Control Volume and Reynolds Transport Theorem

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Reynolds Transport Theorem (RTT)

• An analytical tool to shift from describing the *laws governing fluid motion* using the system concept to using the control volume concept
System vs. Control Volume

• **System**: A collection of matter of fixed identity
  – Always the same atoms or fluid particles
  – A specific, identifiable quantity of matter

• **Control Volume (CV)**: A volume in space through which fluid may flow
  – A geometric entity
  – Independent of mass
Examples of CV

Fixed CV
Control surface

Moving CV
Control surface

Deforming CV
Control surface

CV fixed at a nozzle
CV moving with ship
CV deforming within cylinder
Laws of Mechanics

1. Conservation of mass:
   \[ \frac{dm}{dt} = 0 \]

2. Conservation of linear momentum:
   \[ F = ma = m \frac{dV}{dt} = \frac{d}{dt} (mV) \]

3. Conservation of angular momentum:
   \[ M = \frac{dH}{dt} \]

4. Conservation of Energy:
   \[ \frac{dE}{dt} = \dot{Q} - \dot{W} \]

• The laws apply to either solid or fluid systems
• Ideal for solid mechanics, where we follow the same system
• For fluids, the laws need to be rewritten to apply to a specific region in the neighborhood of our product (i.e., CV)
Extensive vs. Intensive Property

Governing Differential Equations (GDE's):
\[ \frac{d}{dt} \left( \frac{m, mV, E}{B} \right) = (0, F, \dot{Q} - \dot{W}) \]

- \( B \) = The amount of \( m, mV, \) or \( E \) contained in the total mass of a system or a CV; Extensive property – Dependent on mass

- \( \beta \) (or \( b \)) = The amount of \( B \) per unit mass; Intensive property – Independent on mass

\[ \beta \ (\text{or} \ b) = \frac{B}{m} \ (= \frac{dB}{dm} \text{ for nonuniform } B) \]

\[ B = \beta \cdot m \quad \left(= \int_V \beta \frac{\rho dV}{dm} \text{ for nonuniform } \beta \right) \]
Fixed CV

At time $t$: $SYS = CV$

$$B_{sys}(t) = B_{CV}(t)$$

At time $t + \delta t$: $SYS = (CV - I) + II$

$$B_{sys}(t + \delta t) = B_{CV}(t + \delta t) - B_I(t + \delta t) + B_{II}(t + \delta t)$$
Time Rate of Change of $B_{sys}$

$$\frac{\delta B_{sys}}{\delta t} = \frac{B_{sys}(t + \delta t) - B_{sys}(t)}{\delta t}$$

$$= \frac{\{B_{CV}(t + \delta t) - B_I(t + \delta t) + B_{II}(t + \delta t)\} - B_{CV}(t)}{\delta t}$$

$$\therefore \frac{\delta B_{sys}}{\delta t} = \frac{B_{CV}(t + \delta t) - B_{CV}(t)}{\delta t} \quad 1) \text{Change of } B \text{ within CV over } \delta t$$

$$+ \frac{B_{II}(t + \delta t)}{\delta t} \quad 2) \text{Amount of } B \text{ flowing out through CS over } \delta t$$

$$- \frac{B_I(t + \delta t)}{\delta t} \quad 3) \text{Amount of } B \text{ flowing in through CS over } \delta t$$

Now, take limit of $\delta t \rightarrow 0$ to Eq. (1) term by term
LHS of Eq. (1)

\[
\lim_{\delta t \to 0} \frac{\delta B_{sys}}{\delta t} = \lim_{\delta t \to 0} \frac{B_{sys}(t + \delta t) - B_{sys}(t)}{\delta t} = \frac{dB_{sys}}{dt}
\]

Time rate of change of \( B \) within the system

\[
\left( \text{or, } = \frac{DB_{sys}}{Dt}; \text{ material derivative} \right)
\]
First term of RHS of Eq. (1)

\[
\lim_{\delta t \to 0} \frac{B_{CV}(t + \delta t) - B_{CV}(t)}{\delta t} = \frac{dB_{CV}}{dt} = \frac{d}{dt} \int_{CV} \beta \rho dV
\]

