# Chapter 5 Finite Control Volume Analysis

## 5.1 Continuity Equation

RTT can be used to obtain an integral relationship expressing conservation of mass by defining the extensive property B = M such that β = 1.

B = M = mass
β = dB/dM = 1

General Form of Continuity Equation

$$\frac{dM}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho d\forall + \int_{CS} \rho \underline{V} \cdot \underline{dA}$$

or

$$\underbrace{\int_{CS} \rho \underline{V} \cdot \underline{dA}}_{\text{net rate of outflow of mass across CS}} = \underbrace{-\frac{d}{dt}\int_{CV} \rho d\forall}_{\text{rate of decrease of mass within CV}}$$

Simplifications:

1. Steady flow: $-\frac{d}{dt}\int_{CV} \rho d\forall = 0$
2. $\underline{V}$ = constant over discrete $\underline{dA}$ (flow sections):

$$\int_{CS} \rho \underline{V} \cdot \underline{dA} = \sum_{CS} \rho \underline{V} \cdot \underline{A}$$

3. Incompressible fluid (ρ = constant)

$$\int_{CS} \underline{V} \cdot \underline{dA} = -\frac{d}{dt}\int_{CV} d\forall \qquad \text{conservation of volume}$$

4. Steady One-Dimensional Flow in a Conduit:

$$\sum_{CS} \rho \underline{V} \cdot \underline{A} = 0$$

$$-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0$$

for $\rho$ = constant $\quad Q_1 = Q_2$

Some useful definitions:

Mass flux $\quad \dot{m} = \int_A \rho \underline{V} \cdot \underline{dA}$

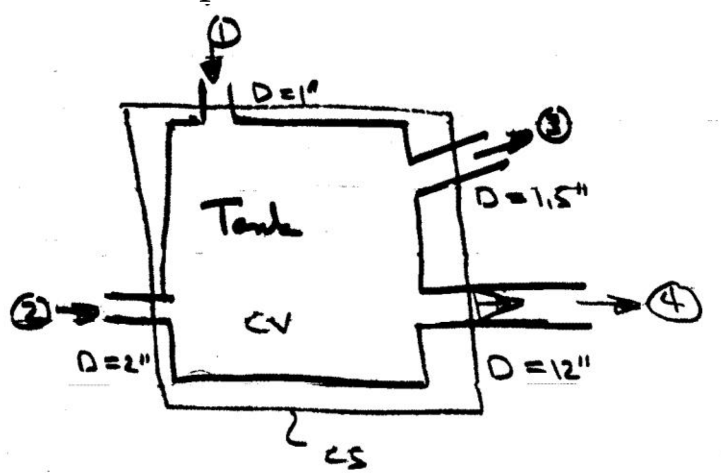
Volume flux $\quad Q = \int_A \underline{V} \cdot \underline{dA}$

Average Velocity $\quad \overline{V} = Q / A$

Average Density $\quad \overline{\rho} = \frac{1}{A}\int \rho dA$

Note: $\dot{m} \neq \overline{\rho} Q$ unless $\rho$ = constant

Example




*Steady flow

*$V_{1,2,3}$ = 50 fps

*At ④, V varies linearly from zero at wall to $V_{max}$ at pipe center

*find $\dot{m}_4$, $Q_4$, $V_{max}$

*water, $\rho_w$ = 1.94 slug/ft$^3$

$$\int_{CS} \rho \underline{V} \cdot \underline{dA} = 0 = -\cancelto{0}{\frac{d}{dt}\int_{CV} \rho d\forall}$$

i.e., $-\rho_1 V_1 A_1 - \rho_2 V_2 A_2 + \rho_3 V_3 A_3 + \underbrace{\boxed{\rho \int_{A_4} V_4 dA_4}}_{}^{\dot{m}_4} = 0$

$\rho$ = const. = 1.94 lb-s$^2$ /ft$^4$ = 1.94 slug/ft$^3$

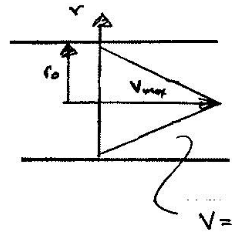
$\dot{m}_4 = \rho \int V_4 dA_4 = \rho V(A_1 + A_2 - A_3)$ $\quad V_1=V_2=V_3=V=50$f/s

$$= \frac{1.94}{144} \times 50 \times \frac{\pi}{4}\left(1^2 + 2^2 - 1.5^2\right)$$
= 1.45 slugs/s

$$Q_4 = \dot{m}_4 / \rho = .75\ \text{ft}^3/\text{s}$$

$$= \int_{A_4} V_4 \, dA_4$$

$$Q_4 = \int_0^{r_o}\int_0^{2\pi} \underbrace{V_{max}\overbrace{\left(1 - \frac{r}{r_o}\right)}^{\text{velocity profile}}}_{V_4 \neq V_4(\theta)} \underbrace{r\,d\theta\,dr}_{dA_4}$$





$V = V_{max}(1 - r/r_o)$ , $dA_4 = r\,dr\,d\theta$

$$= 2\pi \int_0^{r_o} V_{max}\left(1 - \frac{r}{r_o}\right) r\,dr$$

$$= 2\pi V_{max} \int_0^{r_o} \left[ r - \frac{r^2}{r_o} \right] dr$$

$$= 2\pi V_{max} \left[ \left.\frac{r^2}{2}\right|_0^{r_0} - \left.\frac{r^3}{3r_o}\right|_0^{r_o} \right]$$

$$= 2\pi V_{max} r_o^{\,2} \left[ \frac{1}{2} - \frac{1}{3} \right] = \frac{1}{3}\pi r_o^{\,2} V_{max}$$

$$\overline{V}_4 = \frac{Q}{A} = \frac{\frac{1}{3}\pi r_o^2 V_{max}}{\pi r_o^2} = \frac{1}{3} V_{max}$$

$$V_{max} = \frac{Q_4}{\frac{1}{3}\pi r_o^{\,2}} = 2.86\,\text{fps}$$

# 5.2 Momentum Equation

Derivation of the Momentum Equation

Newton's second law of motion for a system is

| | | |
|---|---|---|
| time rate of change of the momentum of the system | = | sum of external forces acting on the system |

Since momentum is mass times velocity, the momentum of a small particle of mass $\rho dV$ is $\underline{V}\rho d\forall$ and the momentum of the entire system is $\int_{sys}\underline{V}\rho d\forall$. Thus,

$$\frac{D}{Dt}\int_{sys}\underline{V}\rho d\forall = \sum \underline{F}_{sys}$$

Recall RTT:

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV}\beta\rho d\forall + \int_{CS}\beta\rho\underline{V}_R\cdot\underline{dA}$$

With $B_{sys} = M\underline{V}$ and $\beta = \frac{dB_{sys}}{dM} = \underline{V}$,

$$\frac{D}{Dt}\int_{sys}\underline{V}\rho d\forall = \frac{\partial}{\partial t}\int_{CV}\underline{V}\rho d\forall + \int_{CS}\underline{V}\rho\underline{V}_R\cdot\underline{dA}$$

Thus, the Newton's second law becomes

$$\underbrace{\frac{\partial}{\partial t}\int_{CV} \underline{V}\rho d\forall}_{\substack{rate\ of\ change\\ of\ momentum\\ in\ the\ CV}} + \underbrace{\int_{CS} \underline{V}\rho \underline{V}_R \cdot \underline{dA}}_{\substack{net\ rate\ of\ flow\\ of\ momentum\\ through\ the\ CS}} = \underbrace{\sum \underline{F}_{CV}}_{\substack{net\ external\\ force\ acting\\ on\ the\ CV}}$$

where,

$\underline{V}$ is fluid velocity referenced to an inertial frame (non-accelerating)

$\underline{V}_S$ is the velocity of CS referenced to the inertial frame

$\underline{V}_R = \underline{V} - \underline{V}_S$ is the relative velocity referenced to CV

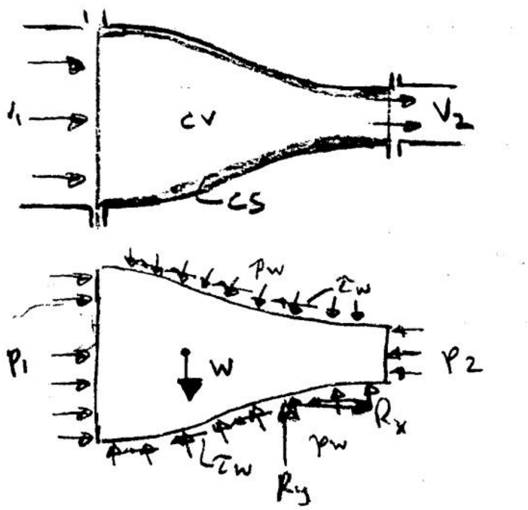
$\sum \underline{F}_{CV} = \sum \underline{F}_B + \sum \underline{F}_S$ is vector sum of all forces acting on the CV

$\underline{F}_B$ is body force such as gravity that acts on the entire mass/volume of CV

$\underline{F}_S$ is surface force such as normal (pressure and viscous) and tangential (viscous) stresses acting on the CS

Note that, when CS cuts through solids, $\underline{F}_S$ may also include reaction force, $\underline{F}_R$

(e.g., the reaction force required to hold nozzle or bend when CS cuts through the bolts that are holding the nozzle/bend in place)




