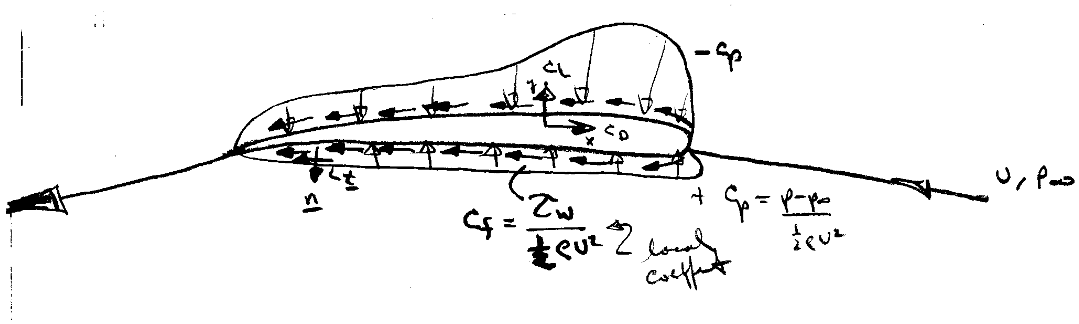
Chapter 9: Surface Resistance

9.1   Introduction: drag and lift on immersed bodies



$$C_D = \underbrace{\frac{1}{\frac{1}{2}\rho V^2 A}\left\{\int_S (p - p_\infty)\underline{n}\cdot\hat{i}\,dA}_{C_{form}\text{ (form drag)}} + \underbrace{\int_S \tau_w \underline{t}\cdot\hat{i}\,dA\right\}}_{C_f\text{ (skin friction drag)}} = \text{drag coefficient}$$

$$C_L = \frac{1}{\frac{1}{2}\rho V^2 A}\left\{\int (p - p_\infty)\underline{n}\cdot\hat{j}\,dA\right\} = \text{lift coefficient}$$

For inviscid fluid, $C_D = 0$ since both the form and skin friction components are zero, which is known as D'Alembert paradox. However, $C_L \neq 0$ and can often be predicted accurately with ideal-flow theory.

In general,

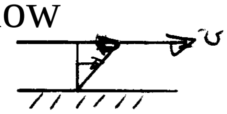
low Re ≤ 1: $C_f > C_{form}$ Stoke's flow (lab 1)

med & high Re: 1. $C_f >> C_{form}$
$t/c << 1$ i.e., streamlined body
2. $C_{form} >> C_f$
$t/c \sim 1$ i.e., bluff body

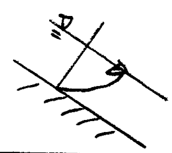
1. = subject of this chapter
2. = subject of chapter 11 along with C_L

Topics of Chapter 9

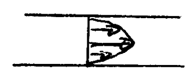
9.2 Surface Resistance with Uniform Laminar Flow
1. parallel plates (internal flow)



2. flow down an inclined plane (open channel flow)



3. parallel plates with pressure gradient (internal flow)



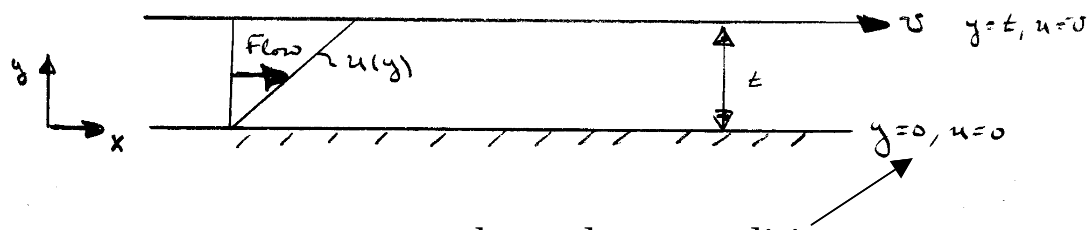
In this class:
1. parallel plates
2. extend as per 3. including inclined flow
3. flow down an inclined plane

external flow:
9.3 Boundary Layer Flow
9.4 Laminar
9.5 Turbulent
9.6 Transition control (brief comments)

9.2 Surface Resistance with Uniform Laminar Flow

We now discuss a couple of exact solutions to the Navier-Stokes equations. Although all known exact solutions (about 80) are for highly simplified geometries and flow conditions, they are very valuable as an aid to our understanding of the character of the NS equations and their solutions. Actually the examples to be discussed are for internal flow (Chapter 10) and open channel flow (Chapter 15), but they serve to underscore and display viscous flow. Finally, the derivations to follow utilize differential analysis. See the text for derivations using CV analysis.

