Chapter 9 Differential Analysis of Fluid Flow

9.1 The Continuity Equation in Differential Form

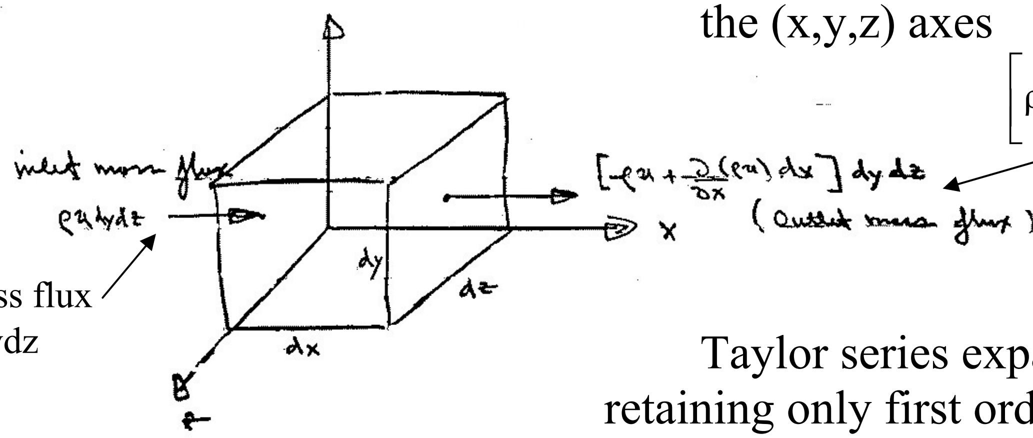
The governing equations can be expressed in both integral and differential form. Integral form is useful for large-scale control volume analysis, whereas the differential form is useful for relatively small-scale point analysis.

Application of RTT to a fixed elemental control volume yields the differential form of the governing equations. For example for conservation of mass

$$\sum_{CS} \rho \underline{V} \cdot \underline{A} = -\int_{CV} \frac{\partial \rho}{\partial t} d\forall$$

net outflow of mass across CS = rate of decrease of mass within CV

Consider a cubical element oriented so that its sides are || to the (x,y,z) axes



inlet mass flux
$\rho u dydz$

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right]dydz$$

outlet mass flux

Taylor series expansion retaining only first order term

We assume that the element is infinitesimally small such that we can assume that the flow is approximately one dimensional through each face.

The mass flux terms occur on all six faces, three inlets, and three outlets. Consider the mass flux on the x faces

$$x_{flux} = \left[\rho u + \frac{\partial}{\partial x}(\rho u)\,dx\right] dydz\Big|_{outflux} - \rho u dydz\Big|_{influx}$$

$$= \frac{\partial}{\partial x}(\rho u)\underbrace{dxdydz}_{\cancel{V}}$$

Similarly for the y and z faces

$$y_{flux} = \frac{\partial}{\partial y}(\rho v)dxdydz$$

$$z_{flux} = \frac{\partial}{\partial z}(\rho w)dxdydz$$

The total net mass outflux must balance the rate of decrease of mass within the CV which is

$$-\frac{\partial \rho}{\partial t}dxdydz$$

Combining the above expressions yields the desired result

$$\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]\underbrace{dxdydz}_{d\cancel{V}}=0$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

per unit $\cancel{V}$
differential form of continuity equations

$$\frac{\partial\rho}{\partial t}+\underbrace{\nabla\cdot(\rho\underline{V})}_{\rho\nabla\cdot\underline{V}+\underline{V}\cdot\nabla\rho}=0$$

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\underline{V}=0 \qquad \frac{D}{Dt}=\frac{\partial}{\partial t}+\underline{V}\cdot\nabla$$

Nonlinear 1st order PDE; (unless ρ = constant, then linear)
Relates $\underline{V}$ to satisfy kinematic condition of mass conservation

Simplifications:

1. Steady flow: $\nabla\cdot(\rho\underline{V})=0$

2. ρ = constant: $\nabla\cdot\underline{V}=0$

i.e., $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ 3D

$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ 2D

4.7 The Stream Function

Powerful tool for 2-D flows in which $\underline{V}$ is obtained by differentiation of a scalar χ which automatically satisfies the continuity equation

2-D cont. $\quad u_x + v_y = 0$
(ρ = const.)

say $\quad u = \chi_y$ and $v = -\chi_x$

then $\quad \frac{\partial}{\partial x}(\chi_y) + \frac{\partial}{\partial y}(-\chi_x) = 0$

$$\chi_{yx} - \chi_{xy} = 0 \text{ by definition!}$$

$$\underline{V} = \chi_y \hat{\imath} - \chi_x \hat{\jmath}$$

$$\text{curl } \underline{V} = \hat{k}\,\omega_z = -\hat{k}\,\nabla^2 \chi$$

Steady

$$\text{curl}\left(\rho \frac{D\underline{V}}{Dt} = -\nabla(p + \gamma z) + \mu \nabla^2 \underline{V}\right)$$