Free body diagram

$$\sum F_x = p_1 A_1 - p_2 A_2 + R_x$$

$$\sum F_y = -W + R_y$$

$\underline{R} = R_x \hat{\imath} + R_y \hat{\jmath}$ = resultant force on fluid in CV due to $p_w$ and $\tau_w$, i.e. reaction force on fluid

Important Features (to be remembered)

1) Vector equation to get component in any direction must use dot product

Carefully define coordinate system with forces positive in positive direction of coordinate axes

<u>x equation</u>

$$\sum F_x = \frac{d}{dt}\int_{CV} \rho u d\forall + \int_{CS} \rho u \underline{V}_R \cdot \underline{dA}$$

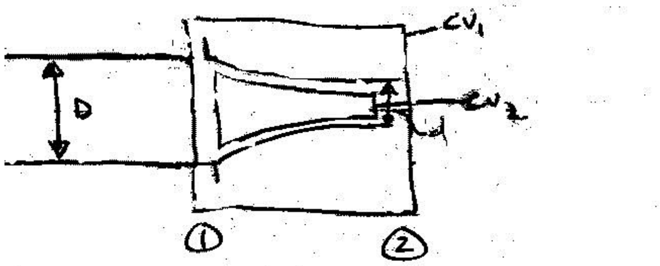
<u>y equation</u>

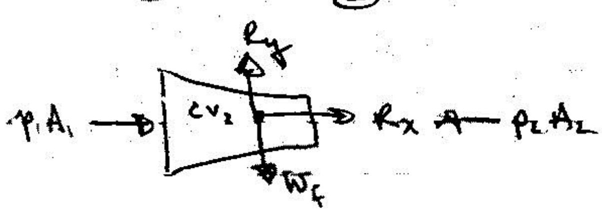
$$\sum F_y = \frac{d}{dt}\int_{CV} \rho v d\forall + \int_{CS} \rho v \underline{V}_R \cdot \underline{dA}$$

<u>z equation</u>

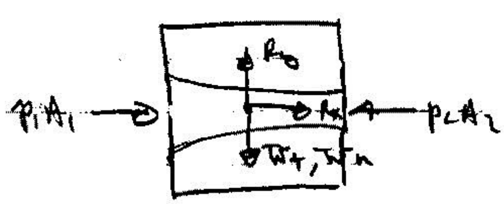
$$\sum F_z = \frac{d}{dt}\int_{CV} \rho w d\forall + \int_{CS} \rho w \underline{V}_R \cdot \underline{dA}$$

2) <u>Carefully</u> define control volume and be sure to include <u>all</u> external body and surface faces acting on it.
For example,




$(R_x,R_y)$ = reaction force on fluid




$(R_x,R_y)$ = reaction force on nozzle

3) Velocity $\underline{V}$ and $\underline{V}_s$ must be referenced to a non-accelerating inertial reference frame. Sometimes it is advantageous to use a moving (at constant velocity) reference frame: relative inertial coordinate. Note $\underline{V}_R = \underline{V} - \underline{V}_s$ is always relative to CS.

4) Steady vs. Unsteady Flow

$$\text{Steady flow} \Rightarrow \frac{d}{dt}\int_{CV} \rho \underline{V} d\forall = 0$$

5) Uniform vs. Nonuniform Flow

$$\int_{CS} \underline{V}\rho\underline{V}_R \cdot \underline{dA} = \text{change in flow of momentum across CS}$$

$$= \Sigma \underline{V}\rho\underline{V}_R \cdot \underline{A} \qquad \text{uniform flow across } \underline{A}$$

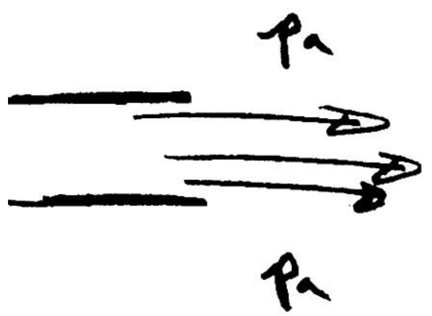
6) $\underline{F}_{pres} = -\int p\underline{n}dA$ $\qquad \int_V \nabla f d\forall = \int_S f\,\underline{n}ds$

f = constant, $\nabla f = 0$

= 0 for p = constant and for a closed surface

i.e., always use gage pressure

7) Pressure condition at a jet exit




at an exit into the atmosphere jet pressure must be $p_a$

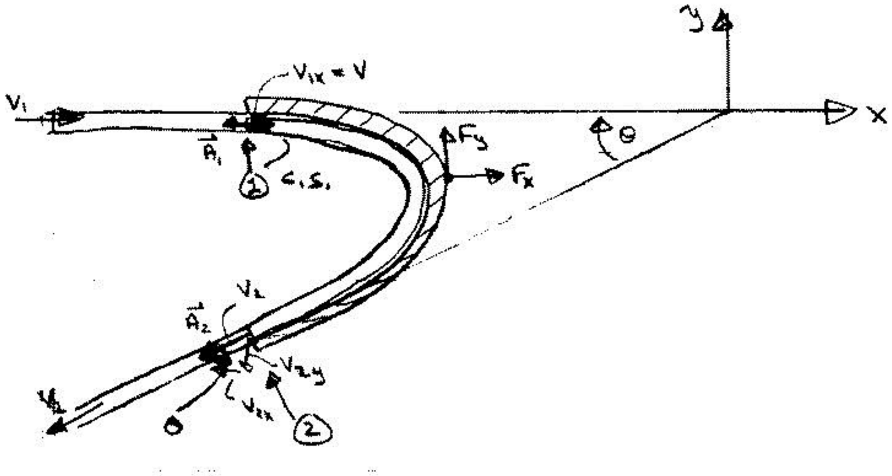
## Applications of the Momentum Equation

Initial Setup and Signs

1. Jet deflected by a plate or a vane
2. Flow through a nozzle
3. Forces on bends
4. Problems involving non-uniform velocity distribution
5. Motion of a rocket
6. Force on rectangular sluice gate
7. Water hammer
8. Steady and unsteady developing and fully developed pipe flow
9. Empting and filling tanks
10. Forces on transitions
11. Hydraulic jump
12. Boundary layer and bluff body drag
13. Rocket or jet propulsion
14. Propeller

## 1. Jet deflected by a plate or vane

Consider a jet of water turned through a horizontal angle




CV and CS are for jet so that $F_x$ and $F_y$ are vane reactions forces on fluid

x-equation: $$\sum F_x = F_x = \frac{d}{dt}\int \rho u \, d\forall + \int_{CS} \rho u \underline{V} \cdot \underline{dA}$$

$$F_x = \sum_{CS} \rho u \underline{V} \cdot \underline{A} \qquad \text{steady flow}$$

$$= \rho V_{1x}(-V_1 A_1) + \rho V_{2x}(V_2 A_2)$$

continuity equation: $\rho A_1 V_1 = \rho A_2 V_2 = \rho Q$ for $A_1 = A_2$, $V_1 = V_2$

$F_x = \rho Q(V_{2x} - V_{1x})$

y-equation: $$\sum F_y = F_y = \sum_{CS} \rho v \underline{V} \cdot \underline{A}$$

$$F_y = \rho V_{1y}(-A_1 V_1) + \rho V_{2y}(-A_2 V_2)$$
$$= \rho Q(V_{2y} - V_{1y})$$

where: $\overbrace{V_{1x} = V_1 \quad V_{2x} = -V_2\cos\theta \quad V_{2y} = -V_2\sin\theta \quad V_{1y} = 0}^{\text{for above geometry only}}$

note: $F_x$ and $F_y$ are force on fluid

$-F_x$ and $-F_y$ are force on vane due to fluid

If the vane is moving with velocity $\underline{V}_v$, then it is convenient to choose CV moving with the vane

i.e., $\underline{V}_R = \underline{V} - \underline{V}_v$ and $\underline{V}$ used for B also moving with vane

x-equation: $F_x = \int_{CS} \rho u \underline{V}_R \cdot \underline{dA}$

$$F_x = \rho V_{1x}[-(V - V_v)_1 A_1] + \rho V_{2x}[(V - V_v)_2 A_2]$$

Continuity: $0 = \int \rho \underline{V}_R \cdot \underline{dA}$

i.e., $\rho(V-V_v)_1 A_1 = \rho(V-V_v)_2 A_2 = \rho\underbrace{(V-V_v)A}_{Q_{rel}}$

$$\underset{\text{on fluid}}{\overset{\uparrow}{F_x}} = \rho\underbrace{(V-V_v)A}_{Q_{rel}}[V_{2x} - V_{1x}]$$

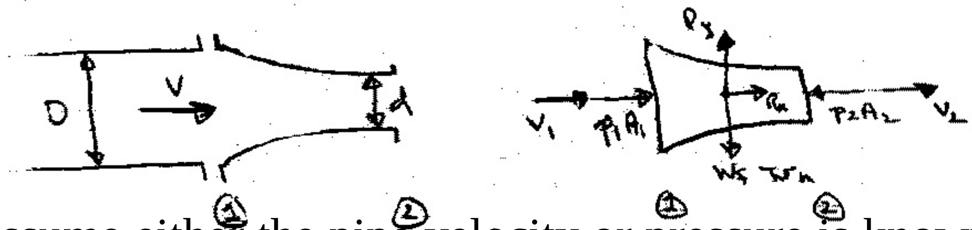
$$\left.\begin{aligned} V_{2x} &= (V - V_v)_{2x} \\ V_{1x} &= (V - V_v)_{1x} \end{aligned}\right\} \text{For coordinate system moving with vane}$$

Power = $-F_x V_v$ i.e., = 0 for $V_v = 0$

$$F_y = \rho Q_{rel}(V_{2y} - V_{1y})$$

## 2. Flow through a nozzle

Consider a nozzle at the end of a pipe (or hose). What force is required to hold the nozzle in place?