1. Couette Flow



boundary conditions

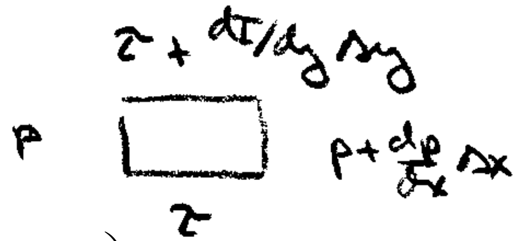
First, consider flow due to the relative motion of two parallel plates

$$\left.\begin{array}{ll} \text{Continuity} & \dfrac{\partial u}{\partial x} = 0 \\ \\ \text{Momentum} & 0 = \mu \dfrac{d^2 u}{dy^2} \end{array}\right\} \quad \begin{array}{l} u = u(y) \\ v = o \\ \dfrac{\partial p}{\partial x} = \dfrac{\partial p}{\partial y} = 0 \end{array}$$

or by CV continuity and momentum equations:

$$\rho u_1 \Delta y = \rho u_2 \Delta y$$

$$u_1 = u_2$$



$$\sum F_x = \sum u \rho \underline{V} \cdot d\underline{A} = \rho Q (u_2 - u_1) = 0$$

$$= p\Delta y - \left(p + \frac{dp}{dx}\Delta x \right)\Delta y - \tau \Delta x + \left(\tau + \frac{d\tau}{dy} dy \right)\Delta x = 0$$

$$\frac{d\tau}{dy} = 0$$

i.e. $$\frac{d}{dy}\left(\mu \frac{du}{dy} \right) = 0$$

$$\mu \frac{d^2 u}{dy^2} = 0$$

from momentum equation

$$\mu \frac{du}{dy} = C$$

$$u = \frac{C}{\mu} y + D$$

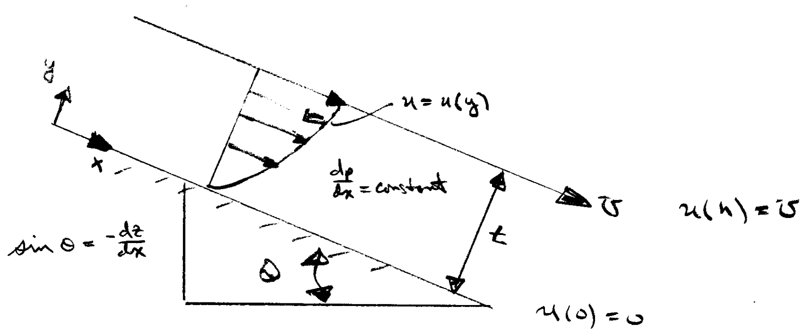
$$u(0) = 0 \Rightarrow D = 0$$

$$u(t) = U \Rightarrow C = \mu \frac{U}{t}$$

$$u = \frac{U}{t} y$$

$$\tau = \mu \frac{du}{dy} = \frac{\mu U}{t} = \text{constant}$$

2. Generalization for inclined flow with a constant pressure gradient



Continutity $$\frac{\partial u}{\partial x} = 0$$

Momentum $$0 = -\frac{\partial}{\partial x}(p + \gamma z) + \mu \frac{d^2 u}{dy^2}$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad u = u(y), \quad v = o, \quad \frac{\partial p}{\partial y} = 0$$

i.e., $$\mu \frac{d^2 u}{dy^2} = \gamma \frac{dh}{dx}$$ h = p/γ +z = constant

plates horizontal $\frac{dz}{dx} = 0$

plates vertical $\frac{dz}{dx}$=-1

which can be integrated twice to yield

$$\mu \frac{du}{dy} = \gamma \frac{dh}{dx} y + A$$

$$\mu u = \gamma \frac{dh}{dx} \frac{y^2}{2} + Ay + B$$

now apply boundary conditions to determine A and B

$$u(y=0)=0 \quad \Rightarrow \quad B=0$$
$$u(y=t)=U$$

$$\mu U=\gamma\frac{dh}{dx}\frac{t^2}{2}+At \Rightarrow A=\frac{\mu U}{t}-\gamma\frac{dh}{dx}\frac{t}{2}$$

$$u(y)=\frac{\gamma}{\mu}\frac{dh}{dx}\frac{y^2}{2}+\frac{1}{\mu}\left[\frac{\mu U}{t}-\gamma\frac{dh}{dx}\frac{t}{2}\right]$$

$$=-\frac{\gamma}{2\mu}\frac{dh}{dx}\left(ty-y^2\right)+\frac{U}{t}y$$

This equation can be put in non-dimensional form:

$$\frac{u}{U}=-\frac{\gamma t^2}{2\mu U}\frac{dh}{dx}\left(1-\frac{y}{t}\right)\frac{y}{t}+\frac{y}{t}$$

define: P = non-dimensional pressure gradient

$$=-\frac{\gamma t^2}{2\mu U}\frac{dh}{dx}$$

$$h=\frac{p}{\gamma}+z$$

$$=-\frac{\gamma z^2}{2\mu U}\left[\frac{1}{\gamma}\frac{dp}{dx}+\frac{dz}{dx}\right]$$