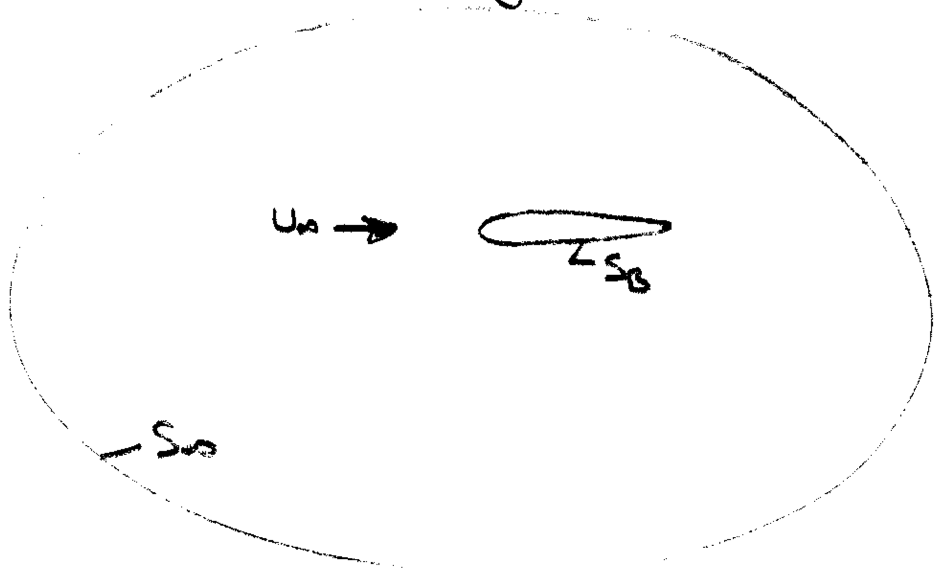
$$\rho \,\text{curl}(\underline{V} \cdot \nabla \underline{V}) = \mu \nabla^2 \,\text{curl}\, \underline{V}$$

$$\rho\left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right)(-\hat{k}\,\nabla^2 \chi) = \mu \nabla^2(-\hat{k}\,\nabla^2 \chi)$$

$$\rho\left[\frac{\partial \chi}{\partial y}\frac{\partial}{\partial x}(\nabla^2 \chi) - \frac{\partial \chi}{\partial x}\frac{\partial}{\partial y}(\nabla^2 \chi)\right] = \mu \nabla^4 \chi$$

single scalar equation; but, 4th order!

Boundary Conditions (4 required)



at infinity: $u = \psi_y = U_\infty$

$v = -\psi_x = 0$

on body: $u = v = 0 = \psi_y = -\psi_x$

Irrotational Flow,

$$\nabla^2 \psi = 0$$

2nd order linear pde
Laplace equation

on S_∞: $\psi = U_\infty y + \text{const.}$

on S_B: $\psi = \text{const.}$

Geometric Interpretation of χ

Besides its importance mathematically χ also has important geometric significance

$$\chi = \text{constant} = \text{streamline}$$

recall definition of a streamline

$$\underline{v} \times \underline{dr} = 0 \qquad \underline{dr} = dx\,\hat{i} + dy\,\hat{j}$$

ie $\qquad \dfrac{dx}{u} = \dfrac{dy}{v}$

or $\qquad u\,dy - v\,dx = 0$

Compare with $\qquad d\chi = \underset{\underset{-v}{\parallel}}{\chi_x}\,dx + \underset{\underset{u}{\parallel}}{\chi_y}\,dy$

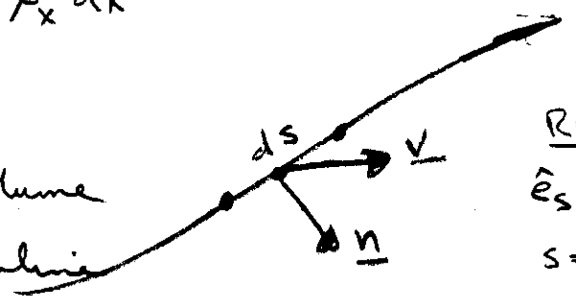
ie $\qquad d\chi = 0$ along a streamline

or $\qquad \chi = $ const along a streamline and curves of constant χ are the flow streamlines

if we know $\chi(x,y)$ then can plot $\chi = $ const curves to show streamlines

Physical interpretation

$$
\begin{aligned}
d Q &= \underline{V} \cdot \underline{n}\, dA \\
&= \left(\hat{\imath} \frac{\partial \psi}{\partial y} - \hat{\jmath} \frac{\partial \psi}{\partial x}\right) \cdot \left(\frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}\right) \cdot ds \cdot 1 \\
&= \psi_y\, dy + \psi_x\, dx \\
&= d\psi
\end{aligned}
$$



$\underline{R} = x\hat{\imath} + y\hat{\jmath}$

$\hat{e}_s = \frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath}$

$s =$ arclength

$ds = \sqrt{dx^2 + dy^2}$

ie change in $d\psi$ is volume flux across streamline

$dQ = 0$

also shows that

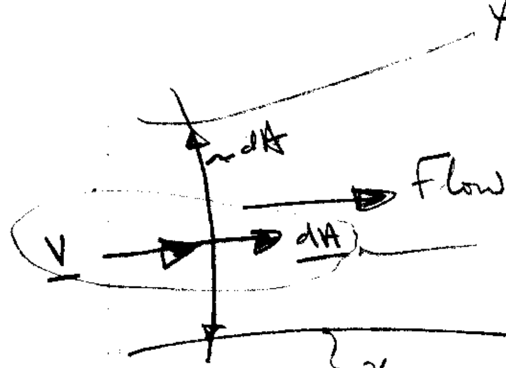
$\hat{e}_n = \hat{k} \times \hat{e}_s = \frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}$

$V_n = \frac{d\psi}{dA}$ $\therefore$ $V \propto \frac{1}{dA}$ ← streamline spacing

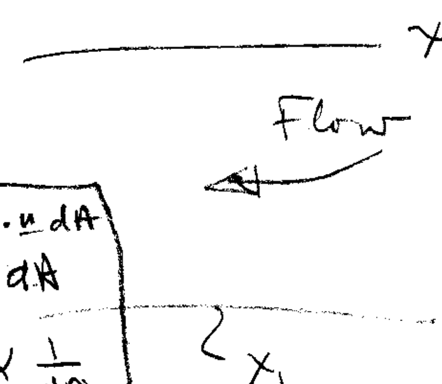
$dA = ds \times 1$

$$
Q_{1\to 2} = \int_1^2 \underline{V} \cdot \underline{n}\, dA = \int_1^2 d\psi = \underbrace{\psi_2 - \psi_1}_{\text{change in } \psi \text{ is volume flux}}
$$

