{n' \in U_n} (s_{n'} - \bar{s}^{(n)})_i (s_{n'} - \bar{s}^{(n)})_j$$

• The eigenvectors of $C$ will then represent the semi-axis of an ellipsoid best approximating the cloud of points $U$
Nonlinear Noise Reduction

- Locally Projective Noise Reduction cont.
  - The eigenvectors of $C$ will be large in the directions of real data and small in the directions of the noise

- Average the projection of the vectors in $U$ created by the largest eigenvectors

- This becomes the new point on the attractor and best approximates the true data with the noise removed.
Noise Reduction Case Study

- Santa Fe Laser

![Original Delay Plot](image)
![Filtered Delay Plot](image)

Nonlinear Prediction

- Stochastic prediction highly error prone due to random nature

- Deterministic nonlinear prediction very accurate for short time length due to rapid divergence (length of time determined by Lyapunov Exponent)

- Performed similarly to the simple noise reduction algorithm
Nonlinear Prediction

- Simple Nonlinear Prediction (Lorenz’s Method of Analogs)
  - Given a continuous map $F$ we can predict $x_{n+1}$ using $x_n$

- Search the list of past states $x^*_n$ with $n < N$ for the closest one to $x_n$ with respect to some norm (Euclidean used here).

- If the discovered state at $x^*_n$ was similar to the current $x_n$ (and thus close in phase space) then we can use its value at $x^*_{n+1}$ for the value of $x_{n+1}$

Nonlinear Prediction Case Study

- Henon Map
  - Starting state
    $x = 0.682$, $y = 0.181$
  - 1000 iterations to be predicted
Nonlinear Prediction Case Study

Data from Original Equations

Data from Prediction

Nonlinear Prediction Case Study

<table>
<thead>
<tr>
<th>Predicted X</th>
<th>Predicted Y</th>
<th>Original X</th>
<th>Original Y</th>
<th>Diff. X</th>
<th>Diff. Y</th>
<th>Error</th>
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</thead>
<tbody>
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<td>0.00271</td>
</tr>
</tbody>
</table>

First 5 values

For all 1000 Predictions
MSPE = 0.87335
RMSPE = 0.9345
Nonlinear Resources


- Simple Chaos Explorer software available on Prof. Kusiak’s webpage

Questions?