$$Y = y/t$$

$$\Rightarrow \quad \frac{u}{U}=P\cdot Y(1-Y)+Y$$

parabolic velocity profile

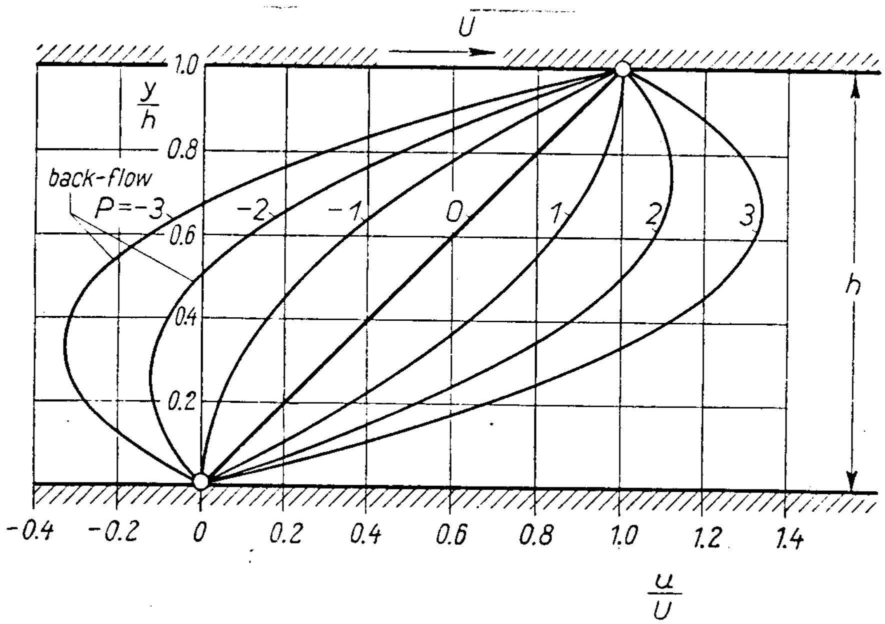


Fig. 5.2. Couette flow between two parallel flat walls
$P > 0$, pressure decrease in direction of wall motion; $P < 0$, pressure increase; $P = 0$, zero pressure gradient

$$\frac{u}{U} = \underbrace{\frac{Py}{t} - \frac{Py^2}{t^2} + \frac{y}{t}}$$

$$q = \int_0^t u\,dy$$

$$\bar{u} = \frac{q}{t} = \frac{\int_0^t U[\quad]dy}{t}$$

$$\frac{t\bar{u}}{U} = \int_0^t \left[\frac{P}{t}y - \frac{P}{t^2}y^2 + \frac{y}{t}\right]dy$$

$$= \frac{Pt}{2} - \frac{Pt}{3} + \frac{t}{2}$$

$$\frac{\bar{u}}{U} = \frac{P}{6} + \frac{1}{2} \Rightarrow \bar{u} = \frac{t^2}{12\mu}\left(-\gamma\frac{dh}{dx}\right) + \frac{U}{2}$$

$$\boxed{\text{For laminar flow } \frac{\bar{u}t}{\nu} < 1000} \text{——} Re_{crit} \sim 1000$$

The maximum velocity occurs at the value of y for which:

$$\frac{du}{dy} = 0 \qquad \frac{d}{dy}\left(\frac{u}{U}\right) = 0 = \frac{P}{t} - \frac{2P}{t^2}y + \frac{1}{t}$$

$$\Rightarrow y = \frac{t}{2P}(P+1) = \frac{t}{2} + \frac{t}{2P} \text{ @ } u_{max}$$

$$\boxed{\text{for } U = 0,\ y = t/2}$$

$$\therefore\ u_{max} = u(y_{max}) = \frac{UP}{4} + \frac{U}{2} + \frac{U}{4P}$$

note: if U = 0: $\frac{\bar{u}}{u_{max}} = \frac{P}{6}\Big/\frac{P}{4} = \frac{2}{3}$

The shape of the velocity profile u(y) depends on P:

1. If P > 0, i.e., $\frac{dh}{dx} < 0$ the pressure decreases in the direction of flow (favorable pressure gradient) and the velocity is positive over the entire width

$$\gamma \frac{dh}{dx} = \gamma \frac{d}{dx}\left(\frac{p}{\gamma} + z\right) = \frac{dp}{dx} - \gamma \sin\theta$$

a) $\frac{dp}{dx} < 0$

b) $\frac{dp}{dx} < \gamma \sin\theta$

2. If P < 0, i.e., $\frac{dh}{dx} > 0$ the pressure increases in the direction of flow (adverse pressure gradient) and the velocity over a portion of the width can become negative (backflow) near the stationary wall. In this case the dragging action of the faster layers exerted on the fluid particles near the stationary wall is insufficient to over come the influence of the adverse pressure gradient

$$\frac{dp}{dx} - \gamma \sin\theta > 0$$

$$\frac{dp}{dx} > \gamma \sin\theta \qquad \text{or} \qquad \gamma \sin\theta < \frac{dp}{dx}$$