Consider flow between two streamlines



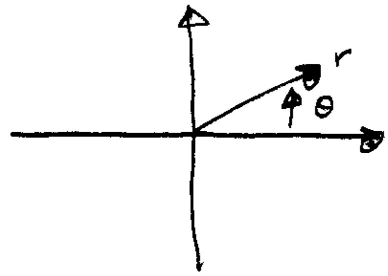
$$
\boxed{
\begin{aligned}
dQ = d\psi &= \underline{V} \cdot \underline{n}\, dA \\
&= V_n\, dA \\
\text{ie } V_n = \frac{d\psi}{dA} &\propto \frac{1}{dA} \\
&\text{proportional to streamline spacing}
\end{aligned}
}
$$



$\psi_y > 0 \Rightarrow u > 0$

$\psi_y < 0 \Rightarrow u < 0$

Incompressible Plane Flow in Polar Coordinates



Cont: $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) = 0$

or $\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial \theta}(v_\theta) = 0$

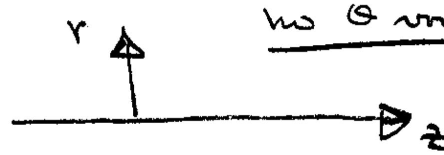
say $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ and $v_\theta = -\frac{\partial \psi}{\partial r}$

then $\frac{\partial}{\partial r}\left(r \cdot \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(-\frac{\partial \psi}{\partial r}\right) = 0$ identically

as before $d\psi = 0$ along a streamline
and $dQ = d\psi$
volume flux = change in stream function

Incompressible Axisymmetric Flow

no θ variation



Cont: $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(v_z) = 0$

Say $v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$ and $v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$

then $\frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot -\frac{1}{r}\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0$

as before $d\psi = 0$ along a streamline
and $dQ = d\psi$

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<u>Generalizations</u>

Steady Plane Compressible Flow

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

define $\quad \rho u = \frac{\partial \chi}{\partial y}$

$\rho v = -\frac{\partial \chi}{\partial x} \qquad \chi$ = compressible flow stream function

$$d\chi = \chi_x dx + \chi_y dy$$

S.L. $\quad u\,dy - v\,dx = 0$

Compare with $\quad \frac{1}{\rho}\chi_y dy + \frac{1}{\rho}\chi_x dx = 0$

$\frac{1}{\rho}(d\chi) = 0 \quad$ ie $\quad d\chi = 0 \quad$ and χ = constant is a streamline

only now

$$d\dot{m} = \rho(\underline{v} \cdot \underline{n})\,dA = d\chi$$

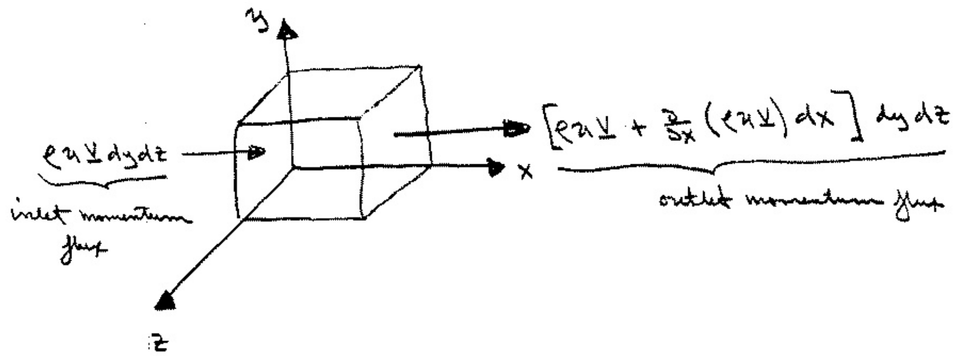
$$\dot{m}_{1\to 2} = \int_1^2 \rho(\underline{v} \cdot \underline{n})\,dA = \chi_2 - \chi_1$$

change in χ is equivalent to the mass flux

9.3 Navier-Stokes Equations

Differential form of momentum equation can be derived by applying control volume form to elemental control volume

The differential equation of linear momentum: elemental fluid volume approach



$$\sum \underline{F} = \underbrace{\frac{d}{dt}\int_{CV} \rho \underline{V} dV}_{\frown} + \underbrace{\int_{CS} \underline{V}\rho \underline{V}\cdot \underline{dA}}_{\ulcorner}$$

1-D flow approximation

$$\ulcorner = \sum\left(\dot{m}_i \underline{V}_i\right)_{out} - \sum\left(\dot{m}_i \underline{V}_i\right)_{in}$$

where $\underbrace{\dot{m} = \rho A V = \rho dydzu}_{\text{mass flux}}$ x-face

$$\frown \sim \frac{d}{dt}\left(\rho \underline{V}\right)dxdydz$$

$$\ulcorner = \left[\underbrace{\frac{\partial}{\partial x}\left(\rho u \underline{V}\right)}_{\text{x-face}} + \underbrace{\frac{\partial}{\partial y}\left(\rho v \underline{V}\right)}_{\text{y-face}} + \underbrace{\frac{\partial}{\partial z}\left(\rho w \underline{V}\right)}_{\text{z-face}}\right]dxdydz$$

combining and making use of the continuity equation yields

$$\sum \underline{F} = \rho \frac{D\underline{V}}{Dt}dxdydz \qquad \frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla \underline{V}$$

where $\sum \underline{F} = \sum \underline{F}_{body} + \sum \underline{F}_{surface}$

