Chapter 6 Momentum Analysis of Flow System

6.1 <u>Momentum Equation</u>

RTT with B = M$\underline{V}$ and β = $\underline{V}$

$$\sum\left[\underline{F}_S + \underline{F}_B\right] = \frac{d}{dt}\int_{CV} \rho \underline{V} d\forall + \int_{CS} \underline{V}\rho \underline{V}_R \cdot \underline{dA}$$

$\underline{V}$ = velocity referenced to an inertial frame (non-accelerating)
$\underline{V}_R$ = velocity referenced to control volume
$\underline{F}_S$ = surface forces + reaction forces (due to pressure and viscous normal and shear stresses)
$\underline{F}_B$ = body force (due to gravity)

6.2 <u>Applications of the Momentum Equation</u>

Initial Setup and Signs

1. Jet deflected by a plate or a vane
2. Flow through a nozzle
3. Forces on bends
4. Problems involving non-uniform velocity distribution
5. Motion of a rocket
6. Force on rectangular sluice gate
7. Water hammer

6.3 <u>Moment of Momentum Equation</u>

6.1 Derivation of the Basic Equation

Recall RTT: $$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV}\beta\rho d\forall + \int_{CS}\beta\rho \underline{V}_R \cdot \underline{dA}$$

General form for moving but non-accelerating reference frame

$\underline{V}_R$=velocity relative to CS=$\underline{V}$ – $\underline{V}_S$=absolute – velocity CS

Subscript not shown in text but implied!

i.e., referenced to CV

Let, B = M$\underline{V}$ = linear momentum

$\beta = \underline{V}$

small v in text and must be referenced to inertial reference frame

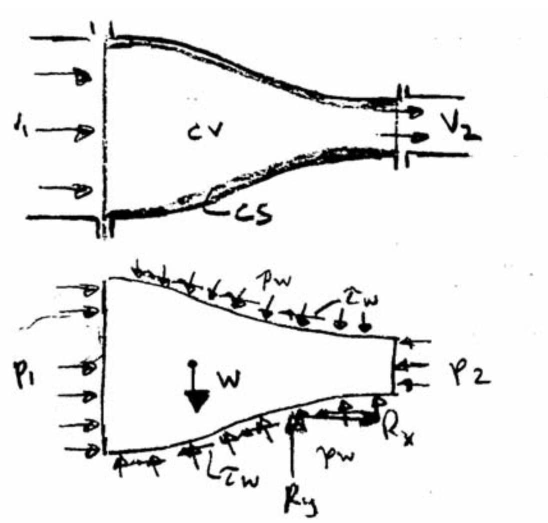
$$\underbrace{\frac{d(M\underline{V})}{dt} = \sum \underline{F}}_{\text{Newton's 2}^{\text{nd}}\text{ law}} = \frac{d}{dt}\int_{CV}\underline{V}\rho d\forall + \int_{CS}\underline{V}\rho \underline{V}_R \cdot \underline{dA}$$

Must be relative to a non-accelerating inertial reference frame

where $\Sigma\underline{F}$ = vector sum of all forces acting on the control volume including both surface and body forces

$= \Sigma\underline{F}_S + \Sigma\underline{F}_B$

$\Sigma\underline{F}_S$ = sum of all external surface forces acting at the CS, i.e., pressure forces, forces transmitted through solids, shear forces,etc.



free body diagram

$\Sigma\underline{F}_B$ = sum of all external body forces, i.e., gravity force

$\Sigma F_x = p_1A_1 - p_2A_2 + R_x$

$\Sigma F_y = -W + R_y$

$\underline{R}$ = resultant force on fluid in CV due to p_w and τ_w

i.e., reaction force on fluid

Important Features (to be remembered)

1) Vector equation to get component in any direction must use dot product

<u>x equation</u>

$$\sum F_x = \frac{d}{dt}\int_{CV} \rho u d\forall + \int_{CS} \rho u \underline{V}_R \cdot \underline{dA}$$