3. If P = 0, i.e., $\frac{dh}{dx} = 0$ the velocity profile is linear

$$u = \frac{U}{t} y$$

a) $\frac{dp}{dx} = 0$ and $\theta = 0$

Note: we derived this special case

b) $\frac{dp}{dx} = \gamma \sin\theta$

For U = 0 the form $\frac{u}{U} = PY(1 - Y) + Y$ is not appropriate

$$u = UPY(1\text{-}Y)\text{+}UY$$

$$= -\frac{\gamma t^2}{2\mu} \frac{dh}{dx} Y(1 - Y) + UY$$

Now let U = 0: $u = -\frac{\gamma t^2}{2\mu} \frac{dh}{dx} Y(1 - Y)$

3. Shear stress distribution

Non-dimensional velocity distribution

$$u^* = \frac{u}{U} = P \cdot Y(1-Y) + Y$$

where $u^* \equiv \frac{u}{U}$ is the non-dimensional velocity,

$P \equiv -\frac{\gamma t^2}{2\mu U}\frac{dh}{dx}$ is the non-dimensional pressure gradient

$Y \equiv \frac{y}{t}$ is the non-dimensional coordinate.

Shear stress

$$\tau = \mu \frac{du}{dy}$$

In order to see the effect of pressure gradient on shear stress using the non-dimensional velocity distribution, we define the non-dimensional shear stress:

$$\tau^* = \frac{\tau}{\frac{1}{2}\rho U^2}$$

Then

$$\begin{aligned}
\tau^* &= \frac{1}{\frac{1}{2}\rho U^2}\mu \frac{Ud(u/U)}{td(y/t)} = \frac{2\mu}{\rho U t}\frac{du^*}{dY} \\
&= \frac{2\mu}{\rho U t}(-2PY + P + 1) \\
&= \frac{2\mu}{\rho U t}(-2PY + P + 1) \\
&= A(-2PY + P + 1)
\end{aligned}$$

where $A \equiv \frac{2\mu}{\rho U t} > 0$ is a positive constant.

So the shear stress always varies linearly with Y across any section.

At the lower wall $(Y = 0)$:

$$\tau_{lw}^{*} = A(1 + P)$$

At the upper wall $(Y = 1)$:

$$\tau_{uw}^{*} = A(1 - P)$$

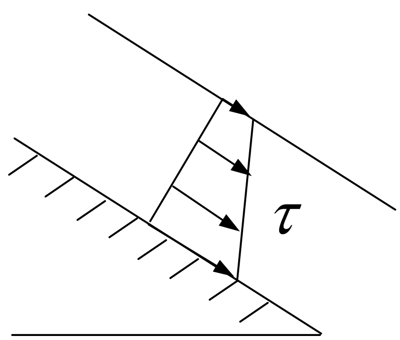
For favorable pressure gradient, the lower wall shear stress is always positive:

1. For small favorable pressure gradient $(0 < P < 1)$:

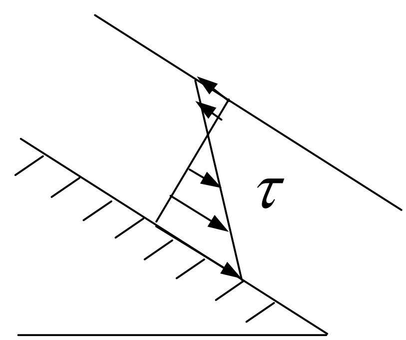
$$\tau_{lw}^{*} > 0 \text{ and } \tau_{uw}^{*} > 0$$

2. For large favorable pressure gradient $(P > 1)$:

$$\tau_{lw}^{*} > 0 \text{ and } \tau_{uw}^{*} < 0$$



$(0 < P < 1)$



$(P > 1)$

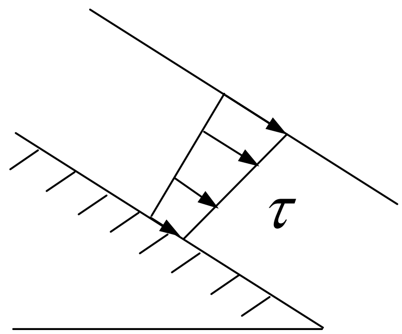
For adverse pressure gradient, the upper wall shear stress is always positive:

1. For small adverse pressure gradient $(-1 < P < 0)$:

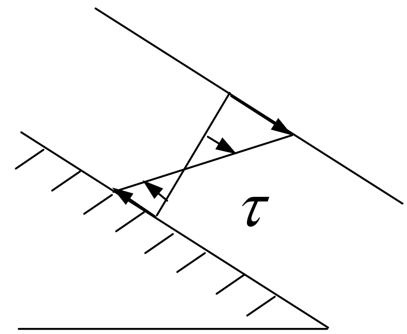
$$\tau_{lw}^{*} > 0 \text{ and } \tau_{uw}^{*} > 0$$

2. For large adverse pressure gradient $(P < -1)$:

$$\tau_{lw}^{*} < 0 \text{ and } \tau_{uw}^{*} > 0$$



$(-1 < P < 0)$



$(P < -1)$

For $U = 0$, i.e., channel flow, the above non-dimensional form of velocity profile is not appropriate. Let's use dimensional form:

$$u = -\frac{\gamma t^2}{2\mu}\frac{dh}{dx}Y(1-Y) = -\frac{\gamma}{2\mu}\frac{dh}{dx}y(t-y)$$

Thus the fluid always flows in the direction of decreasing piezometric pressure or piezometric head because $\frac{\gamma}{2\mu} > 0, \quad y > 0$ and $t - y > 0$. So if $\frac{dh}{dx}$ is negative, u is positive; if $\frac{dh}{dx}$ is positive, u is negative.

