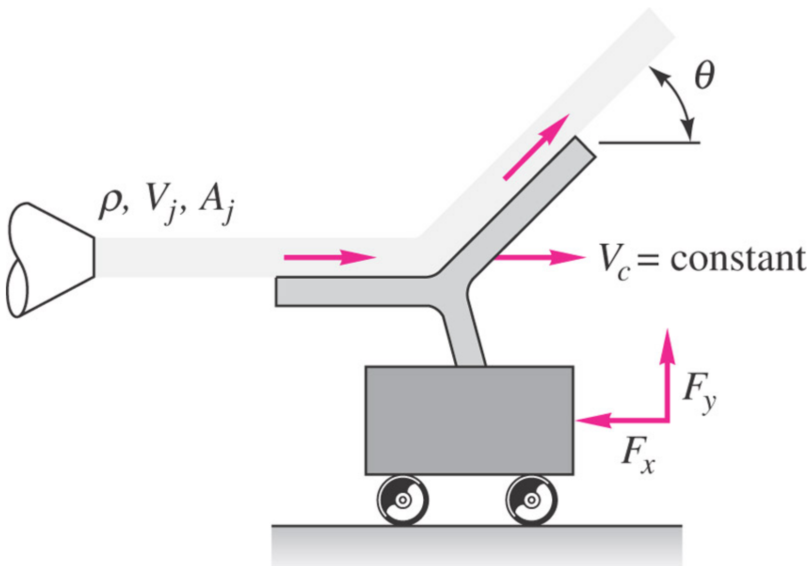
Ex) A jet strikes a vane which moves to the right at constant velocity V_C on a frictionless cart. Compute (a) the force F_x required to restrain the cart and (b) the power P delivered to the cart. Also find the cart velocity for which (c) the force F_x is a maximum and (d) the power P is a maximum.



Solution:

$$\sum \underline{F} = \frac{d}{dt}\int_{CV} \rho \underline{V} d\mathcal{V} + \int_{CS} \rho \underline{V}\, \underline{V_R} \cdot \underline{n} dA$$

where, $\underline{V_R} = \underline{V} - \underline{V_{CS}}$

Assume relative inertial coordinates with non-deforming CV i.e. CV moves at constant translational non-accelerating

$$\underline{V_{CS}} = u_{CS}\hat{\imath} + v_{CS}\hat{\jmath} + w_{CS}\hat{k} = V_C\hat{\imath}$$

as steady flow $\underline{V} \neq \underline{V}(t)$ with $\rho = constant$

Continuity:

$$0 = \frac{d}{dt}\int_{CV} \rho d\mathcal{V} + \int_{CS} \rho \underline{V_R} \cdot \underline{n} dA$$

$$0 = \rho \int_{CS} \underline{V_R} \cdot \underline{n} dA$$

Momentum:

$$\sum \underline{F} = \int_{CS} \rho \underline{V}\, \underline{V_R} \cdot \underline{n} dA$$

$$= \int_{CS} \rho \left(\underline{V_R} + \underline{V_{CS}}\right) \underline{V_R} \cdot \underline{n} dA$$

$$= \rho \int_{CS} \underline{V_R}\, \underline{V_R} \cdot \underline{n} dA + \rho \underline{V_{CS}} \int_{CS} \underline{V_R} \cdot \underline{n} dA$$

0

$$\sum \underline{F} = \rho \int_{CS} \underline{V_R}\, \underline{V_R} \cdot \underline{n} dA$$

$$\sum F_x = -F_x = -\rho \int_{CS} V_{R_x} \underline{V_R} \cdot \underline{n} dA$$

$$-F_x = \rho V_{R_x1}(-V_{R1}A_1) + \rho V_{R_x2}(V_{R2}A_2)$$

$$0 = \rho \int_{CS} \underline{V_R} \cdot \underline{n} dA$$

$$-\rho V_{R1}A_1 + \rho V_{R2}A_2 = 0$$

$$V_{R1}A_1 = V_{R2}A_2 = \underbrace{(V_j - V_C)}_{V_{R1}=V_{R_x1}=V_j-V_C} A_j$$

$$-F_x = \rho(V_j - V_C)[-(V_j - V_C)A_j] + \rho(V_j - V_C)\cos\theta\,(V_j - V_C)A_j$$

$$F_x = \rho(V_j - V_C)^2 A_j[1 - \cos\theta]$$

$$Power = V_C F_x = V_C \rho (V_j - V_C)^2 A_j(1 - \cos\theta)$$

$$F_{x_{max}} = \rho V_j^2 A_j(1 - \cos\theta), \qquad V_C = 0$$

$$P_{max} \Rightarrow \frac{dP}{dV_C} = 0$$

$$P = V_C \rho (V_j^2 - 2V_C V_j + V_C^2) A_j (1 - \cos\theta)$$

$$= \rho(V_j^2 V_C - 2V_C^2 V_j + V_C^3) A_j (1 - \cos\theta)$$

$$\frac{dP}{dV_C} = \rho(V_j^2 - 4V_C V_j + 3V_C^2) A_j (1 - \cos\theta) = 0$$

$$3V_C^2 - 4V_j V_C + V_j^2 = 0$$

$$V_C = \frac{+4V_j \pm \sqrt{16V_j^2 - 12V_j^2}}{6} = \frac{4V_j \pm 2V_j}{6}$$

$$V_C = \frac{V_j}{6}$$

$$P_{max} = \frac{V_j}{3} \rho \left(\frac{2V_j}{3} \right)^2 A_j (1 - \cos\theta)$$

$$= \frac{4}{27} V_j^3 \rho A_j (1 - \cos\theta)$$