# Control Volume and Reynolds Transport Theorem

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### Reynolds Transport Theorem (RTT)

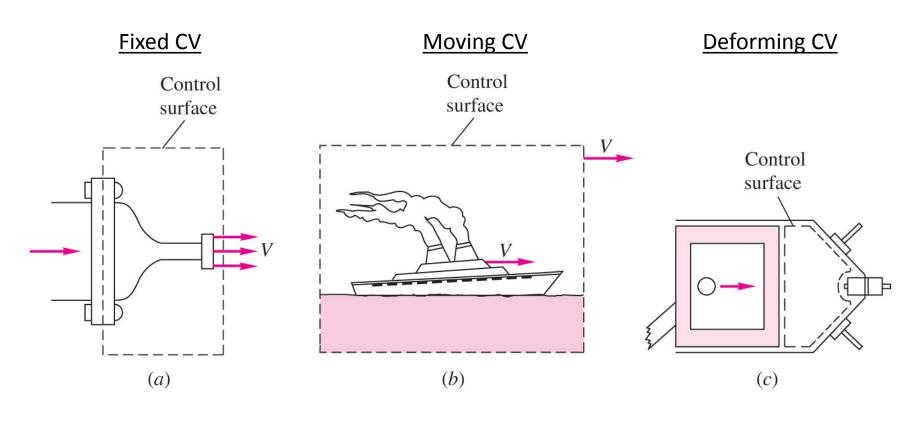
 An analytical tool to shift from describing the laws governing fluid motion using the system concept to using the control volume concept

### System vs. Control Volume

- System: A collection of matter of fixed identity
  - Always the same atoms or fluid particles
  - A specific, identifiable quantity of matter

- Control Volume (CV): A volume in space through which fluid may flow
  - A geometric entity
  - Independent of mass

### **Examples of CV**



CV fixed at a nozzle

CV moving with ship

CV deforming within cylinder

### Laws of Mechanics

1. Conservation of mass:

$$\frac{dm}{dt} = 0$$

2. Conservation of linear momentum:

$$\underline{F} = m\underline{a} = m\frac{d\underline{V}}{dt} = \frac{d}{dt}(m\underline{V})$$

3. Conservation of angular momentum:

$$\underline{M} = \frac{d\underline{H}}{dt}$$

4. Conservation of Energy:

$$\frac{dE}{dt} = \dot{Q} - \dot{W}$$

- The laws apply to either solid or fluid systems
- Ideal for solid mechanics, where we follow the same system
- For fluids, the laws need to be rewritten to apply to a specific region in the neighborhood of our product (i.e., CV)

### Extensive vs. Intensive Property

Governing Differential Equations (GDE's):

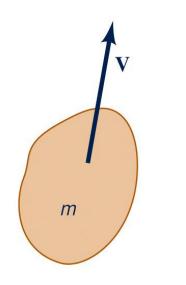
$$\frac{d}{dt}\left(\underbrace{m, m\underline{V}, E}_{B}\right) = \left(0, \underline{F}, \dot{Q} - \dot{W}\right)$$

- B = The amount of m, mV, or E contained in the total mass of a system or a CV; Extensive property – Dependent on mass
- $\beta$  (or b) = The amount of B per unit mass; Intensive property Independent on mass

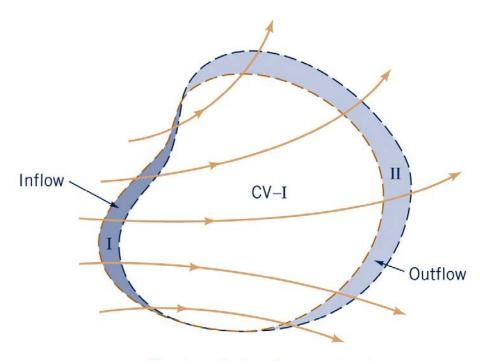
$$\beta$$
 (or  $b$ ) =  $B/m$  (=  $\frac{dB}{dm}$  for nonuniform  $B$ )

$$B = \beta \cdot m \quad \left( = \int_{V} \beta \underbrace{\rho dV}_{=dm} \text{ for nonuniform } \beta \right)$$