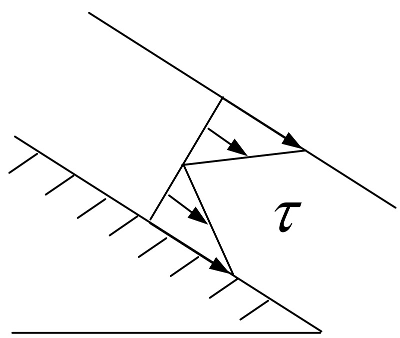
Shear stress:

$$\tau = \mu\frac{du}{dy} = -\frac{\gamma}{2}\frac{dh}{dx}\left(t - \frac{1}{2}y\right)$$

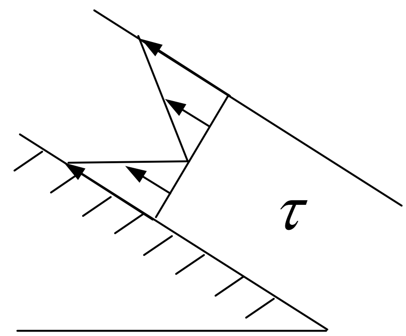
Since $\left(t - \frac{1}{2}y\right) > 0$, the sign of shear stress τ is always opposite to the sign of piezometric pressure gradient $\frac{dh}{dx}$, and the magnitude of τ is always maximum at both walls and zero at centerline of the channel.

For favorable pressure gradient, $\frac{dh}{dx} < 0$, $\tau > 0$

For adverse pressure gradient, $\frac{dh}{dx} > 0$, $\tau < 0$

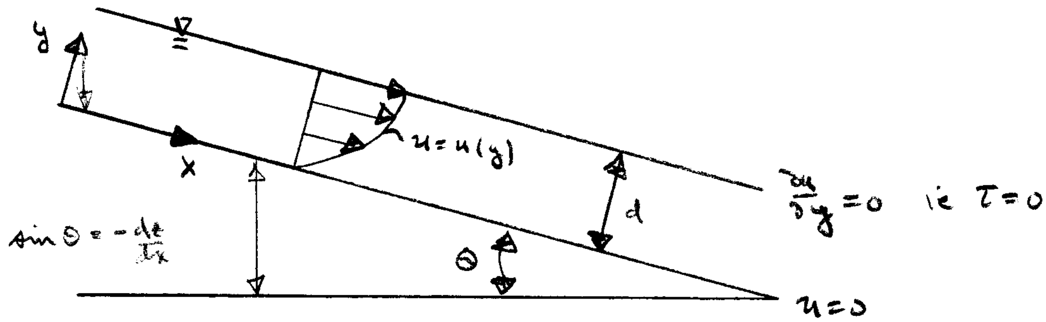


$\frac{dh}{dx} < 0$



$\frac{dh}{dx} > 0$

Flow down an inclined plane



uniform flow $\Rightarrow$ velocity and depth do not change in x-direction

Continuity $\quad \frac{du}{dx} = 0$

x-momentum $\quad 0 = -\frac{\partial}{\partial x}(p + \gamma z) + \mu \frac{d^2 u}{dy^2}$

y-momentum $\quad 0 = -\frac{\partial}{\partial y}(p + \gamma z) \Rightarrow$ hydrostatic pressure variation

$$\Rightarrow \frac{dp}{dx} = 0$$

$$\mu \frac{d^2 u}{dy^2} = -\gamma \sin\theta$$

$$\frac{du}{dy} = -\frac{\gamma}{\mu} \sin\theta y + c$$

$$u = -\frac{\gamma}{\mu}\sin\theta\frac{y^2}{2} + Cy + D$$

$$\left.\frac{du}{dy}\right|_{y=d} = 0 = -\frac{\gamma}{\mu}\sin\theta d + c \Rightarrow c = +\frac{\gamma}{\mu}\sin\theta d$$

$$u(0) = 0 \Rightarrow D = 0$$

$$u = -\frac{\gamma}{\mu}\sin\theta\frac{y^2}{2} + \frac{\gamma}{\mu}\sin\theta dy$$

$$= \frac{\gamma}{2\mu}\sin\theta\, y(2d - y)$$

$$u(y) = \frac{g\sin\theta}{2\nu}y(2d - y)$$

$$q = \int_0^d u\,dy = \frac{\gamma}{2\mu}\sin\theta\left[dy^2 - \frac{y^3}{3}\right]_0^d$$

