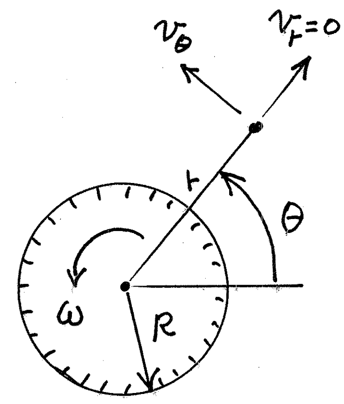
6.94

6.94 An infinitely long, solid, vertical cylinder of radius R is located in an infinite mass of an incompressible fluid. Start with the Navier–Stokes equation in the θ direction and derive an expression for the velocity distribution for the steady flow case in which the cylinder is rotating about a fixed axis with a constant angular velocity ω. You need not consider body forces. Assume that the flow is axisymmetric and the fluid is at rest at infinity.



For this flow field, $v_r = 0$, $v_z = 0$, and from the continuity equation,

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad (\text{Eq. } 6.35)$$

it follows that

$$\frac{\partial v_\theta}{\partial \theta} = 0 \qquad (\text{See figure for notation.})$$

Thus, the Navier-Stokes equation in the θ-direction (Eq. 6.128b) for steady flow reduces to

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2}\right]$$

Due to the symmetry of the flow,

$$\frac{\partial p}{\partial \theta} = 0$$

so that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = 0$$

or

$$\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} = 0 \qquad (1)$$

Since v_θ is a function of only r, Eq. (1) can be expressed as an ordinary differential equation, and re-written as

$$\frac{d^2 v_\theta}{dr^2} + \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0 \qquad (2)$$

Equation (2) can be integrated to yield

$$\frac{d v_\theta}{dr} + \frac{v_\theta}{r} = C_1$$

or

$$r\frac{d v_\theta}{dr} + v_\theta = C_1 r \qquad (3)$$

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Equation (3) can be expressed as

$$\frac{d(rv_\theta)}{dr} = c_1 r$$

and a second integration yields

$$r v_\theta = \frac{c_1 r^2}{2} + c_2$$

or

$$v_\theta = \frac{c_1 r}{2} + \frac{c_2}{r}$$

As $r \to \infty$, $v_\theta \to 0$, (since fluid is at rest at infinity) so that $c_1 = 0$. Thus,

$$v_\theta = \frac{c_2}{r}$$

and since at $r = R$, $v_\theta = R\omega$, it follows that $c_2 = R^2\omega$ and

$$\underline{\underline{v_\theta = \frac{R^2\omega}{r}}}$$