<u>y equation</u>

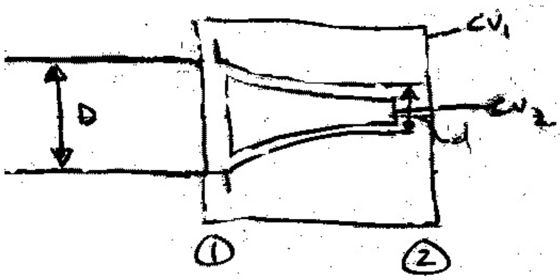
$$\sum F_y = \frac{d}{dt}\int_{CV} \rho v d\forall + \int_{CS} \rho v \underline{V}_R \cdot \underline{dA}$$

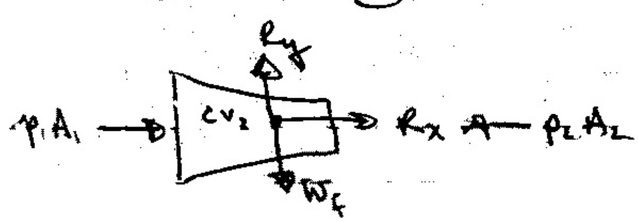
<u>z equation</u>

$$\sum F_z = \frac{d}{dt}\int_{CV} \rho w d\forall + \int_{CS} \rho w \underline{V}_R \cdot \underline{dA}$$

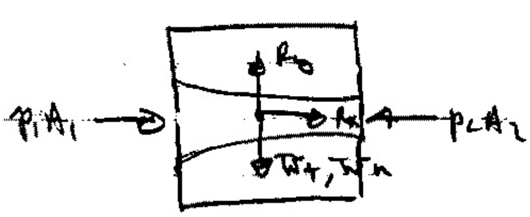
Carefully define coordinate system with forces positive in positive direction of coordinate axes

2) <u>Carefully</u> define control volume and be sure to include <u>all</u> external body and surface faces acting on it.
For example,





(R_x,R_y) = reaction force on fluid



(R_x,R_y) = reaction force on nozzle

3) Velocity $\underline{V}$ must be referenced to a non-accelerating inertial reference frame. Sometimes it is advantageous to use a moving (at constant velocity) reference frame. Note $\underline{V}_R = \underline{V} - \underline{V}_s$ is always relative to CS.

Text uses small $\underline{v}$

i.e., in these cases $\underline{V}$ used for B also referenced to CV (i.e., $\underline{V} = V_{\underline{R}}$)

4) Steady vs. Unsteady Flow

$$\text{Steady flow} \Rightarrow \frac{d}{dt}\int_{CV} \rho \underline{V} d\forall = 0$$

5) Uniform vs. Nonuniform Flow

$$\int_{CS} \underline{V}\rho\underline{V}_R \cdot \underline{dA} = \text{change in flow of momentum across CS}$$

$$= \Sigma\underline{V}\rho\underline{V}_R\cdot\underline{A} \qquad \text{uniform flow across } \underline{A}$$

6) $\underline{F}_{pres} = -\int p\underline{n}dA$

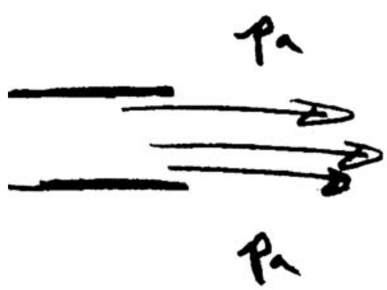
$$\int_V \nabla f d\forall = \int_S f\,\underline{n} ds$$