В	b = B/m		
m	1		
mV	V		
E	e		



### Fixed CV



- Fixed control surface and system boundary at time t
- --- System boundary at time  $t + \delta t$

At time t: SYS = CV

$$B_{sys}(t) = B_{CV}(t)$$

At time  $t + \delta t$ : SYS = (CV – I) + II

$$B_{SyS}(t + \delta t)$$

$$= B_{CV}(t + \delta t) - B_I(t + \delta t)$$

$$+ B_{II}(t + \delta t)$$

# Time Rate of Change of $B_{sys}$

$$\frac{\delta B_{sys}}{\delta t} = \frac{B_{sys}(t + \delta t) - B_{sys}(t)}{\delta t}$$

$$= \frac{\{B_{CV}(t + \delta t) - B_{I}(t + \delta t) + B_{II}(t + \delta t)\} - B_{CV}(t)}{\delta t}$$

$$\therefore \frac{\delta B_{sys}}{\delta t} = \underbrace{\frac{B_{CV}(t + \delta t) - B_{CV}(t)}{\delta t}}_{1) \text{ Change of } B} + \underbrace{\frac{B_{II}(t + \delta t)}{\delta t}}_{2) \text{ Amount of } B} - \underbrace{\frac{B_{I}(t + \delta t)}{\delta t}}_{3) \text{ Amountt of } B}$$
within CV over  $\delta t$  flowing out through CS over  $\delta t$  over  $\delta t$  over  $\delta t$  over  $\delta t$ 

Now, take limit of  $\delta t \rightarrow 0$  to Eq. (1) term by term

### LHS of Eq. (1)

$$\lim_{\delta t \to 0} \frac{\delta B_{sys}}{\delta t} = \lim_{\delta t \to 0} \frac{B_{sys}(t + \delta t) - B_{sys}(t)}{\delta t} = \underbrace{\frac{dB_{sys}}{dt}}_{\begin{subarray}{c} \text{Time rate of change of } B\\ \end{subarray}}_{\begin{subarray}{c} \text{within the system} \end{subarray}}$$

$$\left(\text{or,} = \frac{DB_{sys}}{Dt}; \text{ material derivative}\right)$$

### First term of RHS of Eq.(1)

$$\lim_{\delta t \to 0} \frac{B_{CV}(t+\delta t) - B_{CV}(t)}{\delta t} = \underbrace{\frac{dB_{CV}}{dt} = \frac{d}{dt} \int_{CV} \beta \rho dV}_{\text{Time rate of change of}}$$

$$\underbrace{B \text{ withich CV}}$$

# 2<sup>nd</sup> term of RHS of Eq.(1)

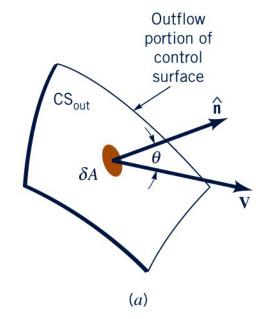
$$\delta m_{out} = \rho \delta \Psi$$

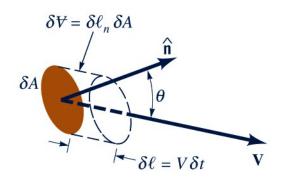
and

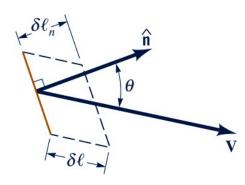
$$\delta \Psi = \delta A \cdot \delta \ell_n = \delta A \cdot \left(\underbrace{\delta \ell}_{=V\delta t} \cos \theta\right) = \delta A \cdot (V\delta t \cos \theta)$$

Thus, the amount of B flowing out of CV through  $\delta A$  over a short time  $\delta t$ :

$$\therefore \delta B_{out} = \beta \delta m_{out} = \beta \rho V \cos \theta \, \delta t \delta A$$







# 2<sup>nd</sup> term of RHS of Eq.(1) – Contd.

By integrating  $\delta B_{out}$  over the entire outflow portion of CS,

$$B_{II}(t + \delta t) = \int_{CS_{out}} dB_{out} = \int_{CS_{out}} \beta \rho V \cos \theta \, \delta t dA$$