CV = nozzle and fluid
$\therefore$ $(R_x, R_y)$ = force required to hold nozzle in place

Assume either the pipe velocity or pressure is known. unknown (velocity or pressure) and the exit velocity $V_2$ can be obtained from combined use of the continuity and Bernoulli equations.

Bernoulli: $$p_1 + \gamma z_1 + \frac{1}{2}\rho V_1^2 = p_2 + \gamma z_2 + \frac{1}{2}\rho V_2^2 \qquad z_1 = z_2$$

$$p_1 + \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V_2^2$$

Continuity: $$A_1 V_1 = A_2 V_2 = Q$$

$$V_2 = \frac{A_1}{A_2} V_1 = \left(\frac{D}{d}\right)^2 V_1$$

$$p_1 + \frac{1}{2}\rho V_1^2 \left(1 - \left(\frac{D}{d}\right)^4\right) = 0$$

Say $p_1$ known: $$V_1 = \left[\frac{-2p_1}{\rho\left(1 - \left(D/d\right)^4\right)}\right]^{1/2}$$

To obtain the reaction force $R_x$ apply momentum equation in x-direction

$$\sum F_x = \frac{d}{dt}\int_{CV} u\rho d\forall + \int_{CS} \rho u \underline{V}\cdot \underline{dA}$$

$$= \sum_{CS} \rho u \underline{V}\cdot \underline{A}$$ steady flow and uniform flow over CS

$$R_x + p_1A_1 - p_2A_2 = \rho V_1(-V_1A_1) + \rho V_2(V_2A_2)$$
$$= \rho Q(V_2 - V_1)$$
$$R_x = \rho Q(V_2 - V_1) - p_1A_1$$

To obtain the reaction force $R_y$ apply momentum equation in y-direction

$$\sum F_y = \sum_{CS} \rho v \underline{V}\cdot \underline{A} = 0$$ since no flow in y-direction

$$R_y - W_f - W_N = 0 \quad \text{i.e., } R_y = W_f + W_N$$

Numerical Example: Oil with S = .85 flows in pipe under pressure of 100 psi. Pipe diameter is 3" and nozzle tip diameter is 1"

$$\rho = \frac{S\gamma}{g} = 1.65$$

D/d = 3

$$Q = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 V_2$$
$$= .716 \text{ ft}^3/\text{s}$$

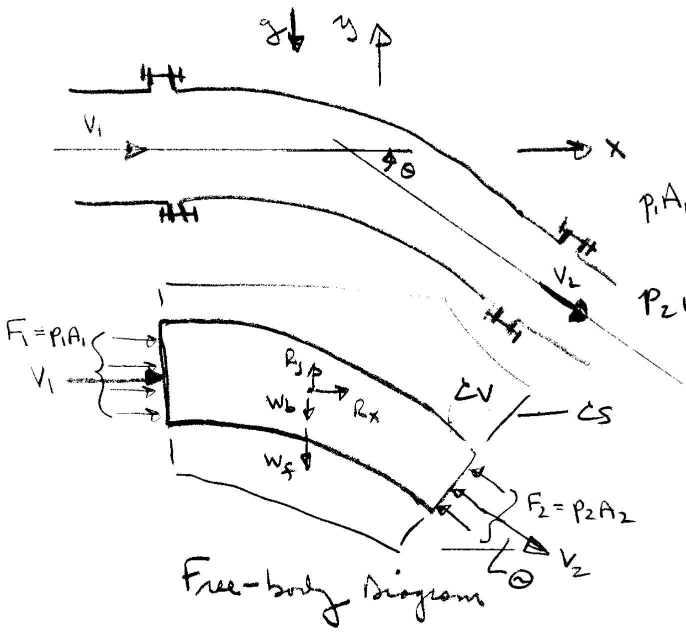
$V_1$ = 14.59 ft/s
$V_2$ = 131.3 ft/s

$R_x$ = 141.48 – 706.86 = –569 lbf
$R_z$ = 10 lbf

This is force on nozzle

## 3. Forces on Bends

Consider the flow through a bend in a pipe. The flow is considered steady and uniform across the inlet and outlet sections. Of primary concern is the force required to hold the bend in place, i.e., the reaction forces $R_x$ and $R_y$ which can be determined by application of the momentum equation.




$A_1 V_1 = A_2 V_2 = Q$

$p_1 A_1 - p_2 A_2 \cos\theta + R_x = \rho Q (V_{2x} - V_{1x})$

$p_2 A_2 \sin\theta + R_y - W_b - W_f = \rho Q (V_{2y} - V_{1y})$

$R_x$, $R_y$ = reaction force on bend i.e., force required to hold bend in place

Continuity: $$0 = \sum \rho \underline{V} \cdot \underline{A} = -\rho V_1 A_1 + \rho V_2 A_2$$
i.e., Q = constant = $V_1 A_1 = V_2 A_2$

x-momentum: $\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$

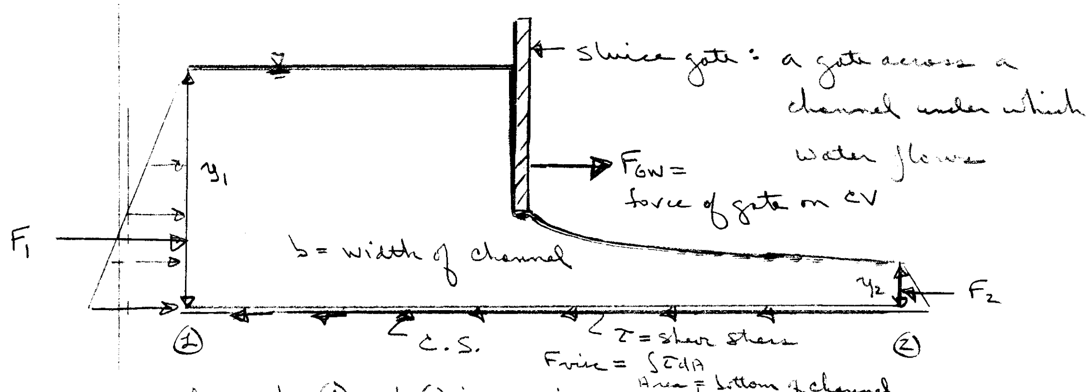
$$p_1 A_1 - p_2 A_2 \cos\theta + R_x = \rho V_{1x}(-V_1 A_1) + \rho V_{2x}(V_2 A_2)$$
$$= \rho Q (V_{2x} - V_{1x})$$

y-momentum: $\sum F_y = \sum \rho v \underline{V} \cdot \underline{A}$

$$p_2 A_2 \sin\theta + R_y - w_f - w_b = \rho V_{1y}\left(-V_1 A_1\right) + \rho V_{2y}\left(V_2 A_2\right)$$
$$= \rho Q\left(V_{2y} - V_{1y}\right)$$

4. Force on a rectangular sluice gate

The force on the fluid due to the gate is calculated from the x-momentum equation:




flow at ① and ② is uniform and steady

* pressure distribution at ① and ② is hydrostatic

if $V_1$ = constant $\quad \underbrace{p + \frac{1}{2}\rho V_1^2 + \rho g y}_{\text{at any } y} = p_{atm} + \frac{1}{2}\rho V_1^2$ at free surface

$p(y) = p_{atm} - \rho g y$

$$\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$$

$$F_1 + F_{GW} - F_{visc} - F_2 = \rho V_1\left(-V_1 A_1\right) + \rho V_2\left(V_2 A_2\right)$$

$$F_{GW} = F_2 - F_1 + \rho Q\left(V_2 - V_1\right) + \cancel{F_{visc}}$$

usually can be neglected

$$= \gamma \frac{y_2}{2} \cdot y_2 b - \gamma \frac{y_1}{2} \cdot y_1 b + \rho Q\left(V_2 - V_1\right)$$

$$F_{GW} = \frac{1}{2} b\gamma\left(y_2^2 - y_1^2\right) + \underbrace{\rho Q\left(V_2 - V_1\right)}$$

$$V_1 = \frac{Q}{y_1 b}$$

$$V_2 = \frac{Q}{y_2 b}$$

$$\frac{\rho Q^2}{b}\left(\frac{1}{y_2}-\frac{1}{y_1}\right)$$

5. Application of relative inertial coordinates for a moving but non-deforming control volume (CV)

The CV moves at a constant velocity $\underline{V}_{CS}$ with respect to the absolute inertial coordinates. If $\underline{V}_R$ represents the velocity in the relative inertial coordinates that move together with the CV, then:

$$\underline{V_R} = \underline{V} - \underline{V_{CS}}$$

Reynolds transport theorem for an arbitrary moving deforming CV:

$$\frac{dB_{SYS}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho \, d\forall + \int_{CS} \beta\rho \underline{V_R} \cdot \underline{n} \; dA$$

For a non-deforming CV moving at constant velocity, RTT for incompressible flow:

$$\frac{dB_{syst}}{dt} = \rho\int_{CV} \frac{\partial \beta}{\partial t} d\forall + \rho\int_{CS} \beta \underline{V}_R \cdot \underline{n} dA$$