f = constant, $\nabla f = 0$

= 0 for p = constant and for a closed surface

i.e., always use gage pressure

7) Pressure condition at a jet exit

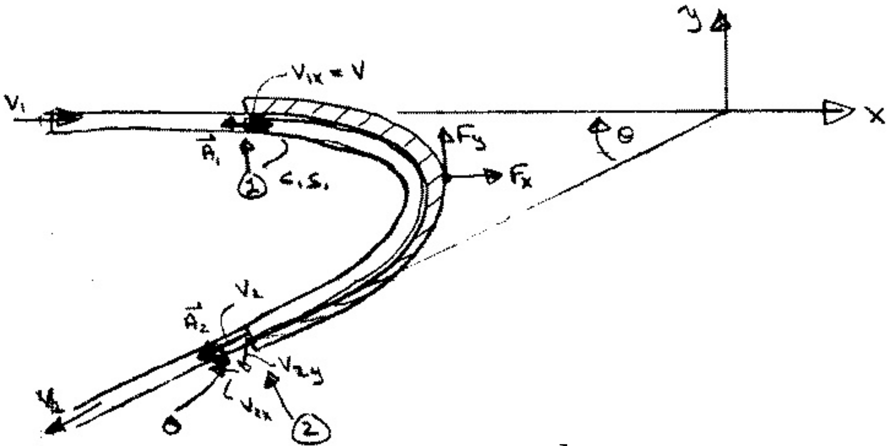


at an exit into the atmosphere jet pressure must be p_a

6.2 Application of the Momentum Equation

1. Jet deflected by a plate or vane

Consider a jet of water turned through a horizontal angle



CV and CS are for jet so that F_x and F_y are vane reactions forces on fluid

x-equation: $\sum F_x = F_x = \frac{d}{dt}\int \rho u d\forall + \int_{CS} \rho u \underline{V} \cdot \underline{dA}$

$$F_x = \sum_{CS} \rho u \underline{V} \cdot \underline{A} \qquad \text{steady flow}$$

$$= \rho V_{1x}(-V_1 A_1) + \rho V_{2x}(V_2 A_2)$$

continuity equation: $\rho A_1 V_1 = \rho A_2 V_2 = \rho Q$ for $A_1 = A_2$, $V_1 = V_2$

$F_x = \rho Q(V_{2x} - V_{1x})$

y-equation: $\sum F_y = F_y = \sum_{CS} \rho v \underline{V} \cdot \underline{A}$

$$F_y = \rho V_{1y}(-A_1 V_1) + \rho V_{2y}(-A_2 V_2)$$
$$= \rho Q(V_{2y} - V_{1y})$$

where: $\underbrace{V_{1x} = V_1 \quad V_{2x} = -V_2\cos\theta \quad V_{2y} = -V_2\sin\theta \quad V_{1y} = 0}_{\text{for above geometry only}}$

note: F_x and F_y are force on fluid
$-F_x$ and $-F_y$ are force on vane due to fluid

If the vane is moving with velocity $\underline{V}_v$, then it is convenient to choose CV moving with the vane

i.e., $\underline{V}_R = \underline{V} - \underline{V}_v$ and $\underline{V}$ used for B also moving with vane

x-equation: $F_x = \int_{CS} \rho u \underline{V}_R \cdot \underline{dA}$

$$F_x = \rho V_{1x}[-(V - V_v)_1 A_1] + \rho V_{2x}[-(V - V_v)_2 A_2]$$

Continuity: $0 = \int \rho \underline{V}_R \cdot \underline{dA}$

i.e., $\rho(V-V_v)_1 A_1 = \rho(V-V_v)_2 A_2 = \rho\underbrace{(V-V_v)A}_{Q_{rel}}$

$F_x = \rho\underbrace{(V-V_v)A}_{Q_{rel}}[V_{2x} - V_{1x}]$

↑

on fluid

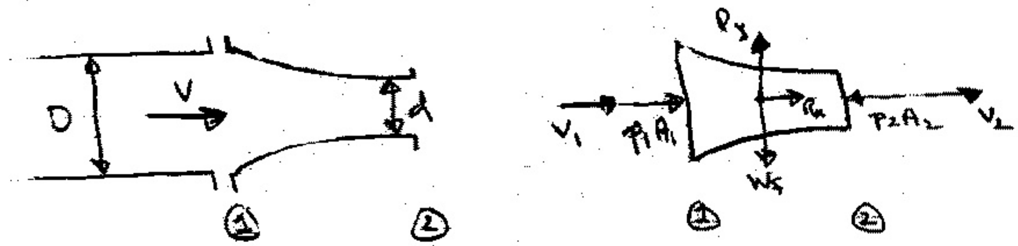
$$\left.\begin{array}{l} V_{2x} = (V - V_v)_{2x} \\ V_{1x} = (V - V_v)_{1x} \end{array}\right\} \text{For coordinate system moving with vane}$$

Power = $-F_x V_v$ i.e., = 0 for $V_v = 0$

$F_y = \rho Q_{rel}(V_{2y} - V_{1y})$

2. Flow through a nozzle

Consider a nozzle at the end of a pipe (or hose). What force is required to hold the nozzle in place?