Thus,

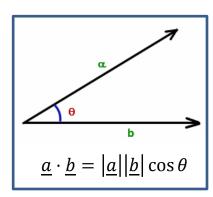
$$\lim_{\delta t \to 0} \frac{B_{II}(t + \delta t)}{\delta t} = \lim_{\delta t \to 0} \frac{1}{\delta t} \left( \delta t \int_{CS_{out}} \beta \rho V \cos \theta \, dA \right)$$

$$= \int_{CS_{out}} \beta \rho \underbrace{V \cos \theta}_{=V_{out}} dA \qquad (\equiv \dot{B}_{out})$$

i.e., Out flux of  $\boldsymbol{B}$  through CS

Note that  $V \cos \theta = \underline{V} \cdot \widehat{\boldsymbol{n}}$ ,

$$\therefore \dot{B}_{\text{out}} = \int_{CS_{out}} \beta \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA$$



### 3<sup>rd</sup> term of RHS of Eq.(1)

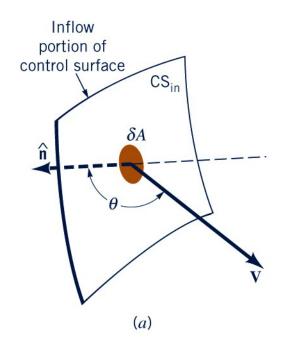
$$\delta m_{in} = \rho \delta \Psi$$

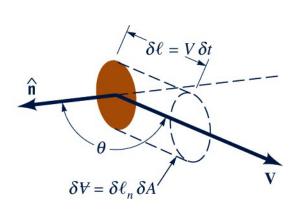
and

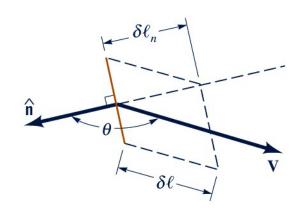
$$\delta \Psi = \delta A \cdot \delta \ell_n = \delta A \cdot \left(\underbrace{\delta \ell}_{=V\delta t} \left( -\underbrace{\cos \theta}_{<0} \right) \right) = \delta A \cdot \left( -V\delta t \cos \theta \right)$$

Thus, the amount of B flowing out of CV through  $\delta A$  over a short time  $\delta t$ :

$$\therefore \delta B_{in} = \beta \delta m_{in} = -\beta \rho V \cos \theta \, \delta t \delta A$$







(b)

(c)

# 3<sup>rd</sup> term of RHS of Eq.(1) – Contd.

By integrating  $\delta B_{out}$  over the entire outflow portion of CS,

$$B_{I}(t + \delta t) = \int_{CS_{in}} dB_{in} = \int_{CS_{in}} (-\beta \rho V \cos \theta) \delta t dA$$

Thus,

$$\lim_{\delta t \to 0} \frac{B_I(t + \delta t)}{\delta t} = \lim_{\delta t \to 0} \frac{1}{\delta t} \left( \delta t \int_{CS_{in}} (-\beta \rho V \cos \theta) dA \right)$$

$$= -\int_{CS_{in}} \beta \rho \underbrace{V \cos \theta}_{=-V_n} dA \quad (\equiv \dot{B}_{in})$$
i.e., influx of  $B$  through CS

Note that  $V \cos \theta = \underline{V} \cdot \widehat{\boldsymbol{n}}$ ,

$$\therefore \dot{B}_{\rm in} = -\int_{CS_{in}} \beta \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA$$

### RTT for Fixed CV

Now the relationship between the time rate of change of B for the system and that for the CV is given by,

$$\frac{DB_{sys}}{Dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \underbrace{\int_{CS_{out}} \beta \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA}_{\dot{B}_{out}} - \underbrace{\left(-\int_{CS_{in}} \beta \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA\right)}_{\dot{B}_{in}}$$

With the fact that  $CS = CS_{out} + CS_{in}$ ,

$$\frac{DB_{sys}}{Dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \int_{CS} \beta \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA$$

Time rate of thange of 
$$B$$
 = change of  $B$  + Net flux of  $B$  through CS =  $\dot{B}_{out} - \dot{B}_{in}$ 

### Example 1

4.72 Water flows through the 2-m-wide rectangular channel shown in Fig. P4.72 with a uniform velocity of 3 m/s. (a) Directly integrate Eq. 4.16 with b = 1 to determine the mass flowrate (kg/s) across section CD of the control volume. (b) Repeat part (a) with  $b = 1/\rho$ , where  $\rho$  is the density. Explain the physical interpretation of the answer to part (b).