<u>Body forces</u> are due to external fields such as gravity or magnetics. Here we only consider a gravitational field; that is,

$$\sum \underline{F}_{body} = d\underline{F}_{grav} = \rho \underline{g} dxdydz$$

and $\underline{g} = -g\hat{k}$ for $g\downarrow$ $z\uparrow$

i.e., $\underline{f}_{body} = -\rho g\hat{k}$

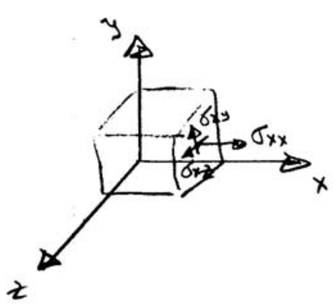
$\delta_{ij} = 1$	$i = j$
$\delta_{ij} = 0$	$i \neq j$

<u>Surface forces</u> are due to the stresses that act on the sides of the control surfaces

$$\sigma_{ij} = \underbrace{-}_{\text{normal pressure}} p\delta_{ij} + \underbrace{\tau_{ij}}_{\text{viscous stress}}$$

symmetric ($\sigma_{ij} = \sigma_{ji}$)
2nd order tensor

normal pressure viscous stress



$$= \begin{bmatrix} -p+\tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p+\tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p+\tau_{zz} \end{bmatrix}$$

As shown before for p alone it is not the stresses themselves that cause a net force but their gradients.

$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{xy}) + \frac{\partial}{\partial z}(\sigma_{xz})\right] dxdydz$$

$$= \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{xy}) + \frac{\partial}{\partial z}(\tau_{xz})\right] dxdydz$$

This can be put in a more compact form by defining

$$\underline{\tau}_x = \tau_{xx}\hat{i} + \tau_{xy}\hat{j} + \tau_{xz}\hat{k} \qquad \text{vector stress on x-face}$$

and noting that

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \nabla \cdot \underline{\tau}_x \right] dxdydz$$

$$f_{x,surf} = -\frac{\partial p}{\partial x} + \nabla \cdot \underline{\tau}_x \qquad \text{per unit volume}$$

similarly for y and z

$$f_{y,surf} = -\frac{\partial p}{\partial y} + \nabla \cdot \underline{\tau}_y \qquad \underline{\tau}_y = \tau_{yx}\hat{i} + \tau_{yy}\hat{j} + \tau_{yz}\hat{k}$$

$$f_{z,surf} = -\frac{\partial p}{\partial z} + \nabla \cdot \underline{\tau}_z \qquad \underline{\tau}_z = \tau_{zx}\hat{i} + \tau_{zy}\hat{j} + \tau_{zz}\hat{k}$$

finally if we define

$$\tau_{ij} = \underline{\tau}_x\hat{i} + \underline{\tau}_y\hat{j} + \underline{\tau}_z\hat{k} \qquad \text{then}$$

$$\underline{f}_{surf} = -\nabla p + \nabla \cdot \tau_{ij} = \nabla \cdot \sigma_{ij} \qquad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

Putting together the above results

$$\sum \underline{f} = \underline{f}_{body} + \underline{f}_{surf} = \rho \frac{D\underline{V}}{Dt}$$

$$\underline{f}_{body} = -\rho g\hat{k}$$

$$\underline{f}_{surface} = -\nabla p + \nabla \cdot \tau_{ij}$$

$$\underline{a} = \frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V}$$

$$\rho \underline{a} = -\rho g \vec{k} - \nabla p + \nabla \cdot \tau_{ij}$$

inertia force	body force due to gravity	surface force due to p	surface force due to viscous shear and normal stresses

For Newtonian fluid the shear stress is proportional to the rate of strain, which for incompressible flow can be written

$$\tau_{ij} = \mu \varepsilon_{ij} \qquad \mu = \text{coefficient of viscosity}$$

ε_{ij} = rate of strain tensor

$$= \begin{bmatrix} \dfrac{\partial u}{\partial x} & \left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) \\ \left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) \\ \left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

$$\tau = \mu \frac{du}{dy} \qquad \text{1-D flow}$$

rate of strain

$$\rho \underline{a} = -\rho g\hat{k} - \nabla p + \underbrace{\nabla \cdot (\mu \varepsilon_{ij})}_{\mu \frac{\partial}{\partial x_i}(\varepsilon_{ij}) = \mu \nabla^2 \underline{V}}$$

$$\rho \underline{a} = -\rho g\hat{k} - \nabla p + \mu \nabla^2 \underline{V}$$

$$\boxed{\begin{array}{l} \rho \underline{a} = -\nabla(p + \gamma z) + \mu \nabla^2 \underline{V} \\ \nabla \cdot \underline{V} = 0 \end{array}}$$

Navier-Stokes Equation
Continuity Equation

Four equations in four unknowns: $\underline{V}$ and p
Difficult to solve since 2^{nd} order nonlinear PDE

$$x: \quad \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$y: \quad \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

$$z: \quad \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Navier-Stokes equations can also be written in other coordinate systems such as cylindrical, spherical, etc.

There are about 80 exact solutions for simple geometries. For practical geometries, the equations are reduced to algebraic form using finite differences and solved using computers.

