Dimensional Analysis

11. 3. 2014

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Dimensions and Units

- **Dimension**: A measure of a physical quantity
  - MLT system: Mass (M), Length (L), Time (T)
  - FLT system: Force ($F = MLT^{-2}$), Length (L), Time (T)

- **Unit**: A way to assign a number to that dimension

Dimension	Symbol	SI Unit	BG Unit
Mass	M	kg (kilogram)	lb (pound)
Length	L	m (meter)	ft (feet)
Time	T	s (second)	s (second)

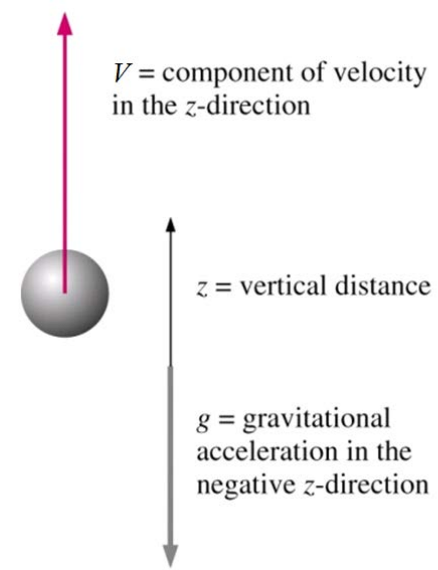
The Principle of Dimensional Homogeneity (PDH)

- Every additive terms in an equation must have the same dimensions

Ex) Displacement of a falling body

$$z = z_0 + V_0 t - \frac{1}{2} g t^2$$

$$(L) = (L) + \left(\frac{L}{T}\right)(T) - \left(\frac{L}{T^2}\right)(T^2)$$



- z_0: Initial distance at t = 0
- V_0: Initial velocity

Nondimensionalization

- **Nondimensionalization**: Removal of units from physical quantities by a suitable substitution of variables
- **Nondimensionalized equation**: Each term in an equation is dimensionless

E.g.) Displacement of a falling body
Let:

$$z^* = \frac{z}{z_0} \doteq \frac{\{L\}}{\{L\}} \qquad t^* = \frac{V_0 t}{z_0} \doteq \frac{\{LT^{-1}\}\{T\}}{\{L\}}$$

Substitute into the equation,

$$z^* z_0 = z_0 + V_0 \frac{t^* z_0}{V_0} - \frac{1}{2} g \left(\frac{t^* z_0}{V_0}\right)^2$$

Then, divide by z_0,

$$z^* = 1 + t^* + \frac{1}{2\alpha} t^{*2}, \qquad \alpha = \frac{V_0^2}{g z_0}$$

Dimensional vs. Non-dimensional Equation

- Dimensional equation

$$z = z_0 + V_0 t - \frac{1}{2} g t^2$$

or

$$F(z, z_0, V_0, g, t) = 0 \quad \Rightarrow \text{ 5 variables}$$

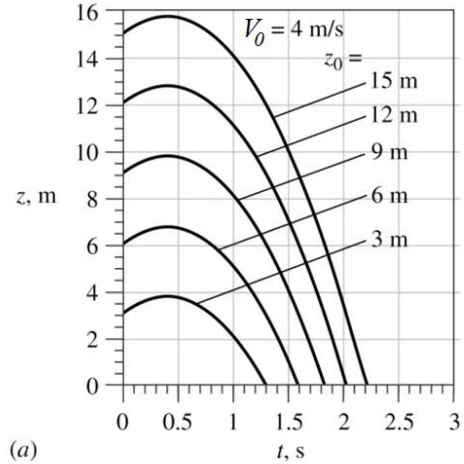
- Non-dimensional equation

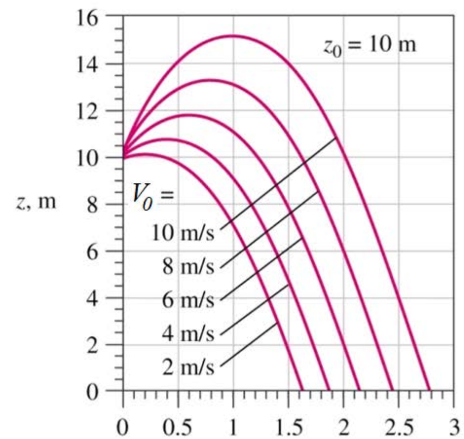
$$z^* = 1 + t^* - \frac{1}{2\alpha} t^{*2}$$