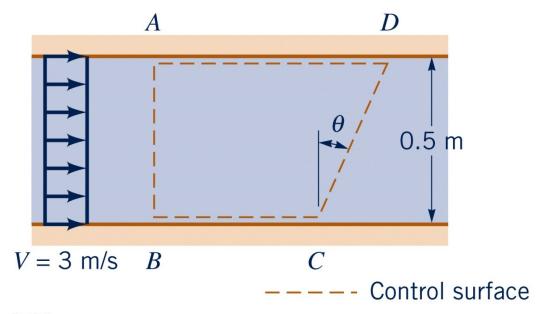


Figure P4.72 © John Wiley & Sons, Inc. All rights reserved.

### Example 1 - Contd.

$$\dot{B}_{out} = \int_{CS_{out}} \rho \beta \mathbf{V} \cdot \hat{\mathbf{n}} dA \quad (4.16)$$

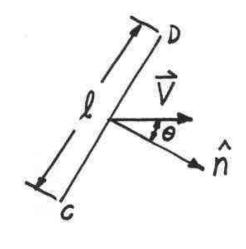
With  $\beta$ = 1 and  ${m V}\cdot {m {\widehat n}} = V\cos heta$  ,

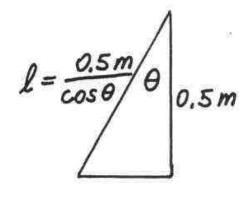
$$\dot{B}_{out} = \int_{CD} \rho V \cos \theta \, dA = \rho V \cos \theta \underbrace{\int_{CD} dA}_{=A_{CD}}$$
$$= \rho V \cos \theta \, A_{CD}$$

where

$$A_{CD} = \ell \times (2 m)$$

$$= \left(\frac{0.5 m}{\cos \theta}\right) (2 m) = \left(\frac{1}{\cos \theta}\right) m^2$$





### Example 1 - Contd.

Thus, with  $\rho$  = 1,000 kg/m<sup>3</sup> for water and V = 3 m/s,

$$\dot{B}_{out} = \left(1,000 \, \frac{\text{kg}}{\text{m}^3}\right) \left(3 \, \frac{\text{m}}{\text{s}}\right) \cos \theta \left(\frac{1}{\cos \theta} \, \text{m}^2\right) = 3,000 \, \text{kg/s}$$

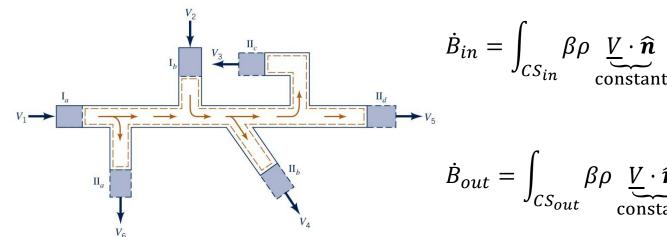
With  $\beta = 1/\rho$ ,

$$\dot{B}_{out} = \int_{CD} V \cos \theta \, dA = V \cos \theta \int_{\underbrace{CD}} dA = V \cos \theta \, A_{CD}$$

$$= \left(3 \frac{m}{s}\right) \cos \theta \left(\frac{1}{\cos \theta} \, m^2\right) = 3 \, m^3/s \quad (i.e., \text{volume flow rate})$$