\underbrace{\text{Time rate of change of } B \text{ within CV}}

\text{Time rate of change of } B \text{ within CV}
2\textsuperscript{nd} term of RHS of Eq.(1)

\[ \delta m_{out} = \rho \delta V \]

and

\[ \delta V = \delta A \cdot \delta \ell_n = \delta A \cdot \left( \frac{\delta \ell}{\delta t} \cos \theta \right) = \delta A \cdot (V \delta t \cos \theta) \]

Thus, the amount of \( B \) flowing out of \( CV \) through \( \delta A \) over a short time \( \delta t \):

\[ \therefore \delta B_{out} = \beta \delta m_{out} = \beta \rho V \cos \theta \delta t \delta A \]
2\textsuperscript{nd} term of RHS of Eq.(1) – Contd.

By integrating $\delta B_{out}$ over the entire outflow portion of CS,

$$B_{II}(t + \delta t) = \int_{CS_{out}} dB_{out} = \int_{CS_{out}} \beta \rho V \cos \theta \delta t dA$$

Thus,

$$\lim_{\delta t \to 0} \frac{B_{II}(t + \delta t)}{\delta t} = \lim_{\delta t \to 0} \frac{1}{\delta t} \left( \delta t \int_{CS_{out}} \beta \rho V \cos \theta dA \right)$$

$$= \int_{CS_{out}} \beta \rho \frac{V \cos \theta}{V_n} dA \quad (\equiv \dot{B}_{out})$$

i.e., Out flux of $B$ through CS

Note that $V \cos \theta = V \cdot \hat{n}$,

$$\therefore \dot{B}_{out} = \int_{CS_{out}} \beta \rho V \cdot \hat{n} dA$$
3rd term of RHS of Eq.(1)

\[ \delta m_{in} = \rho \delta \mathcal{V} \]

and

\[ \delta \mathcal{V} = \delta A \cdot \delta \ell_n = \delta A \cdot \left( \frac{\delta \ell}{\text{in} \delta t} \left( -\frac{\cos \theta}{<0} \right) \right) = \delta A \cdot (-V \delta t \cos \theta) \]

Thus, the amount of \( B \) flowing out of CV through \( \delta A \) over a short time \( \delta t \):

\[ \therefore \delta B_{in} = \beta \delta m_{in} = -\beta \rho V \cos \theta \delta t \delta A \]
3rd term of RHS of Eq.(1) – Contd.

By integrating $\delta B_{out}$ over the entire outflow portion of CS,

$$B_1(t + \delta t) = \int_{CS_{in}} dB_{in} = \int_{CS_{in}} (-\beta \rho V \cos \theta) \delta t dA$$

Thus,

$$\lim_{\delta t \to 0} \frac{B_1(t + \delta t)}{\delta t} = \lim_{\delta t \to 0} \frac{1}{\delta t} \left( \delta t \int_{CS_{in}} (-\beta \rho V \cos \theta) dA \right)$$

$$= - \int_{CS_{in}} \beta \rho V \cos \theta \, dA \quad (\equiv \dot{B}_{in})$$

i.e., influx of $B$ through CS

Note that $V \cos \theta = \underline{V} \cdot \hat{n}$,

$$\therefore \dot{B}_{in} = - \int_{CS_{in}} \beta \rho \underline{V} \cdot \hat{n} dA$$
RTT for Fixed CV

Now the relationship between the time rate of change of $B$ for the system and that for the CV is given by,

$$\frac{DB_{sys}}{Dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \int_{CS_{out}} \beta \rho V \cdot \hat{n}dA - \left( \int_{CS_{in}} \beta \rho V \cdot \hat{n}dA \right)$$

With the fact that $CS = CS_{out} + CS_{in},$

$$\frac{DB_{sys}}{Dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \int_{CS} \beta \rho V \cdot \hat{n}dA$$

Time rate of change of $B$ within a system = Time rate of change of $B$ within CV + Net flux of $B$ through CS