1) Conservation of mass

$B_{syst} = M$, and $\beta = 1$:

$$\frac{dM}{dt} = \rho\int_{CS} \underline{V_R} \cdot \underline{n} dA$$

For steady flow:

$$\int_{CS} \underline{V_R} \cdot \underline{n} dA = 0$$

## 2) Conservation of momentum

$$B_{syst} = M\left(\underline{V}_R + \underline{V}_{CS}\right) \text{ and } \beta = d\underline{B}_{syst}/dM = \underline{V}_R + \underline{V}_{CS}$$

$$\frac{d[M\left(\underline{V}_R + \underline{V}_{CS}\right)]}{dt} = \sum \underline{F} = \rho \int_{CV} \frac{\partial\left(\underline{V}_R + \underline{V}_{CS}\right)}{\partial t} d\forall + \rho \int_{CS} \left(\underline{V}_R + \underline{V}_{CS}\right) \underline{V}_R \cdot \underline{n} dA$$

For steady flow with the use of continuity:

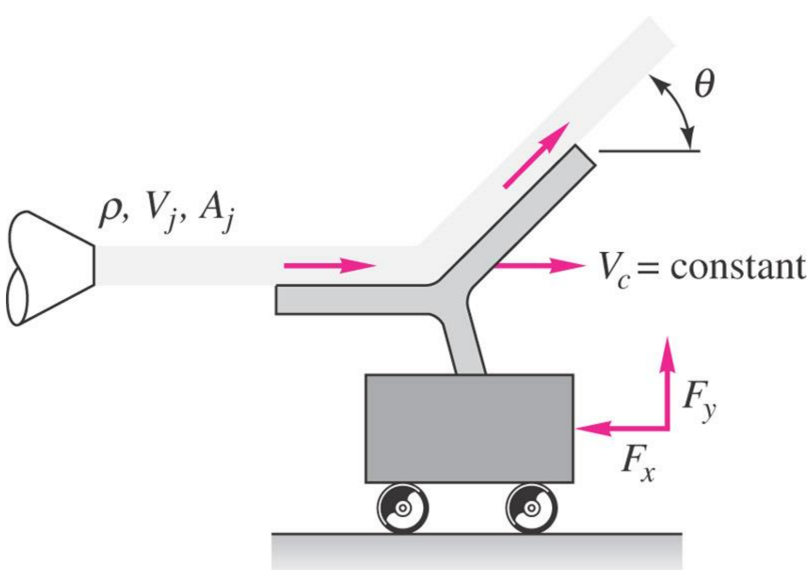
$$\sum \underline{F} = \rho \int_{CS} \left(\underline{V}_R + \underline{V}_{CS}\right) \underline{V}_R \cdot \underline{n} dA$$

$$= \rho \int_{CS} \underline{V}_R \underline{V}_R \cdot \underline{n} dA + \rho \underline{V}_{CS} \cancelto{0}{\int_{CS} \underline{V}_R \cdot \underline{n} dA}$$

$$\boxed{\sum \underline{F} = \rho \int_{CS} \underline{V}_R \underline{V}_R \cdot \underline{n} dA}$$

## Example (use relative inertial coordinates):

Ex) A jet strikes a vane which moves to the right at constant velocity $V_c$ on a frictionless cart. Compute (a) the force $F_x$ required to restrain the cart and (b) the power $P$ delivered to the cart. Also find the cart velocity for which (c) the force $F_x$ is a maximum and (d) the power $P$ is a maximum.



**Solution:**

Assume relative inertial coordinates with non-deforming CV i.e. CV moves at constant translational non-accelerating

$$\underline{V_{CS}} = u_{CS}\hat{\imath} + v_{CS}\hat{\jmath} + w_{CS}\hat{k} = V_C\hat{\imath}$$

then $\underline{V}_R = \underline{V} - \underline{V}_{CS}$ . Also assume steady flow $\underline{V} \neq \underline{V}(t)$ with $\rho = constant$ and neglect gravity effect.

Continuity:

$$\int_{CS} \underline{V_R} \cdot \underline{n} dA = 0$$

Bernoulli without gravity:

$$\cancel{p_1}^{0} + \frac{1}{2}\rho V_{R1}^2 = \cancel{p_2}^{0} + \frac{1}{2}\rho V_{R2}^2$$

$$V_{R1} = V_{R2}$$

Since
$$\rho V_{R1} A_1 = \rho V_{R2} A_2$$

$$A_1 = A_2 = A_j$$

Momentum:

$$\sum \underline{F} = \rho \int_{CS} \underline{V_R}\, \underline{V_R} \cdot \underline{n} dA$$

$$\sum F_x = -F_x = \rho \int_{CS} V_{Rx} \underline{V_R} \cdot \underline{n} dA$$

$$-F_x = \rho V_{R_x 1}(-V_{R1}A_1) + \rho V_{R_x 2}(V_{R2}A_2)$$

$$0 = \rho \int_{CS} \underline{V_R} \cdot \underline{n} dA$$

$$-\rho V_{R1}A_1 + \rho V_{R2}A_2 = 0$$

$$V_{R1}A_1 = V_{R2}A_2 = \underbrace{(V_j - V_C)}_{V_{R1} = V_{R_x 1} = V_j - V_C} A_j$$

$$-F_x = \rho\left(V_j - V_C\right)\left[-\left(V_j - V_C\right)A_j\right] + \rho\left(V_j - V_C\right)\cos\theta\left(V_j - V_C\right)A_j$$

$$F_x = \rho\left(V_j - V_C\right)^2 A_j[1 - \cos\theta]$$

$$Power = V_C F_x = V_C \rho\left(V_j - V_C\right)^2 A_j(1 - \cos\theta)$$

$$F_{x_{max}} = \rho V_j^2 A_j(1 - \cos\theta), \qquad V_C = 0$$

$$P_{max} \Rightarrow \frac{dP}{dV_C} = 0$$

$$P = V_C\rho\left(V_j^2 - 2V_C V_j + V_C^2\right)A_j(1 - \cos\theta)$$

$$= \rho\left(V_j^2 V_C - 2V_C^2 V_j + V_C^3\right)A_j(1 - \cos\theta)$$

$$\frac{dP}{dV_C} = \rho\left(V_j^2 - 4V_C V_j + 3V_C^2\right)A_j(1 - \cos\theta) = 0$$

$$3V_C^2 - 4V_j V_C + V_j^2 = 0$$

$$V_C = \frac{+4V_j \pm \sqrt{16V_j^2 - 12V_j^2}}{6} = \frac{4V_j \pm 2V_j}{6}$$

$$V_C = \frac{V_j}{3}$$

$$P_{max} = \frac{V_j}{3}\rho\left(\frac{2V_j}{3}\right)^2 A_j(1 - \cos\theta)$$

$$= \frac{4}{27}V_j^3 \rho A_j(1 - \cos\theta)$$

# 5.3 Energy Equation

## Derivation of the Energy Equation

### The First Law of Thermodynamics

The difference between the heat added to a system and the work done by a system depends only on the initial and final states of the system; that is, depends only on the change in energy E: principle of conservation of energy

$$\Delta E = Q - W$$

$\Delta E$ = change in energy
Q = heat added to the system
W = work done by the system

$E = E_u + E_k + E_p$ = total energy of the system

potential energy
kinetic energy
Internal energy due to molecular motion

The differential form of the first law of thermodynamics expresses the rate of change of E with respect to time

$$\frac{dE}{dt} = \dot{Q} - \dot{W}$$

rate of work being done by system
rate of heat transfer to system

## Energy Equation for Fluid Flow

The energy equation for fluid flow is derived from Reynolds transport theorem with

$B_{system}$ = E = total energy of the system (extensive property)

$\beta$ = E/mass = e = energy per unit mass (intensive property)
  $= \hat{u} + e_k + e_p$

$$\frac{dE}{dt} = \frac{d}{dt}\int_{CV} \rho e d\mathcal{V} + \int_{CS} \rho e \underline{V} \cdot \underline{dA}$$

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{CV} \rho(\hat{u} + e_k + e_p) d\mathcal{V} + \int_{CS} \rho(\hat{u} + e_k + e_p) \underline{V} \cdot \underline{dA}$$

This can be put in a more useable form by noting the following:

$$e_k = \frac{\text{Total KE of mass with velocity V}}{\text{mass}} = \frac{\Delta M V^2/2}{\Delta M} = \frac{V^2}{2} \qquad V^2 = |\underline{V}|$$

$$e_p = \frac{E_p}{\Delta M} = \frac{\gamma \Delta\mathcal{V} z}{\rho \Delta\mathcal{V}} = gz \qquad \text{(for } E_p \text{ due to gravity only)}$$

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{CV} \rho\left(\frac{V^2}{2} + gz + \hat{u}\right) d\mathcal{V} + \int_{Cs} \rho\left(\frac{V^2}{2} + gz + \hat{u}\right) \underline{V} \cdot \underline{dA}$$

rate of work done by system

rate of change of energy in CV

flux of energy out of CV (ie, across CS)

rate of heat transfer to sysem

<u>Rate of Work Components</u>: $\dot{W} = \dot{W}_s + \dot{W}_f$

For convenience of analysis, work is divided into shaft work $W_s$ and flow work $W_f$