Note: These results are the same for all  $\theta$  values

### **Special Case:** V = constant over discrete CS's

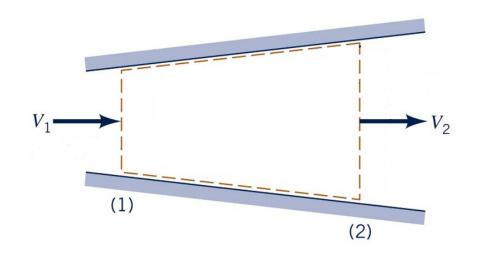


$$\dot{B}_{in} = \int_{CS_{in}} \beta \rho \underbrace{V \cdot \hat{n}}_{\text{constant}} dA = \sum_{i} (\beta_{i} \rho_{i} V_{i} A_{i})_{in}$$

$$\dot{B}_{out} = \int_{CS_{out}} \beta \rho \underbrace{V \cdot \hat{n}}_{constant} dA = \sum_{j} (\beta_{j} \rho_{j} V_{j} A_{j})_{out}$$

$$\therefore \frac{DB_{sys}}{Dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \sum_{j} \left( \beta_{j} \underbrace{\rho_{j} V_{j} A_{j}}_{\dot{m}_{j}} \right)_{out} - \sum_{i} \left( \beta_{i} \underbrace{\rho_{i} V_{i} A_{i}}_{\dot{m}_{i}} \right)_{in}$$

### Example 2



#### Given:

- Water flow ( $\rho$  = constant)
- $D_1 = 10$  cm;  $D_2 = 15$  cm
- $V_1 = 10 \text{ cm/s}$
- Steady flow

Find:  $V_2 = ?$ 

#### Mass conservation:

- $DB_{SYS}/Dt = 0$
- $\beta = 1$
- $\bullet \quad \rho_1 = \rho_2 = \rho$

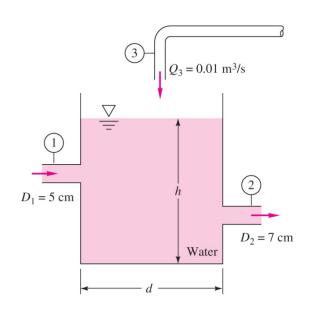
#### Steady flow

$$0 = \frac{d}{dt} \int_{CV} \rho dV + (\rho_2 V_2 A_2) - (\rho_1 V_1 A_1)$$

or, 
$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

$$\therefore V_2 = \frac{\rho_1 A_1}{\rho_2 A_2} V_1 = \left(\frac{\rho}{\rho}\right) \left(\frac{D_1}{D_2}\right)^2 V_1 = (1) \left(\frac{10 \text{ cm}}{15 \text{ cm}}\right)^2 \left(10 \frac{\text{cm}}{\text{s}}\right) = 4.4 \text{ cm/s}$$

### Example 3



#### Given:

- $D_1 = 5$  cm;  $D_2 = 7$  cm
- $V_1 = 3 \text{ m/s}$
- $Q_3 = V_3 A_3 = 0.01 \text{ m}^3/\text{s}$
- *h* = constant (i.e., steady flow)
- $\rho_1 = \rho_2 = \rho_3 = \rho_{water}$

Find:  $V_2 = ?$ 

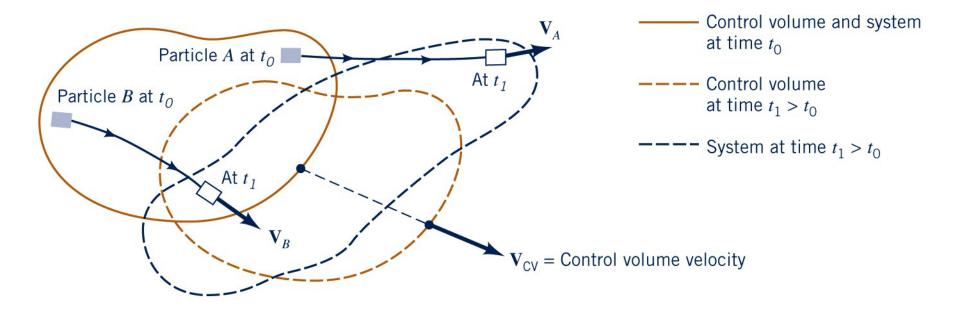
= 0; steady flow

$$0 = \frac{d}{dt} \int_{CV} \rho dV + (\rho_2 V_2 A_2) - (\rho_1 V_1 A_1) - (\rho_3 V_3 A_3)$$

or, 
$$V_2 A_2 = V_1 A_1 + \underbrace{V_3 A_3}_{=Q_3}$$

$$\therefore V_2 = \frac{V_1 A_1 + Q_3}{A_2} = \frac{(3)(\pi)(0.05)^2 / 4 + (0.01)}{(\pi)(0.07)^2 / 4} = 4.13 \text{ m/s}$$