Exact solution for laminar flow in a pipe
(neglect $\underline{g}$ for now)

use cylindrical coordinates: $v_x = u$
$v_r = v$
$u = u(r)$ only $v_\theta = w = 0$

Continuity: $\frac{\partial}{\partial r}(rv) = 0 \Rightarrow$ rv = constant = c

$v = c/r$
$v(r = 0) = 0 \Rightarrow c = 0$
i.e., $v = 0$

Momentum:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\left[\cancel{\frac{\partial^2 u}{\partial x^2}} + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right]$$

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial z}} + \cancel{v\frac{\partial u}{\partial r}} + \cancel{\frac{w}{\partial r}\frac{\partial u}{\partial \theta}}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{\mu}\frac{\partial p}{\partial x}=\lambda$$

$$r\frac{\partial u}{\partial r}=\frac{\lambda}{2}r^2+A$$

$$u(r)=\frac{\lambda}{4}r^2+A\ln r+B$$

$u(r = 0) \neq \infty \Rightarrow A = 0$

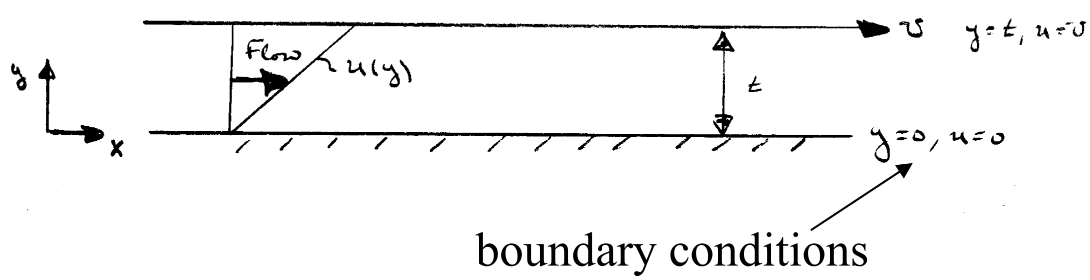
$u(r = r_o) = 0 \Rightarrow u(r)=\frac{\lambda}{4}\left(r^2-r_o^2\right)$

i.e. $u(r)=\frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2-r_o^2\right)$ parabolic velocity profile

9.4 Differential Analysis of Fluid Flow

We now discuss a couple of exact solutions to the Navier-Stokes equations. Although all known exact solutions (about 80) are for highly simplified geometries and flow conditions, they are very valuable as an aid to our understanding of the character of the NS equations and their solutions. Actually the examples to be discussed are for internal flow (Chapter 8) and open channel flow (Chapter 13), but they serve to underscore and display viscous flow. Finally, the derivations to follow utilize differential analysis. See the text for derivations using CV analysis.

1. Couette Flow



First, consider flow due to the relative motion of two parallel plates

Continuity $\frac{\partial u}{\partial x} = 0$

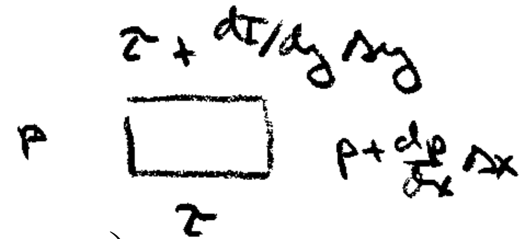
Momentum $0 = \mu \frac{d^2 u}{dy^2}$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \begin{array}{l} u = u(y) \\ v = o \\ \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \end{array}$$

or by CV continuity and momentum equations:

$$\rho u_1 \Delta y = \rho u_2 \Delta y$$

$$u_1 = u_2$$



$$\sum F_x = \sum u\rho \underline{V} \cdot d\underline{A} = \rho Q(u_2 - u_1) = 0$$

$$= p\Delta y - \left(p + \frac{dp}{dx}\Delta x\right)\Delta y - \tau\Delta x + \left(\tau + \frac{d\tau}{dy}dy\right)\Delta x = 0$$

$$\frac{d\tau}{dy} = 0$$

i.e. $\frac{d}{dy}\left(\mu \frac{du}{dy}\right) = 0$

$$\mu \frac{d^2u}{dy^2} = 0$$

from momentum equation

$$\mu \frac{du}{dy} = C$$

$$u = \frac{C}{\mu} y + D$$

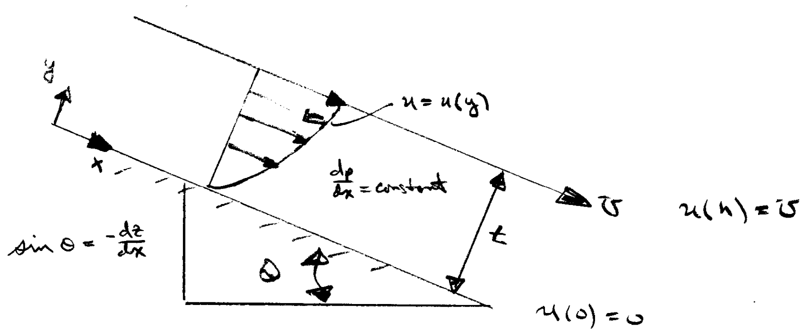
$$u(0) = 0 \Rightarrow D = 0$$

$$u(t) = U \Rightarrow C = \mu \frac{U}{t}$$

$$u = \frac{U}{t} y$$

$$\tau = \mu \frac{du}{dy} = \frac{\mu U}{t} = \text{constant}$$

2. Generalization for inclined flow with a constant pressure gradient



Continutity $\frac{\partial u}{\partial x} = 0$

Momentum $0 = -\frac{\partial}{\partial x}(p + \gamma z) + \mu \frac{d^2 u}{dy^2}$

$u = u(y)$
$v = o$
$\frac{\partial p}{\partial y} = 0$

i.e., $\mu \frac{d^2 u}{dy^2} = \gamma \frac{dh}{dx}$ $h = p/\gamma + z =$ constant

plates horizontal $\frac{dz}{dx} = 0$

plates vertical $\frac{dz}{dx} = -1$

which can be integrated twice to yield

$$\mu \frac{du}{dy} = \gamma \frac{dh}{dx} y + A$$

$$\mu u = \gamma \frac{dh}{dx} \frac{y^2}{2} + Ay + B$$

now apply boundary conditions to determine A and B

$$u(y=0)=0 \quad \Rightarrow \quad B=0$$

$$u(y=t)=U$$

$$\mu U=\gamma\frac{dh}{dx}\frac{t^2}{2}+At \Rightarrow A=\frac{\mu U}{t}-\gamma\frac{dh}{dx}\frac{t}{2}$$