CV = nozzle and fluid
$\therefore$ (R_x, R_y) = force required to hold nozzle in place

Assume either the pipe velocity or pressure is known. Then, the unknown (velocity or pressure) and the exit velocity V_2 can be obtained from combined use of the continuity and Bernoulli equations.

Bernoulli: $$p_1 + \gamma z_1 + \frac{1}{2}\rho V_1^2 = p_2 + \gamma z_2 + \frac{1}{2}\rho V_2^2 \qquad z_1 = z_2$$

$$p_1 + \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V_2^2$$

Continuity: $$A_1 V_1 = A_2 V_2 = Q$$

$$V_2 = \frac{A_1}{A_2} V_1 = \left(\frac{D}{d}\right)^2 V_1$$

$$p_1 + \frac{1}{2}\rho V_1^2 \left(1 - \left(\frac{D}{d}\right)^4\right) = 0$$

Say p_1 known: $$V_1 = \left[\frac{-2p_1}{\rho\left(1 - \left(D/d\right)^4\right)}\right]^{1/2}$$

To obtain the reaction force R_x apply momentum equation in x-direction

$$\sum F_x = \frac{d}{dt}\int_{CV} u\rho d\forall + \int_{CS} \rho u \underline{V}\cdot\underline{dA}$$
$$= \sum_{CS} \rho u \underline{V}\cdot\underline{A} \qquad \text{steady flow and uniform flow over CS}$$

$$R_x + p_1A_1 - p_2A_2 = \rho V_1(-V_1A_1) + \rho V_2(V_2A_2)$$
$$= \rho Q(V_2 - V_1)$$
$$R_x = \rho Q(V_2 - V_1) - p_1A_1$$

To obtain the reaction force R_y apply momentum equation in y-direction

$$\sum F_y = \sum_{CS} \rho v \underline{V}\cdot\underline{A} = 0 \qquad \text{since no flow in y-direction}$$

$$R_y - W_f - W_N = 0 \quad \text{i.e., } R_y = W_f + W_N$$

Numerical Example: Oil with S = .85 flows in pipe under pressure of 100 psi. Pipe diameter is 3" and nozzle tip diameter is 1"

$$\rho = \frac{S\gamma}{g} = 1.65$$

V_1 = 14.59 ft/s
V_2 = 131.3 ft/s

D/d = 3

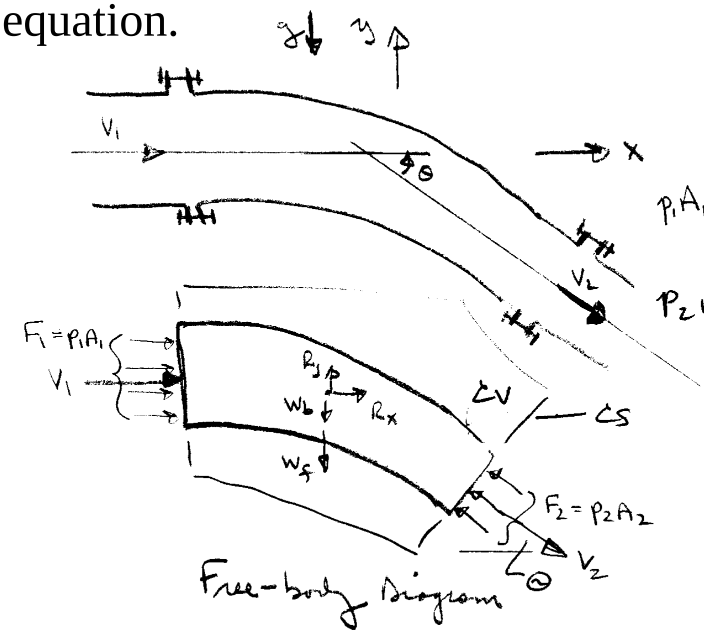
$$Q = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 V_2$$
$$= .716 \text{ ft}^3\text{/s}$$