### Moving CV



### RTT for Moving CV

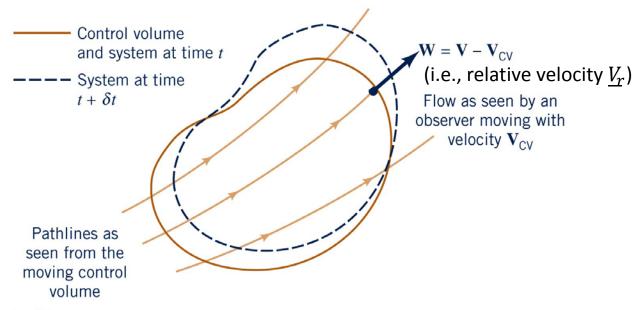


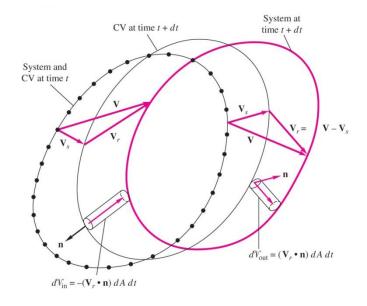
Figure 4.23

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$$\frac{DB_{sys}}{Dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \int_{CS} \beta \rho \underline{V}_r \cdot \widehat{\boldsymbol{n}} dA$$

### RTT for Moving and Deforming CV

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \int_{CS} \beta \rho (\mathbf{V_r} \cdot \widehat{\mathbf{n}}) dA$$



Both CV and CS change their shape and location with time

$$V_r = V(x,t) - V_S(x,t)$$

- $V_S(x,t)$ : Velocity of CS
- V(x, t): Fluid velocity in the coordinate system in which the  $V_s$  is observed
- V<sub>r</sub>: Relative velocity of fluid seen by an observer riding on the CV

\*Ref) Fluid Mechanics by Frank M. White, McGraw Hill

### RTT Summary (1)

General RTT (for moving and deforming CV):

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho dV \right) + \int_{CS} \beta \rho \underline{V_r} \cdot \widehat{\boldsymbol{n}} dA$$

**Special Cases:** 

1) Non-deforming (but moving) CV

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t} (\beta \rho) dV + \int_{CS} \beta \rho \underline{V_r} \cdot \hat{\boldsymbol{n}} dA$$

2) Fixed CV

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t} (\beta \rho) d\Psi + \int_{CS} \beta \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA$$

3) Steady flow:

$$\frac{\partial}{\partial t} = 0$$

4) Flux terms for uniform flow across discrete CS's (steady or unsteady)

$$\int_{CS} \beta \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA = \sum (\beta \dot{\boldsymbol{m}})_{out} - \sum (\beta \dot{\boldsymbol{m}})_{in}$$

### RTT Summary (2)

#### For fixed CV's:

Parameter (B)	$\beta = B/m$	RTT	Remark
Mass (m)	1	$0 = \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA$	Continuity eq. (Ch. 5.1)
Momentum (m <u>V</u> )	<u>V</u>	$\sum \underline{F} = \frac{d}{dt} \int_{CV} \underline{V} \rho d\underline{V} + \int_{CS} \underline{V} \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA$	Linear momentum eq. (Ch. 5.2)
Energy (E)	е	$\dot{Q} - \dot{W} = \frac{d}{dt} \int_{CV} e\rho d\Psi + \int_{CS} e\rho \underline{V} \cdot \hat{\boldsymbol{n}} dA$	Energy eq. (Ch. 5.3)