$$u(y)=\frac{\gamma}{\mu}\frac{dh}{dx}\frac{y^2}{2}+\frac{1}{\mu}\left[\frac{\mu U}{t}-\gamma\frac{dh}{dx}\frac{t}{2}\right]$$

$$=-\frac{\gamma}{2\mu}\frac{dh}{dx}\left(ty-y^2\right)+\frac{U}{t}y$$

This equation can be put in non-dimensional form:

$$\frac{u}{U}=-\frac{\gamma t^2}{2\mu U}\frac{dh}{dx}\left(1-\frac{y}{t}\right)\frac{y}{t}+\frac{y}{t}$$

define: P = non-dimensional pressure gradient

$$=-\frac{\gamma t^2}{2\mu U}\frac{dh}{dx}$$

$$h=\frac{p}{\gamma}+z$$

$$Y=y/t$$

$$=-\frac{\gamma z^2}{2\mu U}\left[\frac{1}{\gamma}\frac{dp}{dx}+\frac{dz}{dx}\right]$$

$$\boxed{\Rightarrow \quad \frac{u}{U}=P\cdot Y(1-Y)+Y}$$

parabolic velocity profile

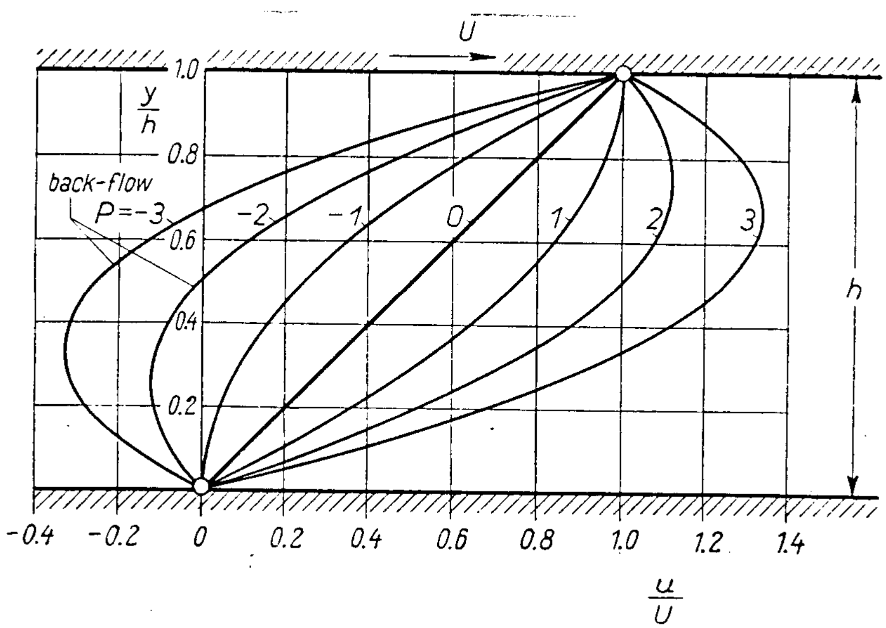


Fig. 5.2. Couette flow between two parallel flat walls
$P > 0$, pressure decrease in direction of wall motion; $P < 0$, pressure increase; $P = 0$, zero pressure gradient

$$\frac{u}{U} = \underbrace{\frac{Py}{t} - \frac{Py^2}{t^2} + \frac{y}{t}}$$

$$q = \int_0^t u\,dy$$

$$\bar{u} = \frac{q}{t} = \frac{\int_0^t U[\quad]dy}{t}$$

$$\frac{t\bar{u}}{U} = \int_0^t \left[\frac{P}{t}y - \frac{P}{t^2}y^2 + \frac{y}{t}\right]dy$$

$$= \frac{Pt}{2} - \frac{Pt}{3} + \frac{t}{2}$$

$$\frac{\bar{u}}{U} = \frac{P}{6} + \frac{1}{2} \Rightarrow \bar{u} = \frac{t^2}{12\mu}\left(-\gamma\frac{dh}{dx}\right) + \frac{U}{2}$$

$$\boxed{\text{For laminar flow } \frac{\bar{u}t}{\nu} < 1000} \text{——} Re_{crit} \sim 1000$$

The maximum velocity occurs at the value of y for which:

$$\frac{du}{dy} = 0 \qquad \frac{d}{dy}\left(\frac{u}{U}\right) = 0 = \frac{P}{t} - \frac{2P}{t^2}y + \frac{1}{t}$$

$$\Rightarrow y = \frac{t}{2P}(P+1) = \frac{t}{2} + \frac{t}{2P} \text{ @ } u_{max} \qquad \boxed{\text{for } U = 0,\ y = t/2}$$

$$\therefore\ u_{max} = u(y_{max}) = \frac{UP}{4} + \frac{U}{2} + \frac{U}{4P}$$

note: if U = 0: $\dfrac{\bar{u}}{u_{max}} = \dfrac{P}{6}\bigg/\dfrac{P}{4} = \dfrac{2}{3}$

The shape of the velocity profile u(y) depends on P:

1. If P > 0, i.e., $\frac{dh}{dx} < 0$ the pressure decreases in the direction of flow (favorable pressure gradient) and the velocity is positive over the entire width

$$\gamma \frac{dh}{dx} = \gamma \frac{d}{dx}\left(\frac{p}{\gamma} + z\right) = \frac{dp}{dx} - \gamma \sin\theta$$

a) $\frac{dp}{dx} < 0$

b) $\frac{dp}{dx} < \gamma \sin\theta$

2. If P < 0, i.e., $\frac{dh}{dx} > 0$ the pressure increases in the direction of flow (adverse pressure gradient) and the velocity over a portion of the width can become negative (backflow) near the stationary wall. In this case the dragging action of the faster layers exerted on the fluid particles near the stationary wall is insufficient to over come the influence of the adverse pressure gradient

$$\frac{dp}{dx} - \gamma \sin\theta > 0$$

$$\frac{dp}{dx} > \gamma \sin\theta \qquad \text{or} \qquad \gamma \sin\theta < \frac{dp}{dx}$$