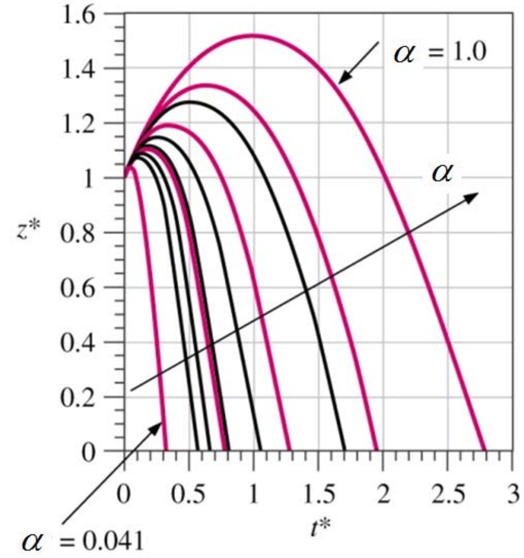
or

$$f(z^*, t^*, \alpha) = 0 \quad \Rightarrow \text{ 3 variables}$$

Advantages of Nondimensionalization







$$\alpha = \frac{V_0^2}{g z_0}$$

Dimensional: (a) V_0 fixed at 4 m/s and (b) z_0 fixed at 10 m

Non-dimensional: (a) and (b) are combined into one plot

Dimensional Analysis

- A process of formulating fluid mechanics problems in terms of non-dimensional variables and parameters

1. Reduction in variables
   $F = (A_1, A_2, \ldots, A_n) = 0,$ A_i = dimensional variables
   $f = (\Pi_1, \Pi_2, \ldots, \Pi_{r<n}) = 0,$ Π_i = non-dimensional parameters
2. Helps in understanding physics
3. Useful in data analysis and modeling
4. Fundamental to concepts of similarity and model testing

Buckingham Pi Theorem

- IF a physical process satisfies the PDH and involves $\boldsymbol{n}$ **dimensional variables**, it can be reduced to a relation between only $\boldsymbol{r}$ **dimensionless variables** or Π's.

- The **reduction,** $\boldsymbol{m = n - r}$, equals the maximum number of variables that do not form a pi among themselves and is always **less than or equal to the number of dimensions** describing the variables.

n = Number of dimensional variables

m = Minimum number of dimensions to describe the variables

r = $n - m$ = Number of non-dimensional variables

Methods for determining Π's

1. Functional relationship method
   a. Inspection (Intuition; Appendix A)
   b. Exponent method (Also called as the method of repeating variables)
   c. Step-by-step method (Appendix B)

2. Non-dimensionalize governing differential equations (GDE's) and initial (IC) and boundary (BC) conditions

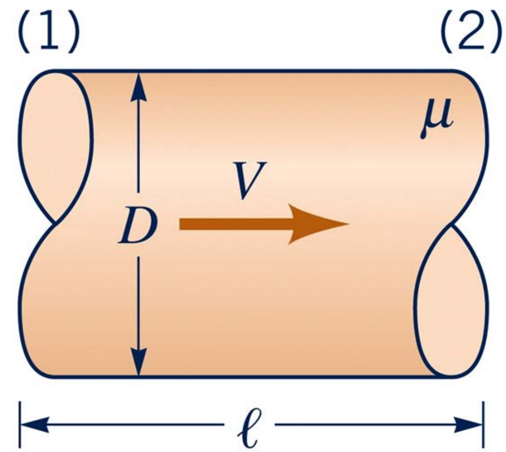
Exponent Method

(or Method of Repeating Variables)

- **Step 1**: List all the variables that are involved in the problem.
- **Step 2**: Express each of the variables in terms of basic dimensions.
- **Step 3**: Determine the required number of pi terms.
- **Step 4**: Select a number of repeating variables, where the number required is equal to the number of reference dimensions.
- **Step 5**: Form a pi term by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.
- **Step 6**: Repeat Step 5 for each of the remaining nonrepeating variables.
- **Step 7**: Check all the resulting pi terms to make sure they are dimensionless.
- **Step 8**: Express the final form as a relationship among the pi terms, and think about what it means.

