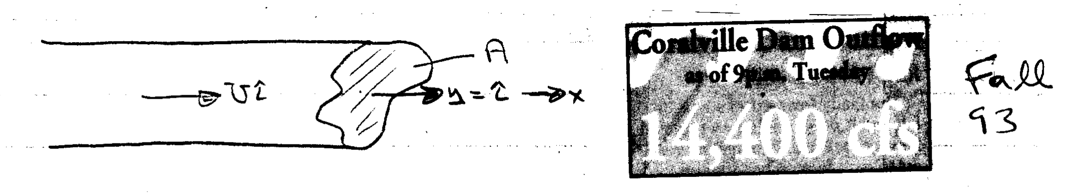
Chapter 5 Finite Control Volume Analysis

5.1 Continuity Equation

1. cross-sectional area oriented normal to velocity vector (simple case where $V \perp A$)

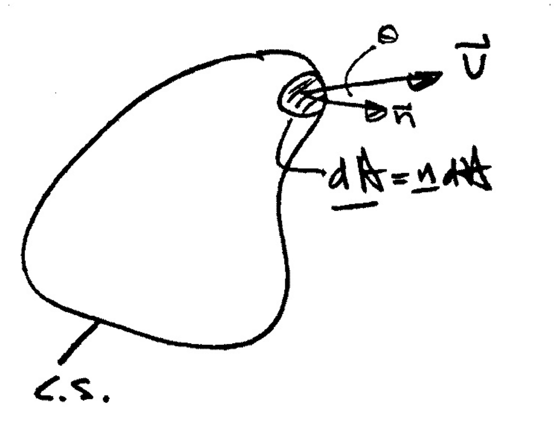


U = constant: Q = volume flux = UA $[\text{m/s} \times \text{m}^2 = \text{m}^3/\text{s}]$

U ≠ constant: $Q = \int_A U dA$

Similarly the mass flux $= \dot{m} = \int_A \rho U dA$

2. general case



$$Q = \int_{CS} \underline{V} \cdot \underline{n} dA$$

$$= \int_{CS} |\underline{V}| \cos\theta dA$$

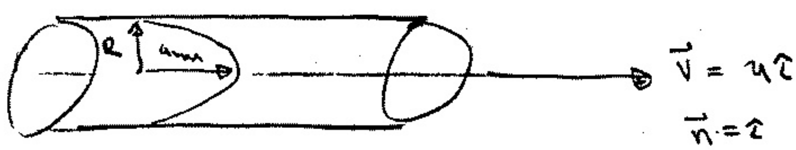
$$\dot{m} = \int_{CS} \rho(\underline{V} \cdot \underline{n}) dA$$

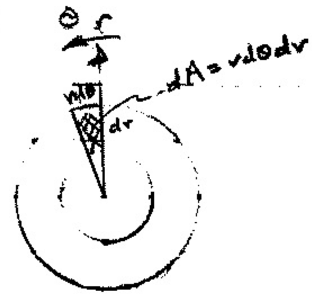
average velocity: $\overline{V} = \dfrac{Q}{A}$

Example:
At low velocities the flow through a long circular tube, i.e. pipe, has a parabolic velocity distribution (actually paraboloid of revolution).

$$u = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

i.e., centerline velocity





a) find Q and $\overline{V}$

$$Q = \int_A \underline{V} \cdot \underline{n} dA = \int_A u dA$$

$$\int_A u dA = \int_0^{2\pi} \int_0^R u(r) r d\theta dr$$

$$= 2\pi \int_0^R u(r) r dr$$

$dA = 2\pi r dr$

$u = u(r)$ and not θ $\therefore$ $\int_0^{2\pi} d\theta = 2\pi$

$$Q = 2\pi \int_0^R u_{max}\left(1-\left(\frac{r}{R}\right)^2\right) r dr = \frac{1}{2}u_{max}\pi R^2$$

$$\overline{V} = \frac{1}{2}u_{max}$$

Continuity Equation

RTT can be used to obtain an integral relationship expressing conservation of mass by defining the extensive property B = M such that β = 1.