discharge per unit width

$$= \frac{1}{3}\frac{\gamma}{\mu}d^3\sin\theta$$

$$\overline{V}_{avg} = \frac{q}{d} = \frac{1}{3}\frac{\gamma}{\mu}d^2\sin\theta = \frac{gd^2}{3\nu}\sin\theta$$

in terms of the slope $S_o = \tan\theta \sim \sin\theta$

$$\overline{V} = \frac{gd^2 S_o}{3\nu}$$

Exp. show $Re_{crit} \sim 500$, i.e., for $Re \geq 500$ the flow will become turbulent

$$\frac{\partial p}{\partial y} = -\gamma\cos\theta \qquad\qquad Re_{crit} = \frac{\overline{V}d}{\nu} \sim 500$$

$$p = -\gamma\cos\theta\, y + C$$

$$p(d) = p_o = -\gamma\cos\theta\, d + C$$

i.e., $$p = \gamma\cos\theta(d - y) + p_o$$

* $p(d) > p_o$

* if $\theta = 0$ $p = \gamma(d - y) + p_o$
entire weight of fluid imposed

if $\theta = \pi/2$ $p = p_o$
no pressure change through the fluid

9.3 Qualitative Description of the Boundary Layer

Recall our previous description of the flow-field regions for high Re flow about slender bodies

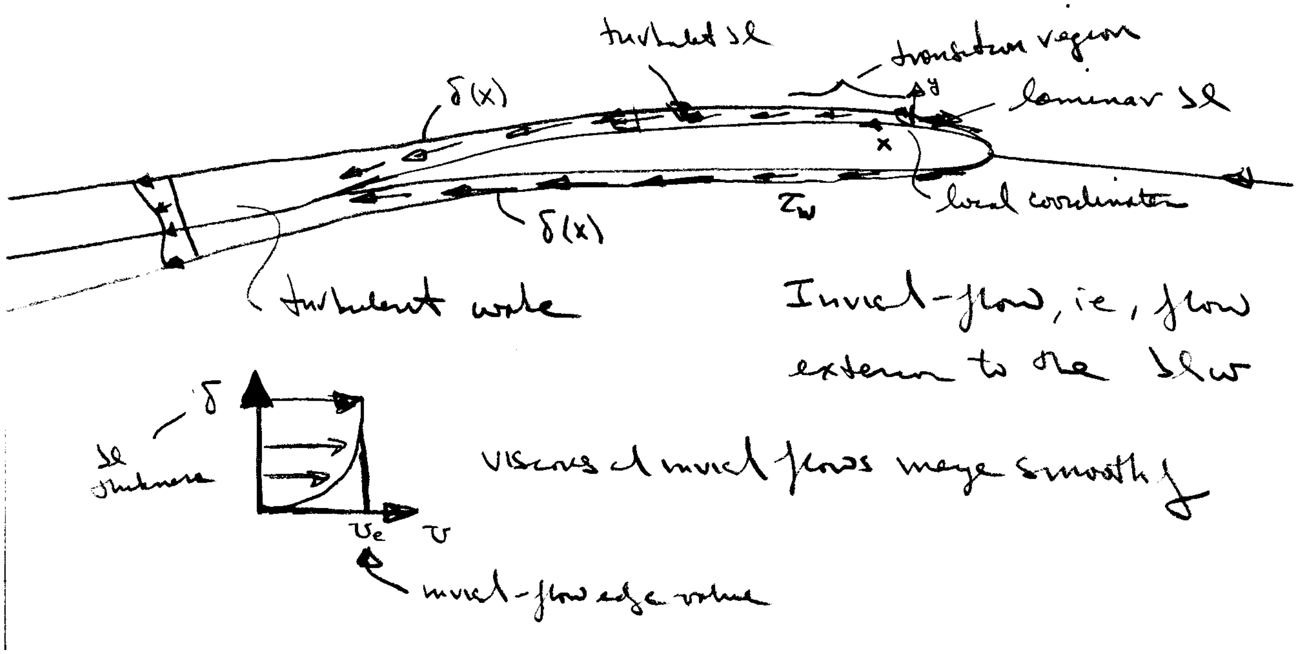
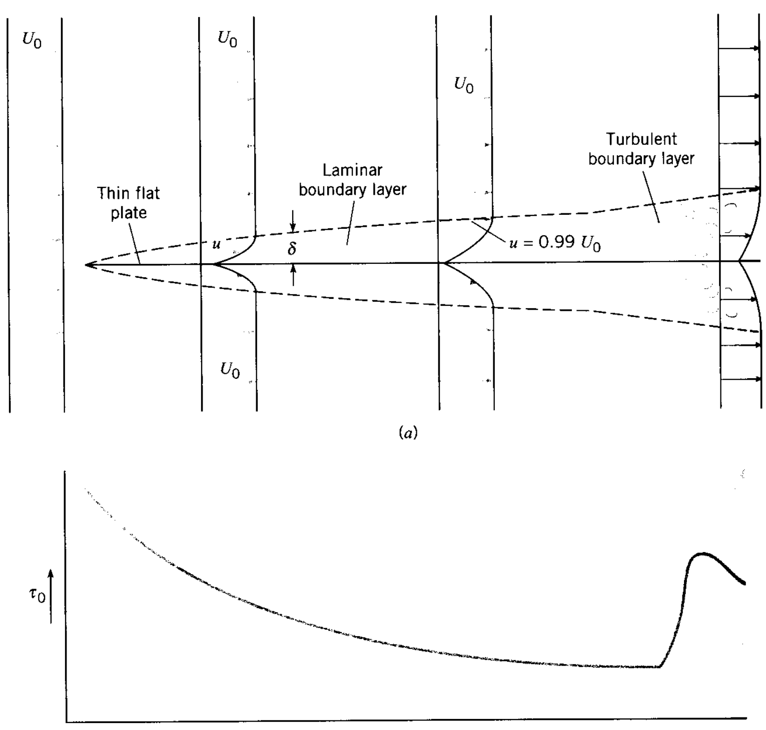