Example 1

7.22 The pressure drop, Δp, along a straight pipe of diameter D has been experimentally studied, and it is observed that for laminar flow of a given fluid and pipe, the pressure drop varies directly with the distance, ℓ, between pressure taps. Assume that Δp is a function of D and ℓ, the velocity, V, and the fluid viscosity, μ. Use dimensional analysis to deduce how the pressure drop varies with pipe diameter.



$$\Delta p = p_1 - p_2$$

Steps 1 through 3

- Step 1: List all the variables that are involved in the problem.

$$\Delta p = f(D, \ell, V, \mu)$$

- Step 2: Express each of the variables in terms of basic dimensions.

Variable	Δp	D	ℓ	V	μ
Unit	N/m²	m	m	m/s	N·s/m²
Dimension	$\{ML^{-1}T^{-2}\}$	$\{L\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-1}T^{-1}\}$

- Step 3: Determine the required number of pi terms.

n = 5 for Δp, D, ℓ, V, and μ

m = 3 for M, L, T

$\therefore r = n - m = 5 - 3 = 2$ (i.e., 2 pi terms)

Step 4

- Select a number of repeating variables, where the number required is equal to the number of reference dimensions (for this example, m = 3).

- All of the required reference dimensions must be included within the group of repeating variables, and each repeating variable must be dimensionally independent of the others (The repeating variables cannot themselves be combined to form a dimensionless product).

- Do NOT choose the dependent variable as one of the repeating variables, since the repeating variables will generally appear in more than one pi term.

$\Rightarrow$ (D, V, μ) for (L, T, M), respectively

Step 5

- Combine D, V, μ with one additional variable (Δp or ℓ), in sequence, to find the two pi products

$$\Pi_1 = D^a V^b \mu^c \Delta p = (L)^a (LT^{-1})^b (ML^{-1}T^{-1})^c (ML^{-1}T^{-2})$$

$$= M^{(c+1)} L^{(a+b-c-1)} T^{-b-c-2} = M^0 L^0 T^0$$

Equate exponents:

Mass(M):	$c + 1 = 0$
Length(L):	$a + b - c - 1 = 0$
Time(T):	$-b - c - 2 = 0$

Solve for,

$$a = 1 \quad b = -1 \quad c = -1$$

Therefore,

$$\Pi_1 = DV^{-1}\mu^{-1}\Delta p = \frac{\Delta p D}{\mu V}$$

Step 6

- Repeat Step 5 for each of the remaining nonrepeating variables.

$$\Pi_2 = D^a V^b \mu^c \ell = (L)^a (LT^{-1})^b (ML^{-1}T^{-1})^c (L)$$

$$= M^c L^{a+b-c+1} T^{-b-c} = M^0 L^0 T^0$$

Equate exponents:

Mass(M): $c = 0$

Length(L): $a + b - c + 1 = 0$

Time(T): $-b - c = 0$

Solve for,

$$a = -1 \quad b = 0 \quad c = 0$$

Therefore,

$$\Pi_2 = D^{-1} V^0 \mu^0 \ell = \frac{\ell}{D}$$

Step 7

- Check all the resulting pi terms.
- One good way to do this is to express the variables in terms of F, L, T if the basic dimensions M, L, T were used initially, or vice versa.

Variable	Δp	D	ℓ	V	μ
Unit	N/m²	m	m	m/s	N·s/m²
Dimension	$\{FL^{-2}\}$	$\{L\}$	$\{L\}$	$\{LT^{-1}\}$	$\{FTL^{-2}\}$

$$\Pi_1 = \frac{\Delta p D}{\mu V} \doteq \frac{(FL^{-2})(L)}{(FTL^{-2})(LT^{-1})} \doteq F^0 L^0 T^0$$

$$\Pi_2 = \frac{\ell}{D} \doteq \frac{(L)}{(L)} \doteq F^0 L^0 T^0$$

Step 8

- Express the final form as a relationship among the pi terms.

$$\Pi_1 = f(\Pi_2)$$

or

$$\frac{\Delta p D}{\mu V} = f\left(\frac{\ell}{D}\right)$$

- Think about what it means.