$= \dot{B}_{out} - \dot{B}_{in}$
Example 1

4.72 Water flows through the 2-m-wide rectangular channel shown in Fig. P4.72 with a uniform velocity of 3 m/s. (a) Directly integrate Eq. 4.16 with \( b = 1 \) to determine the mass flow rate (kg/s) across section \( CD \) of the control volume. (b) Repeat part (a) with \( b = 1/\rho \), where \( \rho \) is the density. Explain the physical interpretation of the answer to part (b).
Example 1 - Contd.

\[ \dot{B}_{out} = \int_{CS_{out}} \rho \beta \mathbf{V} \cdot \hat{n} dA \quad (4.16) \]

With \( \beta = 1 \) and \( \mathbf{V} \cdot \hat{n} = V \cos \theta \),

\[ \dot{B}_{out} = \int_{CD} \rho V \cos \theta \ dA = \rho V \cos \theta \int_{CD} dA \]

\[ = \rho V \cos \theta A_{CD} \]

where

\[ A_{CD} = \ell \times (2 \ m) \]

\[ = \left( \frac{0.5 \ m}{\cos \theta} \right) (2 \ m) = \left( \frac{1}{\cos \theta} \right) m^2 \]
Example 1 - Contd.

Thus, with \( \rho = 1,000 \text{ kg/m}^3 \) for water and \( V = 3 \text{ m/s} \),

\[
\dot{B}_{\text{out}} = \left( 1,000 \frac{\text{kg}}{\text{m}^3} \right) \left( 3 \frac{\text{m}}{\text{s}} \right) \cos \theta \left( \frac{1}{\cos \theta} \text{ m}^2 \right) = 3,000 \text{ kg/s}
\]

With \( \beta = 1/\rho \),

\[
\dot{B}_{\text{out}} = \int_{CD} V \cos \theta \, dA = V \cos \theta \int_{CD} \frac{dA}{A_{CD}} = V \cos \theta \frac{A_{CD}}{1/\cos \theta}
\]

\[
= \left( 3 \frac{\text{m}}{\text{s}} \right) \cos \theta \left( \frac{1}{\cos \theta} \text{ m}^2 \right) = 3 \text{ m}^3/\text{s} \quad (i.e., \text{volume flow rate})
\]

Note: These results are the same for all \( \theta \) values
Special Case:

\[ \mathbf{V} = \text{constant over discrete CS's} \]

\[ \dot{B}_{\text{in}} = \int_{CS_{\text{in}}} \beta \rho \, \mathbf{V} \cdot \hat{n} \, dA = \sum_i (\beta_i \rho_i V_i A_i)_{\text{in}} \]

\[ \dot{B}_{\text{out}} = \int_{CS_{\text{out}}} \beta \rho \, \mathbf{V} \cdot \hat{n} \, dA = \sum_j (\beta_j \rho_j V_j A_j)_{\text{out}} \]

\[ \therefore \frac{DB_{\text{sys}}}{Dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \sum_j \left( \beta_j \frac{\rho_j V_j A_j}{m_j} \right)_{\text{out}} - \sum_i \left( \beta_i \frac{\rho_i V_i A_i}{m_i} \right)_{\text{in}} \]
Example 2

Given:
- Water flow ($\rho = \text{constant}$)
- $D_1 = 10 \text{ cm}$; $D_2 = 15 \text{ cm}$
- $V_1 = 10 \text{ cm/s}$
- Steady flow

Find: $V_2 = ?$

Mass conservation:
- $DB_{sys}/Dt = 0$
- $\beta = 1$
- $\rho_1 = \rho_2 = \rho$

Steady flow

\[
0 = \frac{d}{dt} \int_{CV} \rho dV + (\rho_2 V_2 A_2) - (\rho_1 V_1 A_1)
\]

or, $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$

\[
\therefore V_2 = \frac{\rho_1 A_1}{\rho_2 A_2} V_1 = \left(\frac{\rho}{\rho}\right) \left(\frac{D_1}{D_2}\right)^2 V_1 = (1) \left(\frac{10 \text{ cm}}{15 \text{ cm}}\right)^2 \left(10 \frac{\text{cm}}{\text{s}}\right) = 4.4 \text{ cm/s}
\]
Example 3