$$B = M = \text{mass}$$
$$\beta = dB/dM = 1$$

General Form of Continuity Equation

$$\frac{dM}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho d\forall + \int_{CS} \rho \underline{V} \cdot \underline{dA}$$

or

$$\underbrace{\int_{CS} \rho \underline{V} \cdot \underline{dA}}_{\text{net rate of outflow of mass across CS}} = \underbrace{-\frac{d}{dt}\int_{CV} \rho d\forall}_{\text{rate of decrease of mass within CV}}$$

net rate of outflow of mass across CS — rate of decrease of mass within CV

Simplifications:

1. Steady flow: $-\frac{d}{dt}\int_{CV} \rho d\forall = 0$

2. $\underline{V}$ = constant over discrete $\underline{dA}$ (flow sections):

$$\int_{CS} \rho \underline{V} \cdot \underline{dA} = \sum_{CS} \rho \underline{V} \cdot \underline{A}$$

3. Incompressible fluid (ρ = constant)

$$\int_{CS} \underline{V} \cdot \underline{dA} = -\frac{d}{dt} \int_{CV} d\forall$$ conservation of volume

4. Steady One-Dimensional Flow in a Conduit:

$$\sum_{CS} \rho \underline{V} \cdot \underline{A} = 0$$

$$-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0$$

for ρ = constant $Q_1 = Q_2$

Some useful definitions:

Mass flux $\dot{m} = \int_A \rho \underline{V} \cdot \underline{dA}$

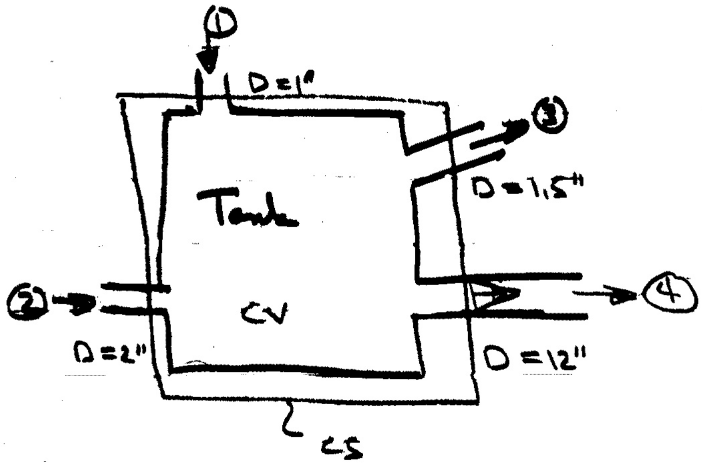
Volume flux $Q = \int_A \underline{V} \cdot \underline{dA}$

Average Velocity $\overline{V} = Q / A$

Average Density $\overline{\rho} = \frac{1}{A} \int \rho dA$

Note: $\dot{m} \neq \overline{\rho} Q$ unless ρ = constant

Example



*Steady flow

*$V_{1,2,3}$ = 50 fps

*@ ℘ V varies linearly from zero at wall to V_{max} at pipe center

*find $\dot{m}_4$, Q_4, V_{max}

*water, $\rho_w = 1.94\ \text{slug/ft}^3$

$$\int_{CS} \rho \underline{V} \cdot d\underline{A} = 0 = -\frac{d}{dt}\int_{CV} \rho d\forall \quad (\text{unsteady term} \to 0)$$

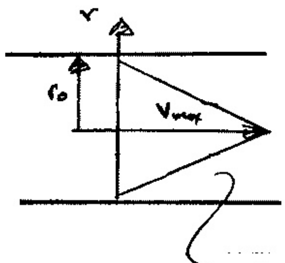
i.e., $-\rho_1 V_1 A_1 - \rho_2 V_2 A_2 + \rho_3 V_3 A_3 + \boxed{\rho \int_{A_4} V_4 dA_4} = 0$ (boxed term $= \dot{m}_4$)