  Since $\Delta p \propto \ell$,

$$\frac{\Delta p D}{\mu V} = C \cdot \frac{\ell}{D}$$

  where C is a constant. Thus,

$$\Delta p \propto \frac{1}{D^2}$$

Problems with One Pi Term

- The functional relationship that must exist for one pi term is

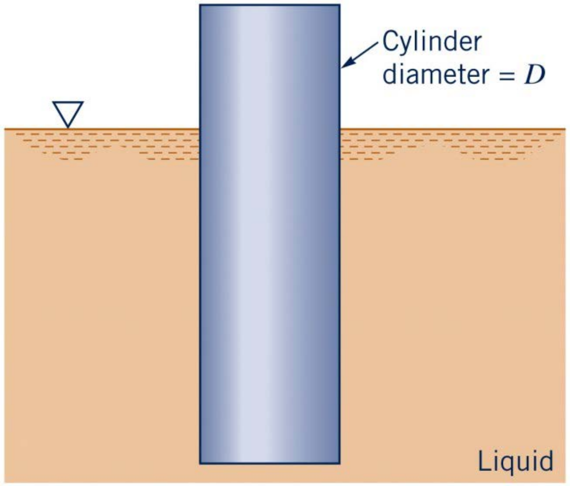
$$\Pi = C$$

where C is a constant.

- In other words, if only one pi term is involved in a problem, it must be equal to a constant.

Example 2

7.23 A cylinder with a diameter, D, floats upright in a liquid as shown in Fig. P7.23. When the cylinder is displaced slightly along its vertical axis it will oscillate about its equilibrium position with a frequency, ω. Assume that this frequency is a function of the diameter, D, the mass of the cylinder, m, and the specific weight, γ, of the liquid. Determine, with the aid of dimensional analysis, how the frequency is related to these variables. If the mass of the cylinder were increased, would the frequency increase or decrease?



$$\omega = f(D, m, \gamma)$$

Steps 1 through 4

Variable	ω	D	m	γ
Unit	1/s	m	kg	N/m^3
Dimension	$\{T^{-1}\}$	$\{L\}$	$\{M\}$	$\{ML^{-2}T^{-2}\}$

n = 4 for ω, D, m, and γ

m = 3 for M, L, T

$\therefore r = n - m = 4 - 3 = 1$ (i.e., 1 pi term)

m repeating variables = D, m, γ

Step 5 (and 6)

$$\Pi = D^a m^b \gamma^c \omega = (L)^a (M)^b (ML^{-2}T^{-2})^c (T^{-1})$$

$$= M^{(b+c)} L^{(a-2c)} T^{-2c-1} = M^0 L^0 T^0$$

Equate exponents:

Mass(M): $b + c = 0$

Length(L): $a - 2c = 0$

Time(T): $-2c - 1 = 0$

or,

$$a = -1 \quad b = \frac{1}{2} \quad c = -\frac{1}{2}$$

$$\therefore \Pi = D^{-1} m^{\frac{1}{2}} \gamma^{-\frac{1}{2}} \omega = \frac{\omega}{D}\sqrt{\frac{m}{\gamma}}$$

Steps 7 through 8

$$\Pi = \frac{\omega}{D}\sqrt{\frac{m}{\gamma}} \doteq \frac{(T^{-1})}{(L)}\sqrt{\frac{FL^{-1}T^2}{FL^{-3}}} \doteq F^0L^0T^0$$

- The dimensionless function is

$$\frac{\omega}{D}\sqrt{\frac{m}{\gamma}} = C$$

where C is a constant. Thus,

$$\omega = C \cdot D\sqrt{\frac{\gamma}{m}}$$

Therefore, if m is increased ω will decrease.

Example 3

7.9 The excess pressure inside a bubble (discussed in Chapter 1) is known to be dependent on bubble radius and surface tension. After finding the pi terms, determine the variation in excess pressure if we (a) double the radius and (b) double the surface tension.

$$\Delta p = f(R, \sigma)$$

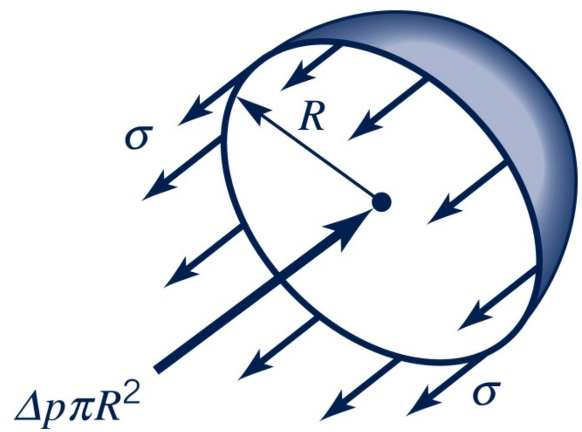


Figure 1.9

Example 3 –Contd.