Given:
- \( D_1 = 5 \text{ cm}; D_2 = 7 \text{ cm} \)
- \( V_1 = 3 \text{ m/s} \)
- \( Q_3 = V_3A_3 = 0.01 \text{ m}^3/\text{s} \)
- \( h = \text{constant (i.e., steady flow)} \)
- \( \rho_1 = \rho_2 = \rho_3 = \rho_{\text{water}} \)

Find: \( V_2 = ? \)

\[
0 = \frac{d}{dt} \int_{CV} \rho dV + (\rho_2 V_2 A_2) - (\rho_1 V_1 A_1) - (\rho_3 V_3 A_3)
\]

or,
\[
V_2 A_2 = V_1 A_1 + \frac{V_3 A_3}{Q_3}
\]

\[
\therefore V_2 = \frac{V_1 A_1 + Q_3}{A_2} = \frac{(3)(\pi)(0.05)^2/4 + (0.01)}{(\pi)(0.07)^2/4} = 4.13 \text{ m/s}
\]
Moving CV

Particle A at $t_0$

Particle B at $t_0$

At $t_1$

$V_A$

$V_B$

$V_{CV} = \text{Control volume velocity}$

Control volume and system at time $t_0$

Control volume at time $t_1 > t_0$

System at time $t_1 > t_0$
RTT for Moving CV

\[ \frac{DB_{sys}}{Dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \int_{CS} \beta \rho \mathbf{V}_r \cdot \hat{n} dA \]

Figure 4.23
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RTT for Moving and Deforming CV

\[ \frac{dB_{sys}}{dt} = \frac{d}{dt} \int_{CV} \beta \rho dV + \int_{CS} \beta \rho (V_r \cdot \hat{n}) dA \]

Both CV and CS change their shape and location with time

\[ V_r = V(x, t) - V_S(x, t) \]

- \( V_S(x, t) \): Velocity of CS
- \( V(x, t) \): Fluid velocity in the coordinate system in which the \( V_S \) is observed
- \( V_r \): Relative velocity of fluid seen by an observer riding on the CV

*Ref) Fluid Mechanics by Frank M. White, McGraw Hill
RTT Summary (1)

General RTT (for moving and deforming CV):
\[
\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho dV \right) + \int_{CS} \beta \rho V_r \cdot \hat{n}dA
\]

Special Cases:

1) Non-deforming (but moving) CV
\[
\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t} (\beta \rho) dV + \int_{CS} \beta \rho V_r \cdot \hat{n}dA
\]

2) Fixed CV
\[
\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t} (\beta \rho) dV + \int_{CS} \beta \rho V \cdot \hat{n}dA
\]

3) Steady flow:
\[
\frac{\partial}{\partial t} = 0
\]

4) Flux terms for uniform flow across discrete CS’s (steady or unsteady)
\[
\int_{CS} \beta \rho V \cdot \hat{n}dA = \sum (\beta \dot{m})_{out} - \sum (\beta \dot{m})_{in}
\]
## RTT Summary (2)

For fixed CV’s:

<table>
<thead>
<tr>
<th>Parameter (B)</th>
<th>( \beta = B/m )</th>
<th>RTT</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (m)</td>
<td>1</td>
<td>0 = ( \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho V \cdot \mathbf{n} dA )</td>
<td>Continuity eq. (Ch. 5.1)</td>
</tr>
<tr>
<td>Momentum (mV)</td>
<td>( V )</td>
<td>( \sum F = \frac{d}{dt} \int_{CV} V \rho dV + \int_{CS} V \rho V \cdot \mathbf{n} dA )</td>
<td>Linear momentum eq. (Ch. 5.2)</td>
</tr>
<tr>
<td>Energy (E)</td>
<td>( e )</td>
<td>( \dot{Q} - \dot{W} = \frac{d}{dt} \int_{CV} e \rho dV + \int_{CS} e \rho V \cdot \mathbf{n} dA )</td>
<td>Energy eq. (Ch. 5.3)</td>
</tr>
</tbody>
</table>
Continuity Equation (Ch. 5.1)