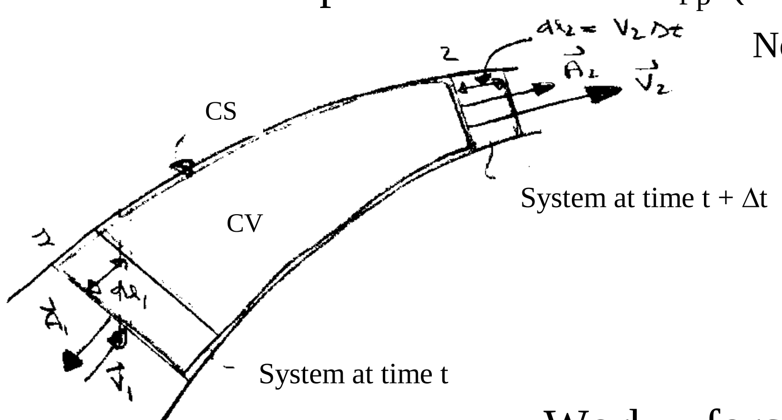
$W_f$ = net work done on the surroundings as a result of normal and tangential stresses acting at the control surfaces

$= W_{f\,pressure} + W_{f\,shear}$

$W_s$ = any other work transferred to the surroundings usually in the form of a shaft which either takes energy out of the system (turbine) or puts energy into the system (pump)

Flow work due to pressure forces $W_{f\,p}$ (for system)

Note: here $\underline{V}$ uniform over $\underline{A}$




Work = force × distance

at 2 $W_2 = p_2A_2 \times V_2\Delta t$ (on surroundings)

rate of work $\Rightarrow \dot{W}_2 = p_2A_2V_2 = p_2\underline{V}_2 \cdot \underline{A}_2$

neg. sign since pressure force on surrounding fluid acts in a direction opposite to the motion of the system boundary

at 1 $W_1 = -p_1A_1 \times V_1\Delta t$

$\dot{W}_1 = p_1\underline{V}_1 \cdot \underline{A}_1$

In general,

$$\dot{W}_{fp} = p\underline{V}\cdot\underline{A}$$

for more than one control surface and $\underline{V}$ not necessarily uniform over $\underline{A}$:

$$\dot{W}_{fp} = \int_{CS} p\underline{V}\cdot\underline{dA} = \int_{CS}\rho\left(\frac{p}{\rho}\right)\underline{V}\cdot\underline{dA}$$

$$\dot{W}_{f} = \dot{W}_{fp} + \dot{W}_{fshear}$$

Basic form of energy equation

$$\dot{Q} - \dot{W}_s - \dot{W}_{fshear} - \int_{CS}\rho\left(\frac{p}{\rho}\right)\underline{V}\cdot\underline{dA}$$
$$= \frac{d}{dt}\int_{CV}\rho\left(\frac{V^2}{2} + gz + \hat{u}\right)d\forall + \int_{CS}\rho\left(\frac{V^2}{2} + gz + \hat{u}\right)\underline{V}\cdot\underline{dA}$$

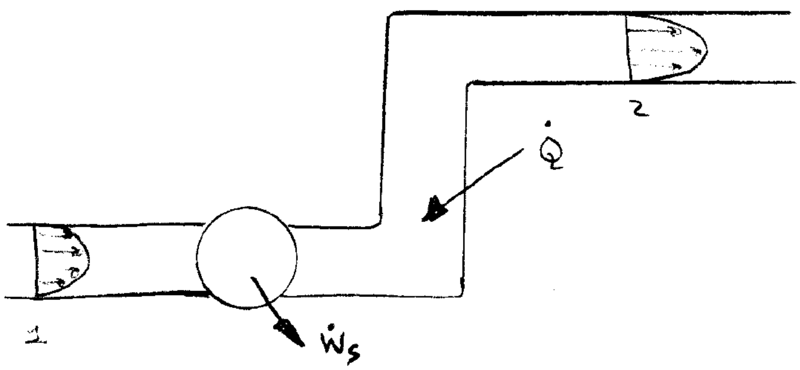
$$\dot{Q} - \dot{W}_s - \cancel{\dot{W}_{fshear}} = \frac{d}{dt}\int_{CV}\rho\left(\frac{V^2}{2} + gz + \hat{u}\right)d\forall$$
$$+ \int_{CS}\rho\left(\frac{V^2}{2} + gz + \underbrace{\hat{u} + \frac{p}{\rho}}_{h=\text{enthalpy}}\right)\underline{V}\cdot\underline{dA}$$

Usually this term can be eliminated by proper choice of CV, i.e. CS normal to flow lines. Also, at fixed boundaries the velocity is zero (no slip condition) and no shear stress flow work is done. Not included or discussed in text!

# Simplified Forms of the Energy Equation

## Energy Equation for Steady One-Dimensional Pipe Flow

Consider flow through the pipe system as shown




Energy Equation (steady flow)

$$\dot{Q} - \dot{W}_s = \int_{CS} \rho\left(\frac{V^2}{2} + gz + \frac{p}{\rho} + \hat{u}\right)\underline{V}\cdot\underline{dA}$$

$$\dot{Q} - \dot{W}_s + \int_{A_1}\left(\frac{p_1}{\rho} + gz_1 + \hat{u}_1\right)\rho_1 V_1 A_1 + \int_{A_1}\frac{\rho_1 V_1^3}{2}dA_1$$

$$= \int_{A_2}\left(\frac{p_2}{\rho} + gz_2 + \hat{u}_2\right)\rho_2 V_2 A_2 + \int_{A_2}\frac{\rho_2 V_2^3}{2}dA_2$$

*Although the velocity varies across the flow sections the streamlines are assumed to be straight and parallel; consequently, there is no acceleration normal to the streamlines and the pressure is hydrostatically distributed, i.e., $p/\rho + gz =$ constant.

*Furthermore, the internal energy u can be considered as constant across the flow sections, i.e. T = constant. These quantities can then be taken outside the integral sign to yield

$$\dot{Q}-\dot{W}_s+\left(\frac{p_1}{\rho}+gz_1+\hat{u}_1\right)\rho\int_{A_1}V_1dA_1+\rho\int_{A_1}\frac{V_1^3}{2}dA_1$$

$$=\left(\frac{p_2}{\rho}+gz_2+\hat{u}_2\right)\rho\int_{A_2}V_2dA_2+\rho\int_{A_2}\frac{V_2^3}{2}dA_2$$

Recall that $\quad \mathrm{Q}=\int \underline{\mathrm{V}}\cdot\underline{\mathrm{dA}}=\overline{\mathrm{V}}\mathrm{A}$

So that $\quad \rho\int \underline{\mathrm{V}}\cdot\underline{\mathrm{dA}}=\rho\overline{\mathrm{V}}\mathrm{A}=\dot{\mathrm{m}}$ $\qquad$ mass flow rate

Define: $\quad \underbrace{\rho\int_{\mathrm{A}}\frac{\mathrm{V}^3}{2}\mathrm{dA}}_{\text{K.E. flux}}=\alpha\underbrace{\frac{\rho\overline{\mathrm{V}}^3\mathrm{A}}{2}}_{\text{K.E. flux for V}=\overline{\mathrm{V}}\text{ =constant across pipe}}=\alpha\frac{\overline{\mathrm{V}}^2}{2}\dot{\mathrm{m}}$

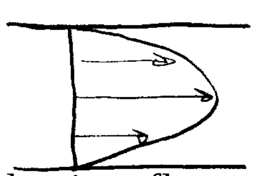
i.e., $\quad \alpha=\frac{1}{\mathrm{A}}\int_{\mathrm{A}}\left(\frac{\mathrm{V}}{\overline{\overline{\mathrm{V}}}}\right)^3\mathrm{dA}$ = kinetic energy correction factor

$$\dot{Q}-\dot{W}+\left(\frac{p_1}{\rho}+gz_1+\hat{u}_1+\alpha_1\frac{\overline{V}_1^2}{2}\right)\dot{m}=\left(\frac{p_2}{\rho}+gz_2+\hat{u}_2+\alpha_2\frac{\overline{V}_2^2}{2}\right)\dot{m}$$

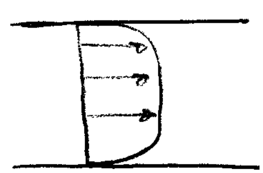
$$\frac{1}{\dot{m}}\left(\dot{Q}-\dot{W}\right)+\frac{p_1}{\rho}+gz_1+\hat{u}_1+\alpha_1\frac{\overline{V}_1^2}{2}=\frac{p_2}{\rho}+gz_2+\hat{u}_2+\alpha_2\frac{\overline{V}_2^2}{2}$$

Note that: $\quad \alpha$ = 1 if V is constant across the flow section

$\alpha$ > 1 if V is nonuniform



laminar flow $\alpha$ = 2



turbulent flow $\alpha$ = 1.05 ~ 1 may be used

## Shaft Work

Shaft work is usually the result of a turbine or a pump in the flow system. When a fluid passes through a turbine, the fluid is doing shaft work on the surroundings; on the other hand, a pump does work on the fluid

$$\dot{W}_s = \dot{W}_t - \dot{W}_p$$

where $\dot{W}_t$ and $\dot{W}_p$ are magnitudes of power $\left(\frac{\text{work}}{\text{time}}\right)$

Using this result in the energy equation and deviding by g results in

$$\underbrace{\frac{\dot{W}_p}{\dot{m}g} + \frac{p_1}{\gamma} + z_1 + \alpha_1\frac{\overline{V}_1^2}{2} = \frac{\dot{W}_t}{\dot{m}g} + \frac{p_2}{\gamma} + z_2 + \alpha_2\frac{\overline{V}_2^2}{2}}_{\text{mechanical part}} + \underbrace{\frac{\hat{u}_2 - \hat{u}_1}{g} - \frac{\dot{Q}}{\dot{m}g}}_{\text{thermal part}}$$

Note: each term has dimensions of length

Define the following:

$$h_p = \frac{\dot{W}_p}{\dot{m}g} = \frac{\dot{W}_p}{\rho Q g} = \frac{\dot{W}_p}{\gamma Q}$$

$$h_t = \frac{\dot{W}_t}{\dot{m}g}$$

$$h_L = \frac{\hat{u}_2 - \hat{u}_1}{g} - \frac{\dot{Q}}{\dot{m}g} = \textit{head loss}$$

## Head Loss

In a general fluid system a certain amount of mechanical energy is converted to thermal energy due to viscous action. This effect results in an increase in the fluid internal energy. Also, some heat will be generated through energy dissipation and be lost (i.e. $-\dot{Q}$). Therefore the term

from 2nd law

$$h_L = \frac{\hat{u}_2 - \hat{u}_1}{g} - \frac{\dot{Q}}{g\dot{m}} > 0$$

represents a loss in mechanical energy due to viscous stresses

Note that adding $\dot{Q}$ to system will not make $h_L = 0$ since this also increases $\Delta u$. It can be shown from 2nd law of thermodynamics that $h_L > 0$.