Variable	Δp	R	σ
Unit	N/m²	m	N/m
Dimension	$\{ML^{-1}T^{-2}\}$	$\{L\}$	$\{MT^{-2}\}$

$m = 3$ for M, L, T

$\therefore r = n - m = 3 - 3 = 0$ $\Rightarrow$ No pi term?

Example 3 –Contd.

- Since the repeating variables form a pi among them:

$$\frac{\Delta p R}{\sigma} \doteq \frac{(ML^{-1}T^{-2})(L)}{(MT^{-2})} \doteq M^0 L^0 T^0$$

m should be reduced to 2 for MT^{-2} and L.

Example 3 –Contd.

Select R and σ as the repeating variables:

$$\Pi = R^a \sigma^b \Delta p = (L)^a (MT^{-2})^b (ML^{-1}T^{-2}) = M^0 L^0 T^0$$

Thus,

$$M: \quad b + 1 = 0$$

$$L: \quad a - 1 = 0$$

$$T: \quad -2b - 2 = 0$$

or,

$$a = 1 \text{ and } b = -1$$

Hence,

$$\Pi = \frac{\Delta p R}{\sigma}$$

Example 3 –Contd.

- Alternatively, by using the FLT system

Variable	Δp	R	σ
Unit	N/m^2	m	N/m
Dimension	$\{FL^{-2}\}$	$\{L\}$	$\{FL^{-1}\}$

$n = 3$ for Δp, R, and σ

$m = 2$ for F and L

$$\therefore r = 3 - 2 = 1 \;\Rightarrow\; 1 \text{ pi term}$$

Example 3 –Contd.

With the FLT system:

$$\Pi_1 = R^a \sigma^b \Delta p = (L)^a (FL^{-1})^b (FL^{-2}) = F^0 L^0 T^0$$

Thus,

$$F: \qquad 1 + b = 0$$

$$L: \qquad -2 + a - b = 0$$

or,

$$a = 1 \text{ and } b = -1$$

Hence,

$$\Pi = \frac{\Delta p R}{\sigma}$$

Common Dimensionless Parameters for Fluid Flow Problems

- Most common physical quantities of importance in fluid flow problems are (without heat transfer):

Length	Velocity	Density	Viscosity	Gravity	Surface tension	Compressibility	Pressure change
L	V	ρ	μ	g	σ	K	Δp
$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$	$\{LT^{-2}\}$	$\{MT^{-2}\}$	$\{ML^{-1}T^{-2}\}$	$\{ML^{-2}T^{-2}\}$

n = 8 variables

m = 3 dimension

$\therefore r = n - m$ = 5 pi terms (Re, Fr, We, Ma, C_p)

1) Reynolds number

$$Re = \frac{\rho V L}{\mu}$$

- Generally of importance in all types of fluid dynamics problems
- A measure of the ratio of the inertia force to the viscous force

$$\frac{\text{Inertia force}}{\text{Viscous force}} = \frac{ma}{\tau A} = \frac{(\rho L^3)\left(V \cdot \frac{V}{L}\right)}{\left(\mu \frac{V}{L}\right)(L^2)} = \frac{\rho V L}{\mu}$$

  – If $Re \ll 1$ (referred to as “creeping flow”), fluid density is less important
  – If Re is large, may neglect the effect of viscosity

- Re_{crit} distinguishes among flow regions: laminar or turbulent value varies depending upon flow situation

2) Froude number

$$Fr = \frac{V}{\sqrt{gL}}$$

- Important in problems involving flows with free surfaces
- A measure of the ratio of the inertia force to the gravity force (i.e., the weight of fluid)

$$\frac{\text{Inertia force}}{\text{Gravity force}} = \frac{ma}{\gamma \not{V}} = \frac{(\rho L^3)\left(V \cdot \frac{V}{L}\right)}{(\rho g)(L^3)} = \frac{V^2}{gL}$$

3) Weber number

$$We = \frac{\rho V^2 L}{\sigma}$$

- Problems in which there is an interface between two fluids where surface tension is important
- An index of the inertial force to the surface tension force

$$\frac{\text{Inertia force}}{\text{Surface tension force}} = \frac{ma}{\sigma L} = \frac{(\rho L^3)\left(V \cdot \frac{V}{L}\right)}{\sigma L} = \frac{\rho V^2 L}{\sigma}$$

- Important parameter at gas-liquid or liquid-liquid interfaces and when these surfaces are in contact with a boundary