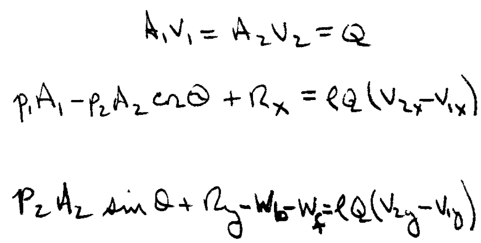
R_x = 141.48 – 706.86 = –569 lbf
R_z = 10 lbf

This is force on nozzle

3. Forces on Bends

Consider the flow through a bend in a pipe. The flow is considered steady and uniform across the inlet and outlet sections. Of primary concern is the force required to hold the bend in place, i.e., the reaction forces R_x and R_y which can be determined by application of the momentum equation.





R_x, R_y = reaction force on bend i.e., force required to hold bend in place

Continuity: $$0 = \sum \rho \underline{V} \cdot \underline{A} = -\rho V_1 A_1 + \rho V_2 A_2$$
i.e., $Q = \text{constant} = V_1A_1 = V_2A_2$

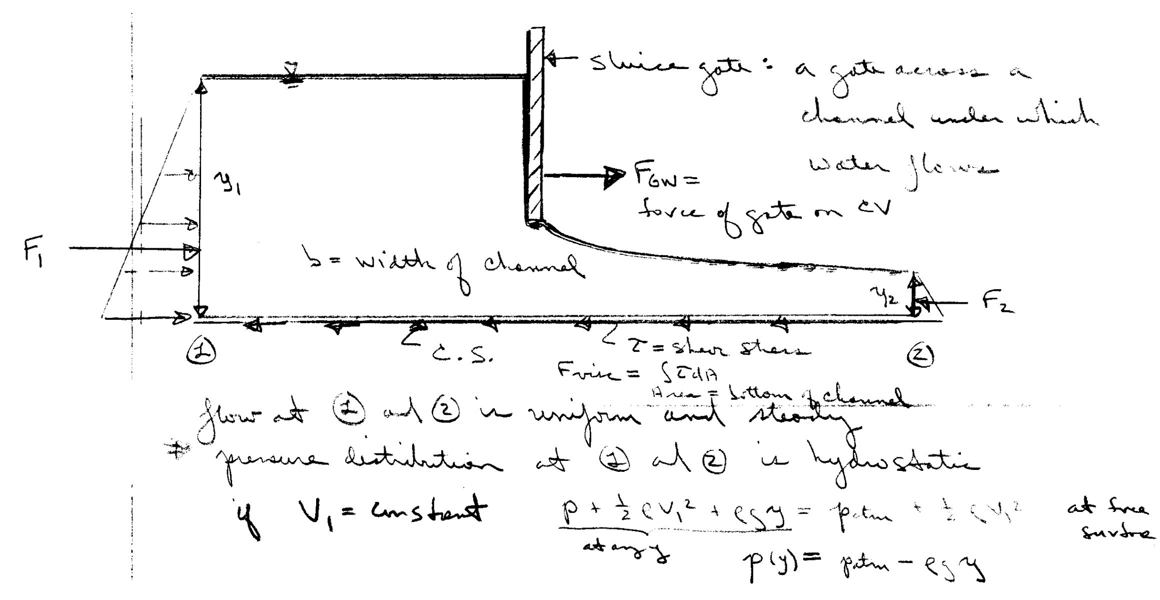
x-momentum: $$\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$$
$$p_1A_1 - p_2A_2\cos\theta + R_x = \rho V_{1x}(-V_1A_1) + \rho V_{2x}(V_2A_2)$$
$$= \rho Q(V_{2x} - V_{1x})$$

y-momentum: $$\sum F_y = \sum \rho v \underline{V} \cdot \underline{A}$$
$$p_2A_2\sin\theta + R_y - w_f - w_b = \rho V_{1y}(-V_1A_1) + \rho V_{2y}(V_2A_2)$$
$$= \rho Q(V_{2y} - V_{1y})$$