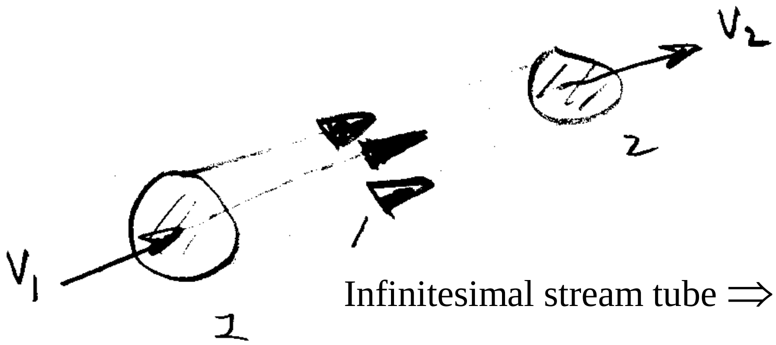
Drop — over $\overline{V}$ and understand that V in energy equation refers to average velocity.

Using the above definitions in the energy equation results in (steady 1-D incompressible flow)

$$\underbrace{\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_L}_{\text{form of energy equation used for this course!}}$$

## Comparison of Energy Equation and Bernoulli Equation

Apply energy equation to a stream tube without any shaft work




Infinitesimal stream tube $\Rightarrow \alpha_1=\alpha_2=1$

Energy eq : $$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2+h_L$$

• If $h_L = 0$ (i.e., $\mu = 0$) we get Bernoulli equation and conservation of mechanical energy along a streamline

• Therefore, energy equation for steady 1-D pipe flow can be interpreted as a modified Bernoulli equation to include viscous effects ($h_L$) and shaft work ($h_p$ or $h_t$)

## Summary of the Energy Equation

The energy equation is derived from RTT with

B = E = total energy of the system

$\beta$ = e = E/M = energy per unit mass

$$= \underbrace{\hat{u}}_{\text{internal}} + \underbrace{\frac{1}{2}V^2}_{\text{KE}} + \underbrace{gz}_{\text{PE}}$$

$$\frac{dE}{dt} = \frac{d}{dt}\int_{CV} \rho e d\forall + \int_{CS} \rho e \underline{V} \cdot \underline{dA} = \underbrace{\dot{Q}}_{\substack{\text{heat}\\ \text{add}}} - \underbrace{\dot{W}}_{\substack{\text{work}\\ \text{done}}}$$ from 1st Law of Thermodynamics

$$\dot{W} = \underbrace{\dot{W}_s}_{\substack{\text{shaft work}\\ \text{done on or}\\ \text{by system}\\ \text{(pump or}\\ \text{turbine)}}} + \underbrace{\dot{W}_p}_{\substack{\text{pressure}\\ \text{work done}\\ \text{on CS}}} + \underbrace{\overbrace{\cancel{\dot{W}_v}}^{\text{Neglected in text presentation}}}_{\substack{\text{Viscous stress}\\ \text{work on CS}}}$$

$$\dot{W}_p = \int_{CV} p\underline{V} \cdot \underline{dA} = \int_{CS} \rho(p/\rho)\underline{V} \cdot \underline{dA}$$

$$\dot{W}_s = \dot{W}_t - \dot{W}_p$$

$$\dot{Q} - \dot{W}_t + \dot{W}_p = \frac{d}{dt}\int_{CV} \rho e d\forall + \int_{CS} \rho(e + p/e)\underline{V} \cdot \underline{dA}$$

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

For steady 1-D pipe flow (one inlet and one outlet):

1) Streamlines are straight and parallel

$\Rightarrow$ p/ρ +gz = constant across CS

2) T = constant $\Rightarrow$ u = constant across CS

3) define $\alpha = \frac{1}{A}\int_{CS}\left(\frac{V}{\overline{\overline{V}}}\right)^3 dA$ = KE correction factor

$$\Rightarrow \quad \frac{\rho}{2}\int V^3 dA = \alpha\frac{\rho\overline{V}^3}{2}A = \alpha\frac{\overline{V}^2}{2}\dot{m}$$

$$\underbrace{\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1+h_p = \frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2+h_t}_{\text{mechanical energy}} + \underbrace{h_L}_{\text{Thermal energy}}$$

$h_p = \dot{W}_p / \dot{m}g$

$h_t = \dot{W}_t / \dot{m}g$

$h_L = \frac{\hat{u}_2 - \hat{u}_1}{g} - \frac{\dot{Q}}{\dot{m}g} =$ head loss

$> 0$ represents loss in mechanical energy due to viscosity

<u>Note: each term has</u> units of length

V is average velocity (vector dropped) and corrected by $\alpha$

# **Concept of Hydraulic and Energy Grade Lines**

$$\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2+h_t+h_L$$

Define $\left.\begin{aligned} \text{HGL} &= \frac{p}{\gamma}+z \\ \text{EGL} &= \frac{p}{\gamma}+z+\alpha\frac{V^2}{2g} \end{aligned}\right\}$ point-by-point application is graphically displayed

<u>HGL corresponds to pressure tap measurement + z</u>
<u>EGL corresponds to stagnation tube measurement + z</u>

$EGL_1 = EGL_2 + h_L$
for $h_p = h_t = 0$

EGL = HGL if V = 0

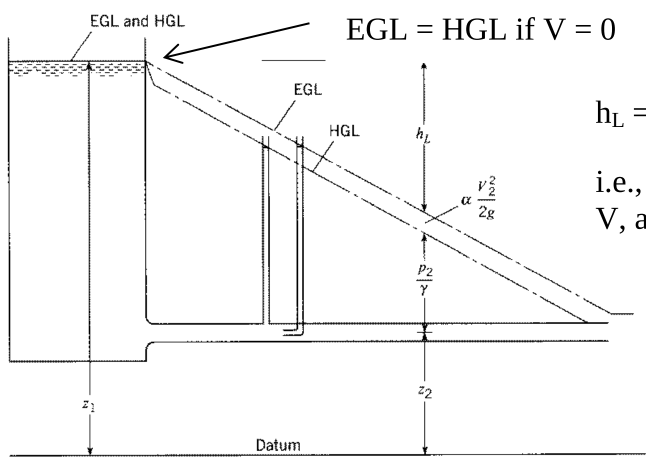



FIGURE 7.4
*EGL and HGL in a straight pipe.*

$h_L = f\frac{L}{D}\frac{V^2}{2g}$

i.e., linear variation in L for D, V, and f constant

f = friction factor
f = f(Re)

pressure tap: $\frac{p_2}{\gamma}=h$

stagnation tube: $\frac{p_2}{\gamma}+\alpha\frac{V_2^2}{2g}=h$

h = height of fluid in tap/tube

$EGL_1 + h_p = EGL_2 + h_t + h_L$

$EGL_2 = EGL_1 + \underbrace{h_p - h_t}_{\text{abrupt change due to } h_p \text{ or } h_t} - h_L$

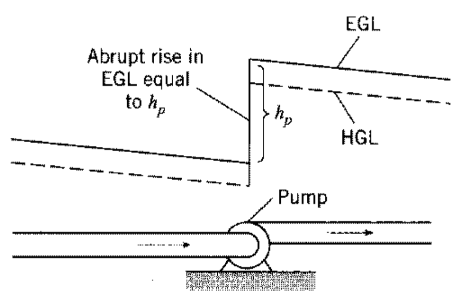
$h_L = f\frac{L}{D}\frac{V^2}{2g}$

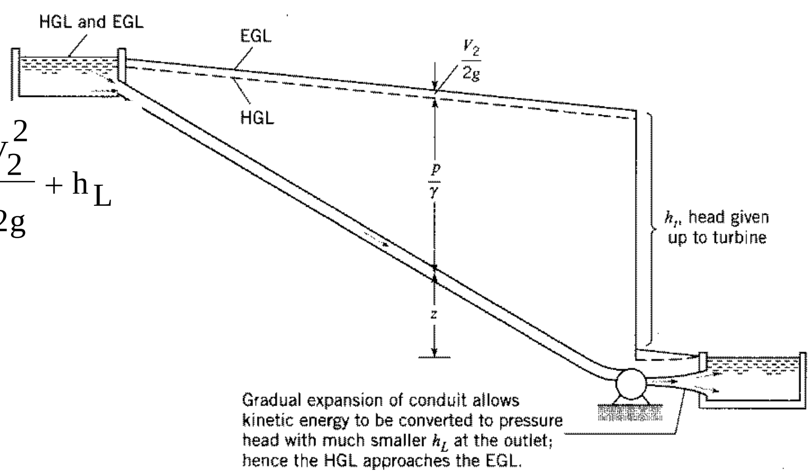
# Helpful hints for drawing HGL and EGL

1. EGL = HGL + $\alpha V^2/2g$ = HGL for V = 0

2.&3. $h_L = f\dfrac{L}{D}\dfrac{V^2}{2g}$ in pipe means EGL and HGL will slope downward, except for abrupt changes due to $h_t$ or $h_p$