4) Mach number

$$Ma = \frac{V}{\sqrt{k/\rho}} = \frac{V}{a}$$

- a: speed of sound in a fluid (a symbol c is also used)
- Problems in which the compressibility of the fluid is important
- An index of the ratio of inertial forces to compressibility forces

$$\frac{\text{Inertia force}}{\text{Compressibility force}} = \frac{ma}{\rho c^2 L^2} = \frac{(\rho L^3)\left(V \cdot \frac{V}{L}\right)}{\rho c^2 L^2} = \frac{V^2}{c^2}$$

(Note: Cauchy number, $Ca = V^2/c^2 = Ma^2$)

- Paramount importance in high speed flow ($V \geq c$)
- If Ma < 0.3, flow can be considered as incompressible

5) Pressure Coefficient

$$C_p = \frac{\Delta p}{\rho V^2}$$

- Problems in which pressure differences, or pressure, are of interest
- A measure of the ratio of pressure forces to inertial forces

$$\frac{\text{Pressure force}}{\text{Inertia force}} = \frac{\Delta p L^2}{ma} = \frac{\Delta p L^2}{(\rho L^3)\left(V \cdot \frac{V}{L}\right)} = \frac{\Delta p}{\rho V^2}$$

- Euler number:

$$Eu = \frac{p}{\rho V^2}$$

- Cavitation number:

$$Ca = \frac{p - p_v}{\frac{1}{2}\rho V^2}$$

Appendix A: Inspection Method

- Steps 1 through 3 of the exponent method are the same:

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Δp_ℓ	D	ρ	μ	V
$\{FL^{-3}\}$	$\{L\}$	$\{FL^{-4}T^2\}$	$\{FL^{-2}T\}$	LT^{-1}

$$r = n - m = 5 - 3 = 2$$

Appendix A: Inspection Method – Contd.

- Let Π_1 contain the dependent variable (Δp_ℓ in this example)
- Then, combine it with other variables so that a non-dimensional product will result:

To cancel F,

$$\frac{\Delta p_\ell}{\rho} \doteq \frac{(FL^{-3})}{(FL^{-4}T^2)} \doteq \frac{L}{T^2}$$

To cancel T,

$$\left(\frac{\Delta p_\ell}{\rho}\right)\frac{1}{V^2} \doteq \left(\frac{L}{T^2}\right)\frac{1}{(LT^{-1})^2} \doteq \frac{1}{L}$$

Then, to cancel L,

$$\left(\frac{\Delta p_\ell}{\rho V^2}\right)D \doteq \left(\frac{1}{L}\right)(L) \doteq L^0$$

$$\therefore \Pi_1 = \frac{\Delta p_\ell D}{\rho V^2}$$

Appendix A: Inspection Method – Contd.

- Select the variable that was not used in Π_1, which in this case μ, and repeat the process:

To cancel F,

$$\frac{\mu}{\rho} \doteq \frac{(FL^{-2}T)}{(FL^{-4}T^2)} \doteq \frac{L^2}{T}$$

To cancel T,

$$\left(\frac{\mu}{\rho}\right)\frac{1}{V} \doteq \left(\frac{L^2}{T}\right)\frac{1}{(LT^{-1})} \doteq L$$

Then, to cancel L,

$$\left(\frac{\mu}{\rho V}\right)\frac{1}{D} \doteq (L)\left(\frac{1}{L}\right) \doteq L^0$$

$$\therefore \Pi_2 = \frac{\mu}{\rho V D}$$

Appendix B: Step-by-step Method*

The pi theorem method, just explained and illustrated, is often called the *repeating variable method* of dimensional analysis. Select the repeating variables, add one more, and you get a pi group. The writer likes it. This method is straightforward and systematically reveals all the desired pi groups. However, there are drawbacks: (1) All pi groups contain the same repeating variables and might lack variety or effectiveness, and (2) one must (sometimes laboriously) check that the selected repeating variables do *not* form a pi group among themselves (see Prob. P5.21).