RTT with \( B = \text{mass} \) and \( \beta = 1, \)

\[
0 = \frac{DM_{sys}}{Dt} = \frac{d}{dt} \int_{CV} \rho d\Psi + \int_{CS} \rho \mathbf{V} \cdot \mathbf{n} dA
\]

or

\[
\int_{CS} \rho \mathbf{V} \cdot \mathbf{n} dA = -\frac{d}{dt} \int_{CV} \rho d\Psi
\]

Net rate of outflow of mass across CS Rate of decrease of mass within CV

Note: Incompressible fluid (\( \rho = \text{constant} \))

\[
\int_{CS} \mathbf{V} \cdot \mathbf{n} dA = -\frac{d}{dt} \int_{CV} d\Psi
\] (Conservation of volume)
Simplifications

1. Steady flow

\[ \int_{CS} \rho V \cdot \hat{n} dA = 0 \]

2. If \( V \) = constant over discrete CS’s (i.e., one-dimensional flow)

\[ \int_{CS} \rho V \cdot \hat{n} dA = \sum_{out} \rho VA - \sum_{in} \rho VA \]

3. Steady one-dimensional flow in a conduit

\( (\rho VA)_{out} - (\rho VA)_{in} = 0 \)

or

\[ \rho_2 V_2 A_2 - \rho_1 V_1 A_1 = 0 \]

For \( \rho = \text{constant} \)

\[ V_1 A_1 = V_2 A_2 \quad \text{(or} \quad Q_1 = Q_2 \text{)} \]
Some useful definitions

Mass flux
\[ \dot{m} = \int_A \rho \underline{V} \cdot dA \]  
(Note: \( \underline{V} \cdot dA = \underline{V} \cdot \hat{n} dA \))

Volume flux
\[ Q = \int_A \underline{V} \cdot dA \]

Average velocity
\[ \bar{A} = \frac{Q}{A} = \frac{1}{A} \int_A \underline{V} \cdot dA \]

Average density
\[ \bar{\rho} = \frac{1}{A} \int_A \rho dA \]

Note: \( \dot{m} \neq \bar{\rho}Q \) unless \( \rho = \) constant
Example 4

Estimate the time required to fill with water a cone-shaped container 5 ft height and 5 ft across at the top if the filling rate is 20 gal/min.

Apply the conservation of mass \((\beta = 1)\)

\[
0 = \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho V \cdot \hat{n} dA
\]

For incompressible fluid (i.e., \(\rho = \text{constant}\)) and one inlet,

\[
0 = \frac{d}{dt} \int_{CV} dV - \left( VA \right)_{in} = Q_{in}
\]
Example 4 – Contd.

Volume of the cone at time t,

\[ V(t) = \frac{\pi D^2}{12} h(t) \]

Flow rate at the inlet,

\[ Q = \left(20 \frac{\text{gal}}{\text{min}} \right) \left(231 \frac{\text{in}^3}{\text{gal}} \right) / \left(1,728 \frac{\text{in}^3}{\text{ft}^3} \right) = 2.674 \text{ ft}^3/\text{min} \]

The continuity eq. becomes

\[ \frac{\pi D^2}{12} \cdot \frac{dh}{dt} = Q \quad \text{or} \quad \frac{dh}{dt} = \frac{12Q}{\pi D^2} \]
Example 4 – Contd.

Solve for $h(t)$,

$$h(t) = \int_0^t \frac{12Q}{\pi D^2} \, dt = \frac{12Q \cdot t}{\pi D^2}$$

Thus, the time for $h = 5$ ft is

$$t = \frac{\pi D^2 h}{12Q} = \frac{\pi (5 \text{ ft})^2 (5 \text{ ft})}{(12)(2.674 \text{ ft}^3/\text{min})} = 12.2 \text{ min}$$