ρ = const. = 1.94 lb-s^2/ft^4 = 1.94 slug/ft^3

$$\dot{m}_4 = \rho \int V_4 dA_4 = \rho V(A_1 + A_2 - A_3) \qquad V_1=V_2=V_3=V=50\text{f/s}$$

$$= \frac{1.94}{144} \times 50 \times \frac{\pi}{4}\left(1^2 + 2^2 - 1.5^2\right)$$

$= 1.45$ slugs/s



$V = V_{max}(1 - r/r_o)$, $dA_4 = r\,dr\,d\theta$

$$Q_4 = \dot{m}_4 / \rho = .75 \text{ ft}^3/\text{s}$$

$$= \int_{A_4} V_4 \, dA_4$$

$$Q_4 = \int_0^{r_o} \int_0^{2\pi} \underbrace{V_{max}\left(1 - \frac{r}{r_o}\right)}_{V_4 \neq V_4(\theta)} \underbrace{r d\theta dr}_{dA_4}$$

(velocity profile)

$$= 2\pi \int_0^{r_o} V_{max}\left(1 - \frac{r}{r_o}\right) r dr$$

$$= 2\pi V_{max} \int_0^{r_o} \left[r - \frac{r^2}{r_o} \right] dr$$

$$\overline{V}_4 = \frac{Q}{A} = \frac{\frac{1}{3}\pi r_o^2 V_{max}}{\pi r_o^2} = \frac{1}{3} V_{max}$$

$$= 2\pi V_{max} \left[\left. \frac{r^2}{2} \right|_0^{r_0} - \left. \frac{r^3}{3r_o} \right|_0^{r_o} \right]$$

$$= 2\pi V_{max} r_o^2 \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{1}{3} \pi r_o^2 V_{max}$$

$$V_{max} = \frac{Q_4}{\frac{1}{3}\pi r_o^2} = 2.86 \text{ fps}$$

5.2 Momentum Equation

RTT with B = M$\underline{V}$ and β = $\underline{V}$

$$\sum\left[\underline{F}_S + \underline{F}_B\right] = \frac{d}{dt}\int_{CV} \rho \underline{V} d\forall + \int_{CS} \underline{V}\rho \underline{V}_R \cdot \underline{dA}$$

$\underline{V}$ = velocity referenced to an inertial frame (non-accelerating)
$\underline{V}_R$ = velocity referenced to control volume
$\underline{F}_S$ = surface forces + reaction forces (due to pressure and viscous normal and shear stresses)
$\underline{F}_B$ = body force (due to gravity)

Applications of the Momentum Equation
Initial Setup and Signs
1. Jet deflected by a plate or a vane
2. Flow through a nozzle
3. Forces on bends
4. Problems involving non-uniform velocity distribution
5. Motion of a rocket
6. Force on rectangular sluice gate
7. Water hammer

Derivation of the Basic Equation

Recall RTT: $$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho d\forall + \int_{CS} \beta\rho \underline{V}_R \cdot \underline{dA}$$

General form for moving but non-accelerating reference frame

$\underline{V}_R$=velocity relative to CS=$\underline{V}$ – $\underline{V}_S$=absolute – velocity CS
Subscript not shown in text but implied!
i.e., referenced to CV

Let, B = M$\underline{V}$ = linear momentum
β = $\underline{V}$

$\underline{V}$ must be referenced to inertial reference frame

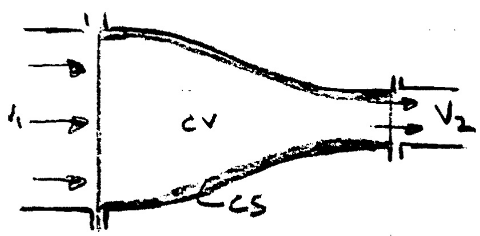
$$\underbrace{\frac{d(M\underline{V})}{dt} = \sum \underline{F}}_{\text{Newton's 2nd law}} = \frac{d}{dt}\int_{CV} \underline{V}\rho d\forall + \int_{CS} \underline{V}\rho \underline{V}_R \cdot \underline{dA}$$