FIGURE 7.5
*Rise in EGL and HGL due to pump.*




$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L$$

$HGL_2 = EGL_1 - h_L$

$$h_L = \frac{V^2}{2g} \text{ for abrupt expansion}$$

*Drop in EGL and HGL due to turbine.*

4. p = 0 ⇒ HGL = z

5. for $h_L = f\frac{L}{D}\frac{V^2}{2g}$ = constant × L

i.e., linearly increased for increasing L with slope $\frac{f}{D}\frac{V^2}{2g}$

EGL/HGL slope downward

6. for change in D ⇒ change in V

i.e.
$$V_1A_1 = V_2A_2$$
$$V_1\frac{\pi D_1^2}{4} = V_2\frac{\pi D_2^2}{4}$$
$$V_1D_1^2 = V_1D_2^2$$

⇒ change in distance between HGL & EGL and slope change due to change in $h_L$

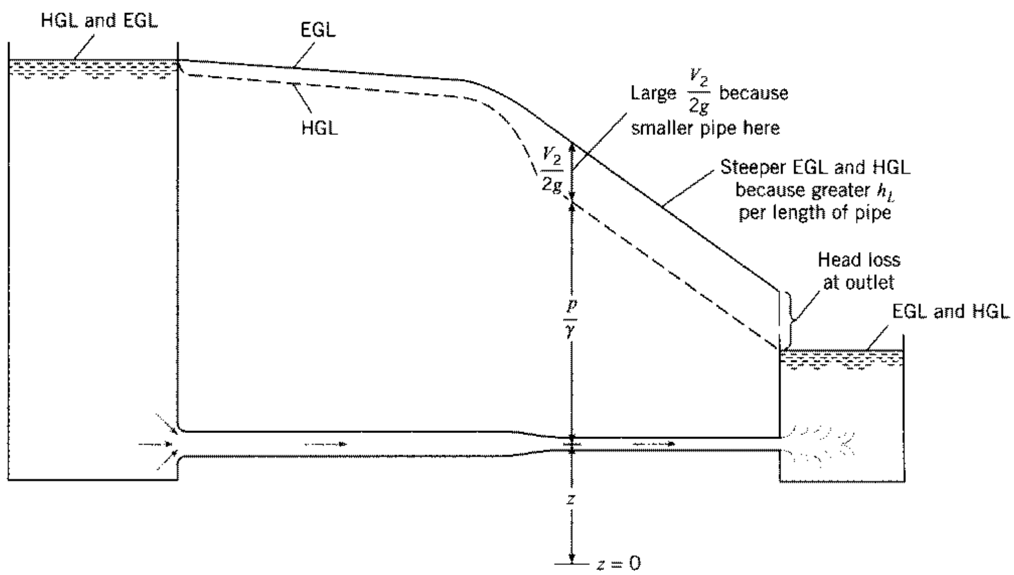



FIGURE 7.8
*Change in EGL and HGL due to change in diameter of pipe.*

7. If HGL < z then $p/\gamma < 0$ i.e., cavitation possible

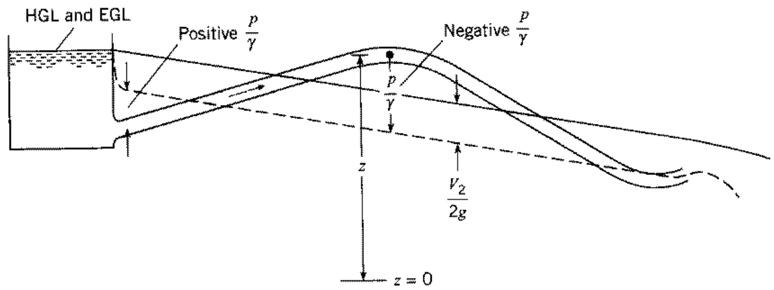



FIGURE 7.9
*Subatmospheric pressure when pipe is above HGL.*

condition for cavitation:

$$p = p_{va} = 2000\frac{N}{m^2}$$

gage pressure $p_{va,g} = p_A - p_{atm} \approx -p_{atm} = -100{,}000\frac{N}{m^2}$

$$\frac{p_{va,g}}{\gamma} \approx -10m$$

9810 N/m$^3$

## 4.15 METHOD OF SOLUTION OF FLOW PROBLEMS

For the solutions of problems of liquid flow there are two fundamental equations, the equation of continuity (3.10) and the energy equation in one of the forms from Eqs. (4.5) to (4.10). The following procedure may be employed:

1. Choose a datum plane through any convenient point.
2. Note at what sections the velocity is known or is to be assumed. If at any point the section area is great compared with its value elsewhere, the velocity head is so small that it may be disregarded.
3. Note at what points the pressure is known or is to be assumed. In a body of liquid at rest with a free surface the pressure is known at every point within the body. The pressure in a jet is the same as that of the medium surrounding the jet.
4. Note whether or not there is any point where all three terms, pressure, elevation, and velocity, are known.
5. Note whether or not there is any point where there is only one unknown quantity.

It is generally possible to write an energy equation that will fulfill conditions 4 and 5. If there are two unknowns in the equation, then the continuity equation must be used also. The application of these principles is shown in the following illustrative examples.

**Illustrative Example 4.7** A pipeline with a pump leads to a nozzle as shown in the accompanying figure. Find the flow rate when the pump develops a head of 80 ft. Assume that the head loss in the 6-in-diameter pipe may be expressed by $h_L = 5V_6^2/2g$, while the head loss in the 4-in-diameter pipe is $h_L = 12V_4^2/2g$. Sketch the energy line and hydraulic grade line, and find the pressure head at the suction side of the pump.

Select the datum as the elevation of the water surface in the reservoir. Note from continuity that

$$V_6 = (\tfrac{3}{6})^2 V_3 = 0.25V_3 \qquad \text{and} \qquad V_4 = (\tfrac{3}{4})^2 V_3 = 0.563V_3$$

where $V_3$ is the jet velocity. Writing an energy equation from the surface of the reservoir to the jet,

$$\left(z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g}\right) - h_{L_6} + h_p - h_{L_4} = z_3 + \frac{p_3}{\gamma} + \frac{V_3^2}{2g}$$

$$0 + 0 + 0 - 5\frac{V_6^2}{2g} + 80 - 12\frac{V_4^2}{2g} = 10 + 0 + \frac{V_3^2}{2g}$$

Express all velocities in terms of $V_3$:

$$-\frac{5(0.25V_3)^2}{2g} + 80 - 12\frac{(0.563V_3)^2}{2g} = 10 + \frac{V_3^2}{2g}$$

$$V_3 = 29.7 \text{ fps}$$

$$Q = A_3V_3 = \frac{\pi}{4}\left(\frac{3}{12}\right)^2 29.7 = 1.45 \text{ cfs}$$

Head loss in suction pipe:

$$h_L = 5\frac{V_6^2}{2g} = \frac{5(0.25V_3)^2}{2g} = \frac{0.312V_3^2}{2g}$$

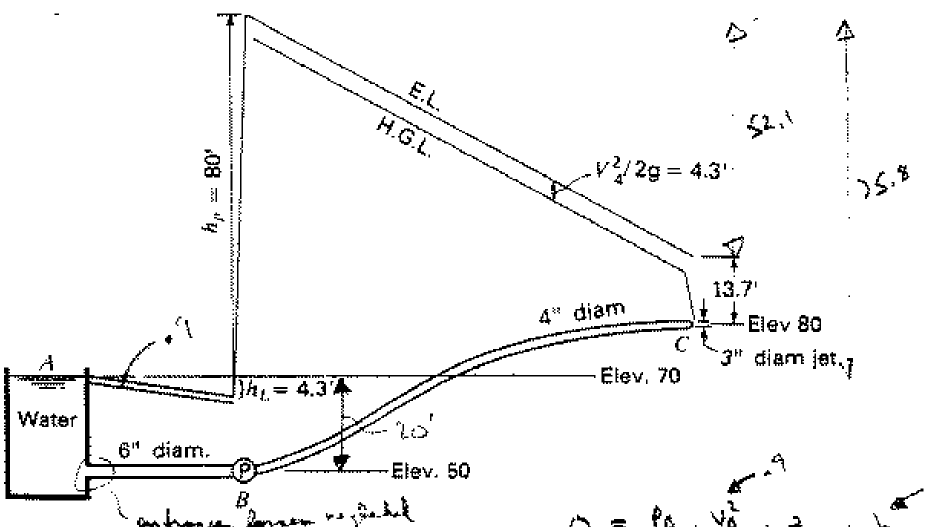
$$= 4.3 \text{ ft}$$

Head loss in discharge pipe:

$$h_L = 12\frac{V_4^2}{2g} = \frac{12(0.563V_3)^2}{2g} = 52.1 \text{ ft}$$

$$\frac{V_3^2}{2g} = 13.7 \text{ ft} \qquad \frac{V_4^2}{2g} = 4.3 \text{ ft} \qquad \frac{V_6^2}{2g} = 0.86 \text{ ft} \approx 0.9 \text{ ft}$$

The energy line and hydraulic grade line are drawn on the figure to scale. Inspection of the figure shows that the pressure head on the suction side of the pump is $p_B/\gamma = 14.8$ ft. Likewise, the pressure head at any point in the pipe may be found if the figure is to scale.