FIGURE 9.4
Development of boundary layer and distribution of shear stress along a thin, flat plate. (a) Flow pattern in boundary layers above and below the plate. (b) Shear-stress distribution on either side of the plate.



τ_w = shear stress

$\tau_w \propto$ rate of strain (velocity gradient)

$$= \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

large near the surface where fluid undergoes large changes to satisfy the no-slip condition

Boundary layer theory is a simplified form of the complete NS equations and provides τ_w as well as a means of estimating C_{form}. Formally, boundary-layer theory represents the asymptotic form of the Navier-Stokes equations for high Re flow about slender bodies. As mentioned before, the NS equations are 2nd order nonlinear PDE and their solutions represent a formidable challenge. Thus, simplified forms have proven to be very useful.

Near the turn of the century (1904), Prandtl put forth boundary-layer theory, which resolved D'Alembert's paradox. As mentioned previously, boundary-layer theory represents the asymptotic form of the NS equations for high Re flow about slender bodies. The latter requirement is necessary since the theory is restricted to unseparated flow. In fact, the boundary-layer equations are singular at separation, and thus, provide no information at or beyond separation. However, the requirements of the theory are met in many practical situations and the theory has many times over proven to be invaluable to modern engineering.

The assumptions of the theory are as follows:

Variable	order of magnitude		
u	U	$O(1)$	
v	$\delta << L$	$O(\varepsilon)$	$\varepsilon = \delta/L$
$\frac{\partial}{\partial x}$	L	$O(1)$	
$\frac{\partial}{\partial y}$	$1/\delta$	$O(\varepsilon^{-1})$	
ν	δ^2	ε^2	

The theory assumes that viscous effects are confined to a thin layer close to the surface within which there is a dominant flow direction (x) such that u ~ U and v << u. However, gradients across δ are very large in order to satisfy the no slip condition.

Next, we apply the above order of magnitude estimates to the NS equations.

$$
\left.
\begin{aligned}
& u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
& 1 \quad 1 \qquad \varepsilon \quad \varepsilon^{-1} \qquad\qquad \varepsilon^{2} \quad 1 \qquad \varepsilon^{-2} \\[1em]
& u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right) \\
& 1 \quad \varepsilon \qquad \varepsilon \quad 1 \qquad\qquad \varepsilon^{2} \quad 1 \qquad \varepsilon^{-1} \\[1em]
& \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
& 1 \qquad 1
\end{aligned}
\right\} \text{elliptic}
$$

Retaining terms of O(1) only results in the celebrated boundary-layer equations

$$
\left.
\begin{aligned}
& u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \\
& \frac{\partial p}{\partial y} = 0 \\
& \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\end{aligned}
\right\} \text{parabolic}
$$

Some important aspects of the boundary-layer equations:

1) the y-momentum equation reduces to

$$\frac{\partial p}{\partial y} = 0$$

i.e., $p = p_e$ = constant across the boundary layer

edge value, i.e., inviscid flow value!

from the Bernoulli equation:

$$p_e + \frac{1}{2}\rho U_e^2 = \text{constant}$$

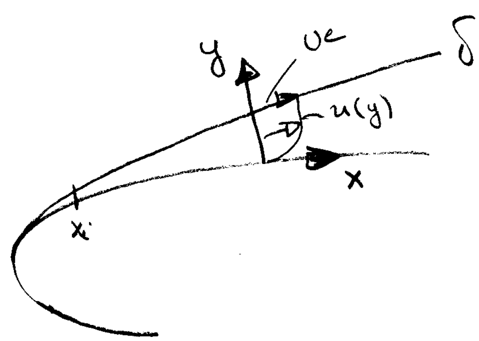
i.e., $\frac{\partial p_e}{\partial x} = -\rho U_e \frac{\partial U_e}{\partial x}$

Thus, the boundary-layer equations are solved subject to a specified inviscid pressure distribution

2) continuity equation is unaffected

3) Although NS equations are fully elliptic, the boundary-layer equations are parabolic and can be solved using marching techniques

4) Boundary conditions



$u = v = 0 \qquad y = 0$

$u = U_e \qquad y = \delta$

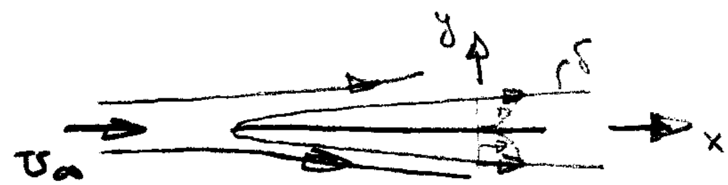
+ appropriate initial conditions @ x_i

There are quite a few analytic solutions to the boundary-layer equations. Also numerical techniques are available for arbitrary geometries, including both two- and three-dimensional flows. Here, as an example, we consider the simple, but extremely important case of the boundary layer development over a flat plate.