4. Problems involving Nonuniform Velocity Distribution
   See text pp. 192 – 194

5. Motion of a Rocket
   See text pp. 194 – 198

6. Force on a rectangular sluice gate



The force on the fluid due to the gate is calculated from the x-momentum equation:

$$\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$$

$$F_1 + F_{GW} - F_{visc} - F_2 = \rho V_1(-V_1 A_1) + \rho V_2(V_2 A_2)$$

$$F_{GW} = F_2 - F_1 + \rho Q(V_2 - V_1) + \cancel{F_{visc}}$$

usually can be neglected

$$= \gamma \frac{y_2}{2} \cdot y_2 b - \gamma \frac{y_1}{2} \cdot y_1 b + \rho Q(V_2 - V_1)$$

$$F_{GW} = \frac{1}{2} b\gamma\left(y_2^2 - y_1^2\right) + \underbrace{\rho Q\left(V_2 - V_1\right)}_{\frac{\rho Q^2}{b}\left(\frac{1}{y_2} - \frac{1}{y_1}\right)}$$

$$V_1 = \frac{Q}{y_1 b}$$

$$V_2 = \frac{Q}{y_2 b}$$

7. Water Hammer
   See text pp. 198 – 204

6.3 Moment of Momentum Equation

See text pp. 248 – 259

6.4 Application of the Energy, Momentum, and Continuity Equations in Combination

In general, when solving fluid mechanics problems, one should use all available equations in order to derive as much information as possible about the flow. For example, consistent with the approximation of the energy equation we can also apply the momentum and continuity equations

Energy:

$$\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2+h_t+h_L$$

Momentum:

$$\sum F_s=\rho V_2^2A_2-\rho V_1^2A_1=\rho Q(V_2-V_1)$$

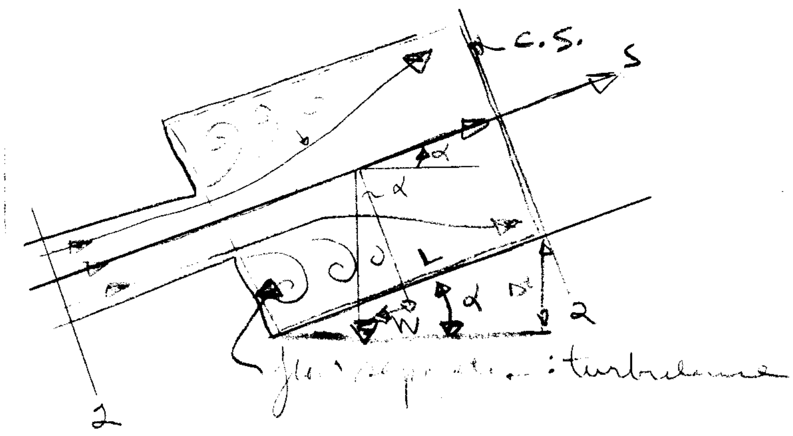
Continuity:

$$A_1V_1 = A_2V_2 = Q = \text{constant}$$

one inlet and
one outlet
ρ = constant

Abrupt Expansion

Consider the flow from a small pipe to a larger pipe. Would like to know $h_L = h_L(V_1,V_2)$. Analytic solution to exact problem is extremely difficult due to the occurrence of flow separations and turbulence. However, if the assumption is made that the pressure in the separation region remains approximately constant and at the value at the point of separation, i.e, p_1, an approximate solution for h_L is possible:



Apply Energy Eq from 1-2 ($\alpha_1 = \alpha_2 = 1$)

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L$$

Momentum eq. For CV shown (shear stress neglected)

$$\sum F_s = p_1A_2 - p_2A_2 - \underbrace{W\sin\alpha}_{\underbrace{\gamma A_2 L}_{W}\underbrace{\frac{\Delta z}{L}}_{\sin\alpha}} = \sum \rho u \underline{V}\cdot\underline{A}$$

$$= \rho V_1(-V_1A_1) + \rho V_2(V_2A_2)$$

$$= \rho V_2^2 A_2 - \rho V_1^2 A_1$$

next divide momentum equation by γA_2

$$+\gamma A_2 \qquad \underbrace{\frac{p_1}{\gamma}-\frac{p_2}{\gamma}-(z_1-z_2)}_{\text{from energy equation}}=\frac{V_2^2}{g}-\frac{V_1^2}{g}\frac{A_1}{A_2}=\underbrace{\frac{V_1^2}{g}\frac{A_1}{A_2}\left(\frac{A_1}{A_2}-1\right)}_{\text{using continuity}}$$