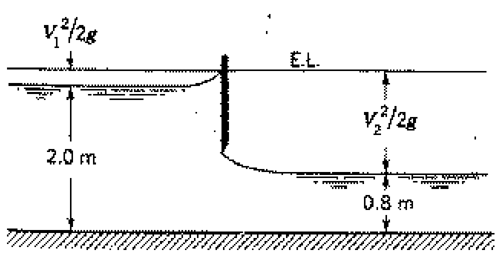

**Illustrative Example 4.7**

$$0 = \frac{p_B}{\gamma} + \frac{V_6^2}{2g} + z_B + h_L \Rightarrow \frac{p_B}{\gamma} = 14.8$$

.9 4.3 −20 $z_A$

**Illustrative Example 4.8** Given the two-dimensional flow as shown in the accompanying figure. Determine the flow rate. Assume no head loss.

across pump:




$$h_p + \frac{p_B}{\gamma} + \frac{V_6^2}{2g} + z_B = \frac{p}{\gamma} + \frac{V^2}{2g} + z + h_L$$

14.8 + .9 − 20

−4.3

75.7

**Illustrative Example 4.8**

# Application of the Energy, Momentum, and Continuity Equations in Combination

In general, when solving fluid mechanics problems, one should use all available equations in order to derive as much information as possible about the flow. For example, consistent with the approximation of the energy equation we can also apply the momentum and continuity equations

Energy:

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

Momentum:

$$\sum F_s = \rho V_2^2 A_2 - \rho V_1^2 A_1 = \rho Q (V_2 - V_1)$$

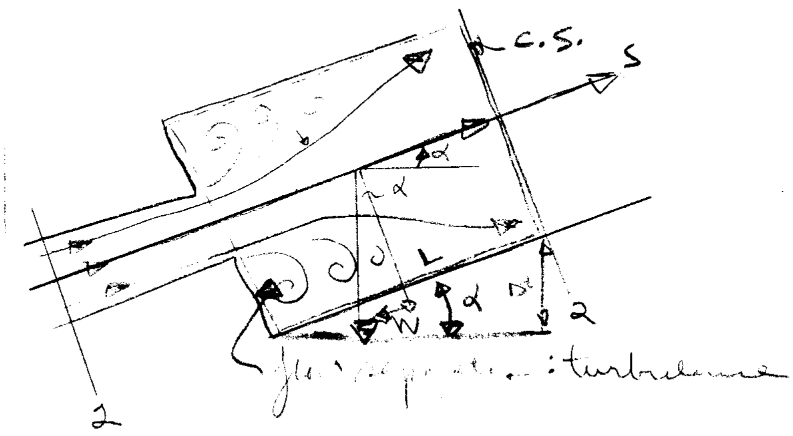
Continuity:

$$A_1V_1 = A_2V_2 = Q = \text{constant}$$

one inlet and
one outlet
$\rho$ = constant

Abrupt Expansion

Consider the flow from a small pipe to a larger pipe. Would like to know $h_L = h_L(V_1, V_2)$. Analytic solution to exact problem is extremely difficult due to the occurrence of flow separations and turbulence. However, if the assumption is made that the pressure in the separation region remains approximately constant and at the value at the point of separation, i.e, $p_1$, an approximate solution for $h_L$ is possible:




Apply Energy Eq from 1-2 ($\alpha_1 = \alpha_2 = 1$)

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L$$

Momentum eq. For CV shown (shear stress neglected)

$$\sum F_s = p_1 A_2 - p_2 A_2 - \underbrace{W \sin\alpha} = \sum \rho u \underline{V} \cdot \underline{A}$$

$$= \rho V_1(-V_1 A_1) + \rho V_2 (V_2 A_2)$$

$$\underbrace{\gamma A_2 L}_{W} \underbrace{\frac{\Delta z}{L}}_{\sin\alpha} \qquad = \rho V_2^2 A_2 - \rho V_1^2 A_1$$

next divide momentum equation by $\gamma A_2$

$$\div\ \gamma A_2 \qquad \underbrace{\frac{p_1}{\gamma}-\frac{p_2}{\gamma}-(z_1-z_2)}_{\text{from energy equation}}=\frac{V_2^2}{g}-\frac{V_1^2}{g}\frac{A_1}{A_2}=\frac{V_1^2}{g}\frac{A_1}{A_2}\left(\frac{A_1}{A_2}-1\right)$$

$$\Downarrow$$

$$\frac{V_2^2}{2g}-\frac{V_1^2}{2g}+h_L=\frac{V_2^2}{g}-\frac{V_1^2}{g}\frac{A_1}{A_2}$$

$$h_L=\frac{V_2^2}{2g}+\frac{V_1^2}{2g}\left(1-\frac{2A_1}{A_2}\right)$$

$$h_L=\frac{1}{2g}\left[V_2^2+V_1^2-\underbrace{2V_1^2\frac{A_1}{A_2}}_{-2V_1V_2}\right] \qquad \left\{\begin{array}{l}\text{continuity eq.}\\ V_1A_1=V_2A_2\\ \dfrac{A_1}{A_2}=\dfrac{V_2}{V_1}\end{array}\right.$$

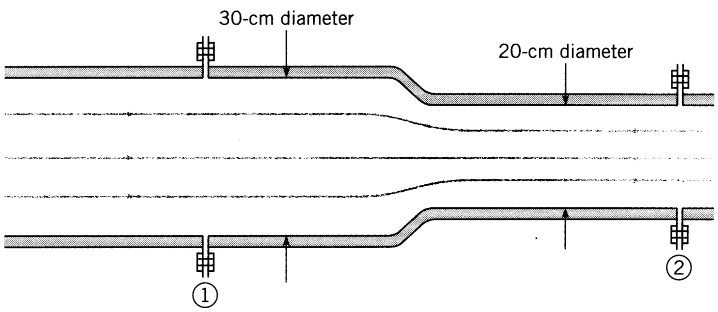
$$\boxed{h_L=\frac{1}{2g}\left[V_2-V_1\right]^2}$$

If $V_2 << V_1$,

$$\boxed{h_L=\frac{1}{2g}V_1^2}$$

# Forces on Transitions

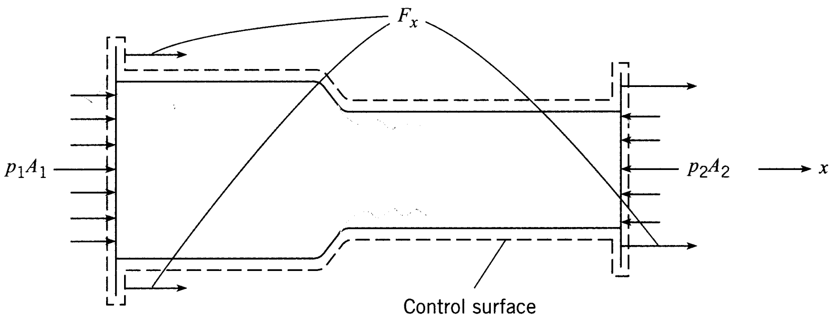
Example 7-6




$Q = .707\ m^3/s$

head loss $= .1\dfrac{V_2^2}{2g}$

(empirical equation)




Fluid = water
$p_1 = 250$ kPa
$D_1 = 30$ cm
$D_2 = 20$ cm
$F_x = ?$

First apply momentum theorem

$$\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$$

$$F_x + p_1A_1 - p_2A_2 = \rho V_1(-V_1A_1) + \rho V_2(V_2A_2)$$

$$F_x = \rho Q(V_2 - V_1) - p_1A_1 + p_2A_2$$

force required to hold transition in place

The only unknown in this equation is $p_2$, which can be obtained from the energy equation.

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + h_L$$

note: $z_1 = z_2$ and $\alpha = 1$

$$p_2 = p_1 - \gamma\left[\frac{V_2^2}{2g} - \frac{V_1^2}{2g} + h_L\right]$$

drop in pressure

$$\Rightarrow F_x = \rho Q(V_2 - V_1) + A_2 \underbrace{\left[p_1 - \gamma\left(\frac{V_2^2}{2g} - \frac{V_1^2}{2g} + h_L\right)\right]}_{p_2} - p_1 A_1$$

(note: if $p_2 = 0$ same as nozzle)

In this equation,

$V_1 = Q/A_1 = 10$ m/s
$V_2 = Q/A_2 = 22.5$ m/s

$$h_L = .1\frac{V_2^2}{2g} = 2.58\text{m}$$

continuity $A_1V_1 = A_2V_2$

$$V_2 = \frac{A_1}{A_2}V_1$$

i.e. $V_2 > V_1$

$F_x = -8.15$ kN is negative x direction to hold transition in place