CS ← Must be relative to a non-accelerating inertial reference frame

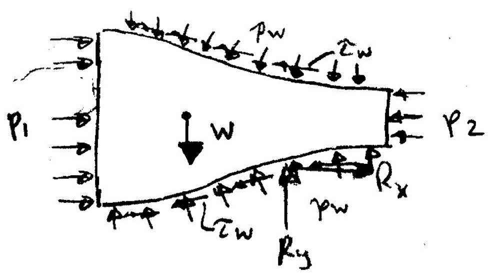
where $\Sigma\underline{F}$ = vector sum of all forces acting on the control volume including both surface and body forces

$= \Sigma\underline{F}_S + \Sigma\underline{F}_B$

$\Sigma\underline{F}_S$ = sum of all external surface forces acting at the CS, i.e., pressure forces, forces transmitted through solids, shear forces, etc.



$\Sigma\underline{F}_B$ = sum of all external body forces, i.e., gravity force



free body diagram

$\Sigma F_x = p_1A_1 - p_2A_2 + R_x$
$\Sigma F_y = -W + R_y$

$\underline{R}$ = resultant force on fluid in CV due to p_w and τ_w

i.e., reaction force on fluid

Important Features (to be remembered)

1) Vector equation to get component in any direction must use dot product

<u>x equation</u>

$$\sum F_x = \frac{d}{dt}\int_{CV} \rho u d\forall + \int_{CS} \rho u \underline{V}_R \cdot \underline{dA}$$

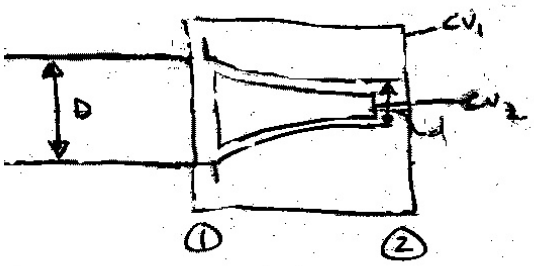
Carefully define coordinate system with forces positive in positive direction of coordinate axes y equation

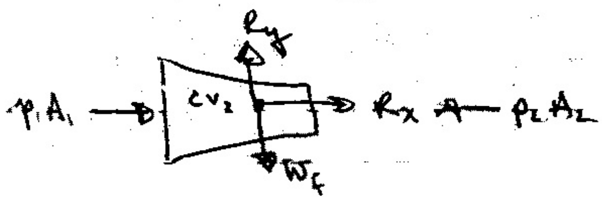
$$\sum F_y = \frac{d}{dt}\int_{CV} \rho v dV\!\!\!\!- + \int_{CS} \rho v \underline{V}_R \cdot \underline{dA}$$

z equation

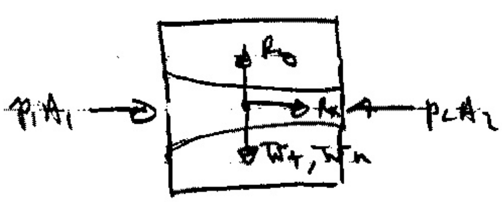
$$\sum F_z = \frac{d}{dt}\int_{CV} \rho w dV\!\!\!\!- + \int_{CS} \rho w \underline{V}_R \cdot \underline{dA}$$

2) Carefully define control volume and be sure to include all external body and surface faces acting on it.
For example,





(R_x,R_y) = reaction force on fluid



(R_x,R_y) = reaction force on nozzle

3) Velocity $\underline{V}$ must be referenced to a non-accelerating inertial reference frame. Sometimes it is advantageous to use a moving (at constant velocity) reference frame. Note $\underline{V}_R = \underline{V} - \underline{V}_s$ is always relative to CS.