$$\Downarrow$$

$$\frac{V_2^2}{2g}-\frac{V_1^2}{2g}+h_L=\frac{V_2^2}{g}-\frac{V_1^2}{g}\frac{A_1}{A_2}$$

$$h_L=\frac{V_2^2}{2g}+\frac{V_1^2}{2g}\left(1-\frac{2A_1}{A_2}\right)$$

$$h_L=\frac{1}{2g}\left[V_2^2+V_1^2-\underbrace{2V_1^2\frac{A_1}{A_2}}_{-2V_1V_2}\right]$$

continutity eq.

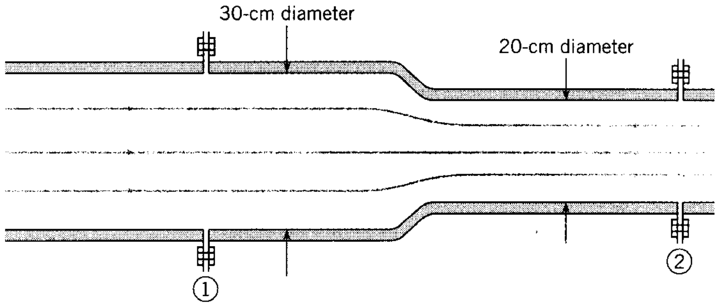
$$V_1A_1 = V_2A_2$$

$$\frac{A_1}{A_2}=\frac{V_2}{V_1}$$

$$\boxed{h_L=\frac{1}{2g}[V_2-V_1]^2}$$

Forces on Transitions

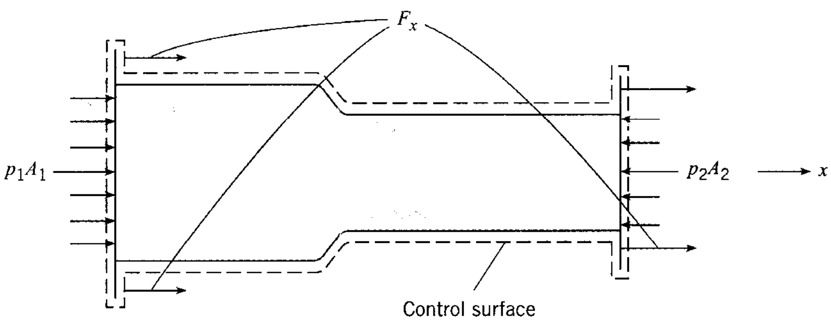
Example 7-6



$Q = .707\ m^3/s$

$$\text{head loss} = .1\frac{V_2^2}{2g}$$

(empirical equation)



Fluid = water
$p_1 = 250$ kPa
$D_1 = 30$ cm
$D_2 = 20$ cm
$F_x = ?$

First apply momentum theorem

$$\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$$

$$F_x + p_1A_1 - p_2A_2 = \rho V_1(-V_1A_1) + \rho V_2(V_2A_2)$$

$$F_x = \rho Q(V_2 - V_1) - p_1A_1 + p_2A_2$$

force required to hold transition in place

The only unknown in this equation is p_2, which can be obtained from the energy equation.

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+h_L$$

note: $z_1 = z_2$ and $\alpha = 1$

$$p_2 = p_1-\gamma\left[\frac{V_2^2}{2g}-\frac{V_1^2}{2g}+h_L\right]$$

drop in pressure

$$\Rightarrow F_x=\rho Q(V_2-V_1)+A_2\underbrace{\left[p_1-\gamma\left(\frac{V_2^2}{2g}-\frac{V_1^2}{2g}+h_L\right)\right]}_{p_2}-p_1A_1$$

(note: if $p_2 = 0$ same as nozzle)

In this equation,

$V_1 = Q/A_1 = 10$ m/s
$V_2 = Q/A_2 = 22.5$ m/s

$$h_L=.1\frac{V_2^2}{2g}=2.58\text{m}$$

continuity $A_1V_1 = A_2V_2$

$$V_2=\frac{A_1}{A_2}V_1$$

i.e. $V_2 > V_1$

$F_x = -8.15$ kN is negative x direction to hold transition in place