3. If P = 0, i.e., $\frac{dh}{dx} = 0$ the velocity profile is linear

$$u = \frac{U}{t} y$$

a) $\frac{dp}{dx} = 0$ and $\theta = 0$

Note: we derived this special case

b) $\frac{dp}{dx} = \gamma \sin\theta$

For U = 0 the form $\frac{u}{U} = PY(1 - Y) + Y$ is not appropriate

$$u = UPY(1\text{-}Y)+UY$$

$$= -\frac{\gamma t^2}{2\mu} \frac{dh}{dx} Y(1 - Y) + UY$$

Now let U = 0: $u = -\frac{\gamma t^2}{2\mu} \frac{dh}{dx} Y(1 - Y)$

3. Shear stress distribution

Non-dimensional velocity distribution

$$u^* = \frac{u}{U} = P \cdot Y(1-Y) + Y$$

where $u^* \equiv \frac{u}{U}$ is the non-dimensional velocity,

$P \equiv -\frac{\gamma t^2}{2\mu U}\frac{dh}{dx}$ is the non-dimensional pressure gradient

$Y \equiv \frac{y}{t}$ is the non-dimensional coordinate.

Shear stress

$$\tau = \mu \frac{du}{dy}$$

In order to see the effect of pressure gradient on shear stress using the non-dimensional velocity distribution, we define the non-dimensional shear stress:

$$\tau^* = \frac{\tau}{\frac{1}{2}\rho U^2}$$

Then

$$\tau^* = \frac{1}{\frac{1}{2}\rho U^2}\mu \frac{Ud(u/U)}{td(y/t)} = \frac{2\mu}{\rho U t}\frac{du^*}{dY}$$

$$= \frac{2\mu}{\rho U t}(-2PY + P + 1)$$

$$= \frac{2\mu}{\rho U t}(-2PY + P + 1)$$

$$= A(-2PY + P + 1)$$

where $A \equiv \frac{2\mu}{\rho U t} > 0$ is a positive constant.

So the shear stress always varies linearly with Y across any section.

At the lower wall $(Y=0)$:

$$\tau_{lw}^{*} = A(1+P)$$

At the upper wall $(Y=1)$:

$$\tau_{uw}^{*} = A(1-P)$$

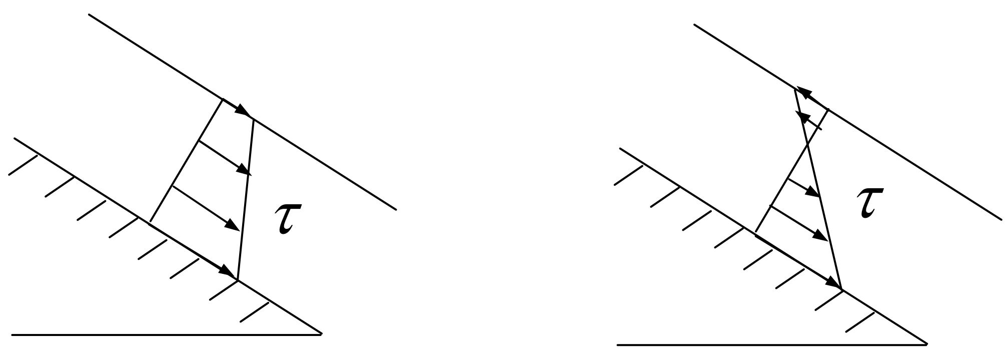
For favorable pressure gradient, the lower wall shear stress is always positive:

1. For small favorable pressure gradient $(0<P<1)$:

$$\tau_{lw}^{*} > 0 \text{ and } \tau_{uw}^{*} > 0$$

2. For large favorable pressure gradient $(P>1)$:

$$\tau_{lw}^{*} > 0 \text{ and } \tau_{uw}^{*} < 0$$



$(0<P<1)$ $(P>1)$

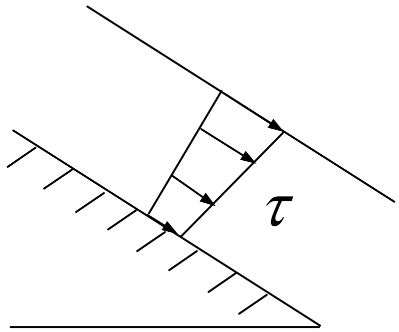
For adverse pressure gradient, the upper wall shear stress is always positive:

1. For small adverse pressure gradient $(-1<P<0)$:

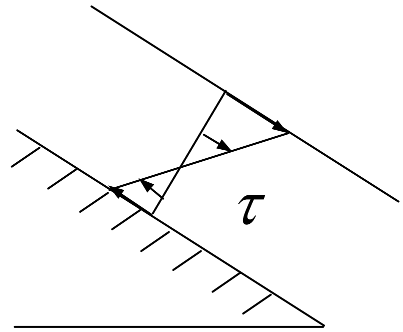
$$\tau_{lw}^{*} > 0 \text{ and } \tau_{uw}^{*} > 0$$

2. For large adverse pressure gradient $(P<-1)$:

$$\tau_{lw}^{*} < 0 \text{ and } \tau_{uw}^{*} > 0$$



$(-1 < P < 0)$



$(P < -1)$

For $U = 0$, i.e., channel flow, the above non-dimensional form of velocity profile is not appropriate. Let's use dimensional form:

$$u = -\frac{\gamma t^2}{2\mu}\frac{dh}{dx}Y(1-Y) = -\frac{\gamma}{2\mu}\frac{dh}{dx}y(t-y)$$

Thus the fluid always flows in the direction of decreasing piezometric pressure or piezometric head because $\frac{\gamma}{2\mu} > 0,\ \ y > 0$ and $t - y > 0$. So if $\frac{dh}{dx}$ is negative, u is positive; if $\frac{dh}{dx}$ is positive, u is negative.