9.4 Quantitative Relations for the Laminar Boundary Layer

Laminar boundary-layer over a flat plate: Blasius solution (1908) student of Prandtl

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$



Note: $\frac{\partial p}{\partial x} = 0$ for a flat plate

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$u = v = 0 \quad @\ y = 0 \qquad\qquad u = U_\infty \quad @\ y = \delta$$

We now introduce a dimensionless transverse coordinate and a stream function, i.e.,

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \propto \frac{y}{\delta}$$

$$\psi = \sqrt{\nu x U_\infty}\, f(\eta)$$

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial y} = U_\infty f'(\eta) \qquad f' = u/U_\infty$$

$$v = -\frac{\partial\psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}(\eta f' - f)$$

substitution into the boundary-layer equations yields

$$ff'' + 2f''' = 0 \qquad \text{Blasius Equation}$$

$$f = f' = 0 \quad @\ \eta = 0 \qquad f' = 1 \quad @\ \eta = 1$$

The Blasius equation is a 3rd order ODE which can be solved by standard methods (Runge-Kutta). Also, series solutions are possible. Interestingly, although simple in appearance no analytic solution has yet been found. Finally, it should be recognized that the Blasius solution is a similarity solution, i.e., the non-dimensional velocity profile f' vs. η is independent of x. That is, by suitably scaling all the velocity profiles have neatly collapsed onto a single curve.

Now, lets consider the characteristics of the Blasius solution:

$$\frac{u}{U_\infty} \text{ vs. } y$$

$$\frac{v}{U_\infty}\sqrt{\frac{U_\infty}{V}} \text{ vs. } y$$

FIGURE 9.5
Velocity distribution in laminar boundary layer. [After Blasius (3)].

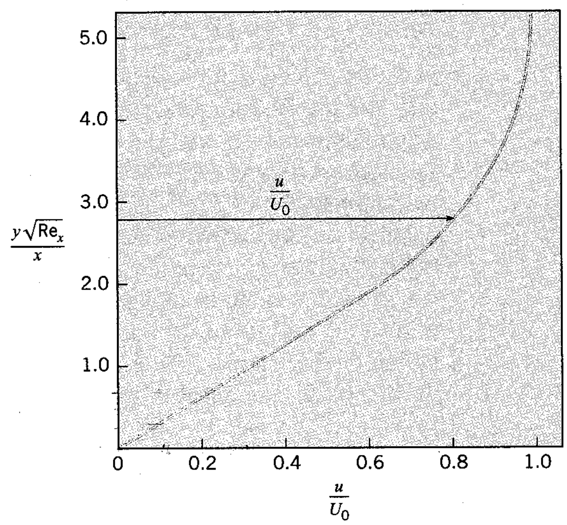


TABLE 9.1 RESULTS—δ AND τ_0 FOR DIFFERENT VALUES OF x

	x = 0.1 ft	x = 1.0 ft	x = 2 ft	x = 4 ft	x = 6 ft
$x^{1/2}$	0.316	1.00	1.414	2.00	2.45
τ_0, psf	0.552	0.174	0.123	0.087	0.071
δ, ft	0.005	0.016	0.022	0.031	0.039
δ, in.	0.060	0.189	0.270	0.380	0.466

$$\delta = \frac{5x}{\sqrt{\mathrm{Re}}}$$

value of y where $u/U_\infty = .99$

$$\mathrm{Re}_x = \frac{U_\infty x}{\nu}$$

$$\tau_w = \frac{\mu U_\infty f''(0)}{\sqrt{2\nu x / U_\infty}}$$

i.e.,
$$c_f = \frac{2\tau_w}{\rho U_\infty^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}$$ see below

$$C_f = \frac{1}{L}\int_0^L c_f\,dx = 2c_f(L)$$
$$= \frac{1.328}{\sqrt{Re_L}}$$

$$Re_L = \frac{U_\infty L}{\nu}$$

Other:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy = 1.7208\frac{x}{\sqrt{Re_x}}$$ displacement thickness

measure of displacement of inviscid flow to due boundary layer

$$\theta = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right)\frac{u}{U_\infty} dy = 0.664\frac{x}{\sqrt{Re_x}}$$ momentum thickness

measure of loss of momentum due to boundary layer

H = shape parameter = $\dfrac{\delta^*}{\theta}$ =2.5916