### Continuity Equation (Ch. 5.1)

RTT with B = mass and  $\beta = 1$ ,

$$\underbrace{0 = \frac{DM_{Sys}}{Dt}}_{\text{mass conservation}} = \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA$$

or

$$\underbrace{\int_{CS} \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA}_{\text{Net rate of outflow of mass across CS}} = \underbrace{-\frac{d}{dt} \int_{CV} \rho d\underline{V}}_{\text{Rate of decrease of mass within CV}}$$

Note: Incompressible fluid ( $\rho$  = constant)

$$\int_{CS} \underline{V} \cdot \widehat{\boldsymbol{n}} dA = -\frac{d}{dt} \int_{CV} dV \qquad \text{(Conservation of volume)}$$

### Simplifications

1. Steady flow

$$\int_{CS} \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA = 0$$

2. If  $\underline{V}$  = constant over discrete CS's (i.e., one-dimensional flow)

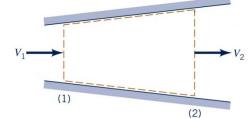
$$\int_{CS} \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA = \sum_{out} \rho VA - \sum_{in} \rho VA$$

3. Steady one-dimensional flow in a conduit

$$(\rho VA)_{out} - (\rho VA)_{in} = 0$$

or

$$\rho_2 V_2 A_2 - \rho_1 V_1 A_1 = 0$$



For  $\rho$  = constant

$$V_1 A_1 = V_2 A_2$$
 (or  $Q_1 = Q_2$ )

### Some useful definitions

Mass flux

$$\dot{m} = \int_{A} \rho \underline{V} \cdot \underline{dA}$$

(Note:  $\underline{V} \cdot \underline{dA} = \underline{V} \cdot \widehat{\boldsymbol{n}} dA$ )

Volume flux

$$Q = \int_A \ \underline{V} \cdot \underline{dA}$$

Average velocity

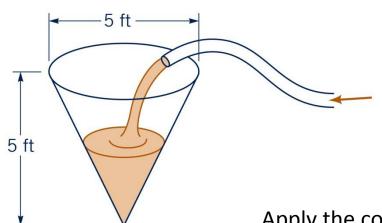
$$\bar{A} = \frac{Q}{A} = \frac{1}{A} \int_{A} \underline{V} \cdot \underline{dA}$$

Average density

$$\bar{\rho} = \frac{1}{A} \int_{A} \rho dA$$

Note:  $\dot{m} \neq \bar{\rho}Q$  unless  $\rho$  = constant

### Example 4



Estimate the time required to fill with water a cone-shaped container 5 ft hight and 5 ft across at the top if the filling rate is 20 gal/min.

Apply the conservation of mass ( $\beta = 1$ )

$$0 = \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho \underline{V} \cdot \widehat{\boldsymbol{n}} dA$$

For incompressible fluid (i.e.,  $\rho$  = constant) and one inlet,

$$0 = \frac{d}{dt} \underbrace{\int_{CV} dV - \underbrace{(VA)_{in}}_{=Q_{in}}}$$

### Example 4 – Contd.

Volume of the cone at time t,

$$V(t) = \frac{\pi D^2}{12} h(t)$$

Flow rate at the inlet,

$$Q = \left(20 \frac{\text{gal}}{\text{min}}\right) \left(231 \frac{\text{in}^3}{\text{gal}}\right) / \left(1,728 \frac{\text{in}^3}{\text{ft}^3}\right) = 2.674 \text{ ft}^3/\text{min}$$

The continuity eq. becomes

$$\frac{\pi D^2}{12} \cdot \frac{dh}{dt} = Q \qquad \text{or} \qquad \frac{dh}{dt} = \frac{12Q}{\pi D^2}$$

### Example 4 – Contd.

Solve for h(t),

$$h(t) = \int_0^t \frac{12Q}{\pi D^2} dt = \frac{12Q \cdot t}{\pi D^2}$$

Thus, the time for h = 5 ft is

$$t = \frac{\pi D^2 h}{12Q} = \frac{\pi (5 \text{ ft})^2 (5 \text{ ft})}{(12)(2.674 \text{ ft}^3/\text{min})} = 12.2 \text{ min}$$