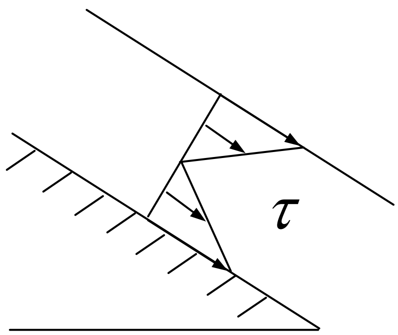
Shear stress:

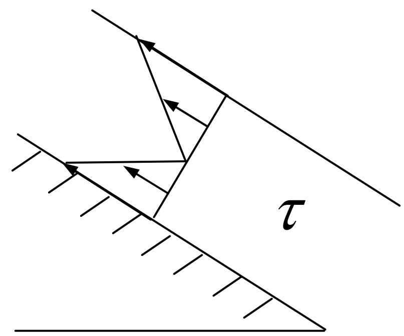
$$\tau = \mu\frac{du}{dy} = -\frac{\gamma}{2}\frac{dh}{dx}\left(t - \frac{1}{2}y\right)$$

Since $\left(t - \frac{1}{2}y\right) > 0$, the sign of shear stress τ is always opposite to the sign of piezometric pressure gradient $\frac{dh}{dx}$, and the magnitude of τ is always maximum at both walls and zero at centerline of the channel.

For favorable pressure gradient, $\frac{dh}{dx} < 0$, $\tau > 0$

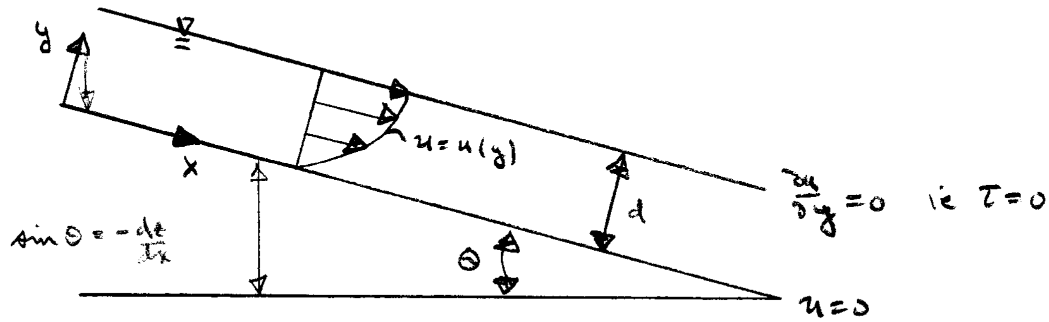
For adverse pressure gradient, $\frac{dh}{dx} > 0$, $\tau < 0$





$\frac{dh}{dx} < 0$ $\frac{dh}{dx} > 0$

Flow down an inclined plane



uniform flow ⇒ velocity and depth do not change in x-direction

Continuity $\frac{du}{dx} = 0$

x-momentum $0 = -\frac{\partial}{\partial x}(p + \gamma z) + \mu \frac{d^2u}{dy^2}$

y-momentum $0 = -\frac{\partial}{\partial y}(p + \gamma z) \Rightarrow$ hydrostatic pressure variation

$$\Rightarrow \frac{dp}{dx} = 0$$

$$\mu \frac{d^2u}{dy^2} = -\gamma \sin\theta$$

$$\frac{du}{dy} = -\frac{\gamma}{\mu}\sin\theta y + c$$

$$u = -\frac{\gamma}{\mu}\sin\theta\frac{y^2}{2} + Cy + D$$

$$\left.\frac{du}{dy}\right|_{y=d} = 0 = -\frac{\gamma}{\mu}\sin\theta d + c \Rightarrow c = +\frac{\gamma}{\mu}\sin\theta d$$

$$u(0) = 0 \ \Rightarrow D = 0$$

$$u = -\frac{\gamma}{\mu}\sin\theta\frac{y^2}{2} + \frac{\gamma}{\mu}\sin\theta dy$$

$$= \frac{\gamma}{2\mu}\sin\theta y(2d - y)$$

$$u(y) = \frac{g\sin\theta}{2\nu}y(2d - y)$$

$$q = \int_0^d udy = \frac{\gamma}{2\mu}\sin\theta\left[dy^2 - \frac{y^3}{3}\right]_0^d$$

discharge per unit width

$$= \frac{1}{3}\frac{\gamma}{\mu}d^3\sin\theta$$

$$\overline{V}_{avg} = \frac{q}{d} = \frac{1}{3}\frac{\gamma}{\mu}d^2\sin\theta = \frac{gd^2}{3\nu}\sin\theta$$

in terms of the slope $S_o = \tan\theta \sim \sin\theta$

$$\overline{V} = \frac{gd^2 S_o}{3\nu}$$

Exp. show $Re_{crit} \sim 500$, i.e., for $Re \geq 500$ the flow will become turbulent

$$\frac{\partial p}{\partial y} = -\gamma\cos\theta \qquad\qquad Re_{crit} = \frac{\overline{V}d}{\nu} \sim 500$$

$$p = -\gamma\cos\theta\, y + C$$

$$p(d) = p_o = -\gamma\cos\theta\, d + C$$

i.e., $$p = \gamma\cos\theta(d - y) + p_o$$

* $p(d) > p_o$

* if $\theta = 0$ $\quad p = \gamma(d - y) + p_o$
entire weight of fluid imposed

if $\theta = \pi/2$ $\quad p = p_o$
no pressure change through the fluid