i.e., in these cases $\underline{V}$ used for B also referenced to CV (i.e., $\underline{V} = V_{\underline{R}}$)

4) Steady vs. Unsteady Flow

Steady flow $\Rightarrow \frac{d}{dt}\int_{CV} \rho \underline{V} d\forall = 0$

5) Uniform vs. Nonuniform Flow

$\int_{CS} \underline{V}\rho\underline{V}_R \cdot \underline{dA}$ = change in flow of momentum across CS

$= \Sigma \underline{V}\rho\underline{V}_R \cdot \underline{A}$ uniform flow across $\underline{A}$

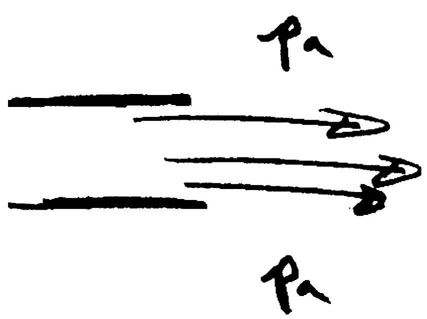
6) $\underline{F}_{pres} = -\int p\underline{n}dA$ $\int_V \nabla f d\forall = \int_S f \underline{n} ds$

f = constant, $\nabla f = 0$

= 0 for p = constant and for a closed surface

i.e., always use gage pressure

7) Pressure condition at a jet exit

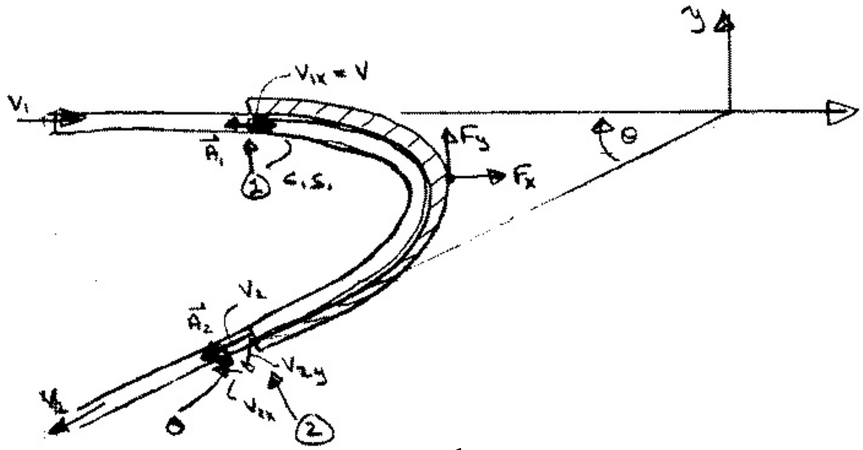


at an exit into the atmosphere jet pressure must be p_a

<u>Application of the Momentum Equation</u>

1. Jet deflected by a plate or vane

Consider a jet of water turned through a horizontal angle



CV and CS are for jet so that F_x and F_y are vane reactions forces on fluid

x-equation: $\sum F_x = F_x = \frac{d}{dt}\int \rho u d\forall + \int_{CS} \rho u \underline{V} \cdot \underline{dA}$

$F_x = \sum_{CS} \rho u \underline{V} \cdot \underline{A}$ steady flow

$= \rho V_{1x}(-V_1 A_1) + \rho V_{2x}(V_2 A_2)$

continuity equation: $\rho A_1 V_1 = \rho A_2 V_2 = \rho Q$ for $A_1 = A_2$, $V_1 = V_2$

$F_x = \rho Q(V_{2x} - V_{1x})$

y-equation: $\sum F_y = F_y = \sum_{CS} \rho v \underline{V} \cdot \underline{A}$