Ipsen [5] suggests an entirely different procedure, a step-by-step method that obtains all of the pi groups at once, without any counting or checking. One simply successively eliminates each dimension in the desired function by division or multiplication. Let us illustrate with the same classical drag function proposed in Eq. (5.1). Underneath the variables, write out the dimensions of each quantity.

$$\begin{array}{cccccc} F & = \text{fcn}(L, & V, & \rho, & \mu) \\ \{MLT^{-2}\} & \{L\} & \{LT^{-1}\} & \{ML^{-3}\} & \{ML^{-1}T^{-1}\} \end{array} \tag{5.1}$$

There are three dimensions, $\{MLT\}$. Eliminate them successively by division or multiplication by a variable. Start with mass $\{M\}$. Pick a variable that contains mass and divide it into all the other variables with mass dimensions. We select ρ, divide, and rewrite the function (5.1):

$$\begin{array}{cccccc} \dfrac{F}{\rho} & = \text{fcn}\Big(L, & V, & \cancel{\rho}, & \dfrac{\mu}{\rho}\Big) \\ \{L^4T^{-2}\} & \{L\} & \{LT^{-1}\} & \{ML^{-3}\} & \{L^2T^{-1}\} \end{array} \tag{5.1a}$$

*by Ipsen (1960). The pi theorem and Ipsen method are quite different. Both are useful and interesting.

Appendix B: Step-by-step Method – Contd.

We did not divide into L or V, which do not contain $\{M\}$. Equation (5.1a) at first looks strange, but it contains five distinct variables and the same information as Eq. (5.1).

We see that ρ is no longer important because no other variable contains $\{M\}$. Thus *discard* ρ, and now there are only four variables. Next, eliminate time $\{T\}$ by dividing the time-containing variables by suitable powers of, say, V. The result is

$$\underset{\{L^2\}}{\frac{F}{\rho V^2}} = \mathrm{fcn}\left(\underset{\{L\}}{L}, \quad \underset{\{LT^{-1}\}}{\cancel{V},} \quad \underset{\{L\}}{\frac{\mu}{\rho V}}\right) \tag{5.1b}$$

Now we see that V is no longer relevant since only V contains time $\{T\}$. Finally, eliminate $\{L\}$ through division by, say, appropriate powers of L itself:

$$\underset{\{1\}}{\frac{F}{\rho V^2 L^2}} = \mathrm{fcn}\left(\underset{\{L\}}{\cancel{L},} \quad \underset{\{1\}}{\frac{\mu}{\rho V L}}\right) \tag{5.1c}$$

Now L by itself is no longer relevant and so discard it also. The result is equivalent to Eq. (5.2):

$$\frac{F}{\rho V^2 L^2} = \mathrm{fcn}\left(\frac{\mu}{\rho V L}\right) \tag{5.2}$$

In Ipsen's step-by-step method, we find the force coefficient is a function solely of the Reynolds number. We did no counting and did not find j. We just successively eliminated each primary dimension by division with the appropriate variables.

Appendix B: Step-by-step Method – Contd.

Recall Example 5.5, where we discovered, awkwardly, that the number of repeating variables was *less* than the number of primary dimensions. Ipsen's method avoids this preliminary check. Recall the beam-deflection problem proposed in Example 5.5 and the various dimensions:

$$\underset{\{L\}}{\delta} = f(\underset{\{MLT^{-2}\}}{P}, \quad \underset{\{L\}}{L}, \quad \underset{\{L^4\}}{I}, \quad \underset{\{ML^{-1}T^{-2}\}}{E})$$

For the first step, let us eliminate $\{M\}$ by dividing by E. We only have to divide into P:

$$\underset{\{L\}}{\delta} = f\left(\underset{\{L^2\}}{\frac{P}{E}}, \quad \underset{\{L\}}{L}, \quad \underset{\{L^4\}}{I}, \quad \underset{\{ML^{-1}T^{-2}\}}{\cancel{E}}\right)$$

We see that we may discard E as no longer relevant, and the dimension $\{T\}$ has vanished along with $\{M\}$. We need only eliminate $\{L\}$ by dividing by, say, powers of L itself:

$$\underset{\{1\}}{\frac{\delta}{L}} = \text{fcn}\left(\underset{\{1\}}{\frac{P}{EL^2}}, \quad \underset{\{L\}}{\cancel{L}}, \quad \underset{\{1\}}{\frac{I}{L^4}}\right)$$

Discard L itself as now irrelevant, and we obtain *Answer* (1) to Example 5.5:

$$\frac{\delta}{L} = \text{fcn}\left(\frac{P}{EL^2}, \quad \frac{I}{L^4}\right)$$

Ipsen's approach is again successful. The fact that $\{M\}$ and $\{T\}$ vanished in the same division is proof that there are only *two* repeating variables this time, not the three that would be inferred by the presence of $\{M\}$, $\{L\}$, and $\{T\}$.