Summary continuity, momentum, and energy equations for pipe flow

Continuity: $Q_1 = Q_2$ i.e., $V_1 = V_2$

Use continuity and momentum: $\tau = \frac{r}{2}\left[-\frac{d}{ds}(p+\gamma z)\right]$

Energy: $$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1 = \frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2+h_L$$

$$h_L = \frac{p_1 - p_2}{\gamma}+(z_1 - z_2) = \Delta h = -(h_2 - h_1)$$

$$h_L = L\left(-\frac{dh}{ds}\right)$$ where $L = ds =$ length of pipe

$$h = \frac{p}{\gamma}+z$$

$$h_L = \frac{L}{\gamma}\left[-\frac{d}{ds}(p+\gamma z)\right]$$

Combine Energy with Momentum evaluated at

$r = R =$ radius of pipe $\tau_w = \frac{R}{2}\left[-\frac{d}{ds}(p+\gamma z)\right]$

$$h_L = \frac{L}{\gamma}\left(\frac{2\tau_w}{R}\right)$$ $h_L = h_f =$ loss due to pipe friction

Define friction factor, i.e., nondimensional wall shear stress

$f = 8\tau_w / \rho V^2$

$$h_L = h_f = f\,\frac{L}{D}\frac{V^2}{2g}$$ Darcy-Weisbach equation

Laminar Flow: $\mathrm{Re}=\frac{VD}{\nu}\leq 2000$

Exact solution Navier-Stokes

$$u(r)=\frac{R^2-r^2}{4\mu}\left[-\frac{d}{ds}(p+\gamma z)\right]$$

$$Q=\frac{\pi R^4}{8\mu}\left[-\frac{d}{ds}(p+\gamma z)\right]$$

$$V=\frac{Q}{A}=\frac{R^2}{8\mu}\left[-\frac{d}{ds}(p+\gamma z)\right]=\frac{u_{\max}}{2}=\frac{u(r=0)}{2}$$

$$h_L=h_f=\frac{32\mu LV}{\gamma D^2}\propto V$$

i.e. $f=\frac{64}{\mathrm{Re}}$ friction factor for laminar pipe flow

Turbulent Flow: $\mathrm{Re}\geq 2000\sim 3000$

$f=f(\mathrm{Re},k/D)$ $k=$roughness

f determined from $V=\frac{Q}{A}=\frac{\int_0^R u(r)2\pi r dr}{\pi R^2}$

with $u(r)$ from log-law including pipe roughness.

$$u^+=\frac{u(r)}{u^*}=\frac{1}{\kappa}\ln y^+ + B^*$$

$$y^+=\frac{yu^*}{\nu},\ y=R-r,\ u^*=\sqrt{\frac{\tau_w}{\rho}},\ B^*=5-\frac{1}{\kappa}\ln\left(1+0.3k^+\right)$$

$f^{-1/2}=-2\log\left[\frac{k}{D}+\frac{9.35}{\mathrm{Re}\, f^{1/2}}\right]+1.14$ Moody Diagram

Three Cononical Types of Problems

1. Determine The Head Loss

$$h_f = f\,\frac{L}{D}\frac{V^2}{2g} = \Delta h = \left(\frac{p_1}{\gamma} + z_1\right) - \left(\frac{p_2}{\gamma} + z_2\right)$$

$$f = f\left(\mathrm{Re}, k/D\right)$$

2. Determine The Flow Rate

$$V = \underbrace{\left[\frac{2gh_f}{L/D}\right]^{1/2}}_{\text{known from problem statement}} f^{-1/2}$$

Given $f \to \underbrace{V \to \mathrm{Re} \to f}$, repeat to convergence

2. Determine The Pipe Diameter Rate

$$D = \underbrace{\left[\frac{8LQ^2}{\pi^2 g h_f}\right]^{1/5}}_{\text{known from problem statement}} f^{-1/5}$$

Given $f \to \underbrace{D \to \mathrm{Re}, k/D \to f}$, repeat to convergence