$F_y = \rho V_{1y}(-A_1 V_1) + \rho V_{2y}(-A_2 V_2)$

$= \rho Q(V_{2y} - V_{1y})$

where: $\underbrace{V_{1x} = V_1 \quad V_{2x} = -V_2\cos\theta \quad V_{2y} = -V_2\sin\theta \quad V_{1y} = 0}_{\text{for above geometry only}}$

note: F_x and F_y are force on fluid
$-F_x$ and $-F_y$ are force on vane due to fluid

If the vane is moving with velocity $\underline{V}_v$, then it is convenient to choose CV moving with the vane i.e., $\underline{V}_R = \underline{V} - \underline{V}_v$ and $\underline{V}$ used for B also moving with vane

x-equation: $F_x = \int_{CS} \rho u \underline{V}_R \cdot \underline{dA}$

$$F_x = \rho V_{1x}[-(V - V_v)_1 A_1] + \rho V_{2x}[(V - V_v)_2 A_2]$$

Continuity: $0 = \int \rho \underline{V}_R \cdot \underline{dA}$

i.e., $\rho(V-V_v)_1 A_1 = \rho(V-V_v)_2 A_2 = \rho \underbrace{(V-V_v)A}_{Q_{rel}}$

$F_x = \rho \underbrace{(V-V_v)A}_{Q_{rel}} [V_{2x} - V_{1x}]$

↑ on fluid

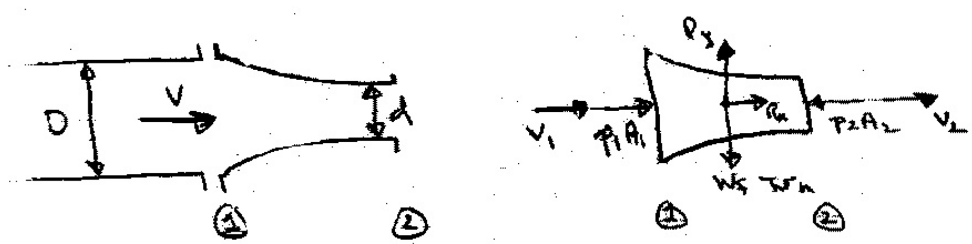
$$\left.\begin{array}{l} V_{2x} = (V - V_v)_{2x} \\ V_{1x} = (V - V_v)_{1x} \end{array}\right\}$$ For coordinate system moving with vane

Power = $-F_x V_v$ i.e., = 0 for $V_v = 0$

$F_y = \rho Q_{rel}(V_{2y} - V_{1y})$

2. Flow through a nozzle

Consider a nozzle at the end of a pipe (or hose). What force is required to hold the nozzle in place?



CV = nozzle and fluid
∴ (R_x, R_y) = force required to hold nozzle in place

Assume either the pipe velocity or pressure is known. Then, the unknown (velocity or pressure) and the exit velocity V_2 can be obtained from combined use of the continuity and Bernoulli equations.

Bernoulli: $$p_1 + \gamma z_1 + \frac{1}{2}\rho V_1^2 = p_2 + \gamma z_2 + \frac{1}{2}\rho V_2^2 \qquad z_1 = z_2$$

$$p_1 + \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V_2^2$$

Continuity: $$A_1 V_1 = A_2 V_2 = Q$$

$$V_2 = \frac{A_1}{A_2} V_1 = \left(\frac{D}{d}\right)^2 V_1$$

$$p_1 + \frac{1}{2}\rho V_1^2 \left(1 - \left(\frac{D}{d}\right)^4\right) = 0$$

Say p_1 known: $$V_1 = \left[\frac{-2p_1}{\rho\left(1 - \left(D/d\right)^4\right)}\right]^{1/2}$$

To obtain the reaction force R_x apply momentum equation in x-direction

$$\sum F_x = \frac{d}{dt}\int_{CV} u\rho d\forall + \int_{CS} \rho u \underline{V} \cdot \underline{dA}$$

$$= \sum_{CS} \rho u \underline{V} \cdot \underline{A}$$ steady flow and uniform flow over CS

$$R_x + p_1 A_1 - p_2 A_2 = \rho V_1(-V_1 A_1) + \rho V_2 (V_2 A_2)$$
$$= \rho Q (V_2 - V_1)$$

$$R_x = \rho Q(V_2 - V_1) - p_1 A_1$$

To obtain the reaction force R_y apply momentum equation in y-direction

$$\sum F_y = \sum_{CS} \rho v \underline{V} \cdot \underline{A} = 0 \qquad \text{since no flow in y-direction}$$

$$R_y - W_f - W_N = 0 \quad \text{i.e., } R_y = W_f + W_N$$

Numerical Example: Oil with S = .85 flows in pipe under pressure of 100 psi. Pipe diameter is 3" and nozzle tip diameter is 1"

$$\rho = \frac{S\gamma}{g} = 1.65$$

$$D/d = 3$$

$$V_1 = 14.59 \text{ ft/s}$$

$$V_2 = 131.3 \text{ ft/s}$$

$$Q = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 V_2 = .716 \text{ ft}^3/\text{s}$$

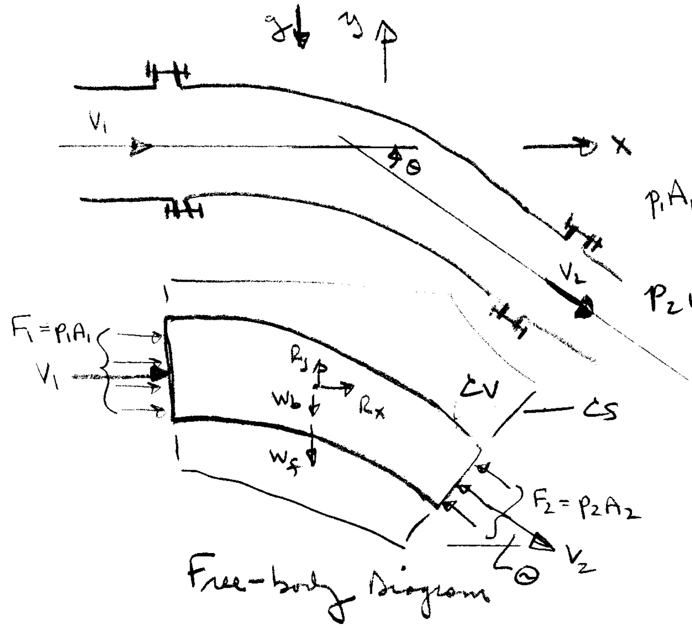
$$R_x = 141.48 - 706.86 = -569 \text{ lbf}$$

$$R_z = 10 \text{ lbf}$$

This is force on nozzle

3. Forces on Bends

Consider the flow through a bend in a pipe. The flow is considered steady and uniform across the inlet and outlet sections. Of primary concern is the force required to hold the bend in place, i.e., the reaction forces R_x and R_y which can be determined by application of the momentum equation.



$A_1V_1 = A_2V_2 = Q$

$p_1A_1 - p_2A_2\cos\theta + R_x = \rho Q(V_{2x} - V_{1x})$

$p_2A_2\sin\theta + R_y - W_b - W_f = \rho Q(V_{2y} - V_{1y})$

R_x, R_y = reaction force on bend i.e., force required to hold bend in place

Continuity: $0 = \sum \rho \underline{V} \cdot \underline{A} = -\rho V_1 A_1 + \rho V_2 A_2$

i.e., Q = constant = $V_1A_1 = V_2A_2$

x-momentum: $\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$

$$p_1A_1 - p_2A_2\cos\theta + R_x = \rho V_{1x}(-V_1A_1) + \rho V_{2x}(V_2A_2)$$
$$= \rho Q(V_{2x} - V_{1x})$$

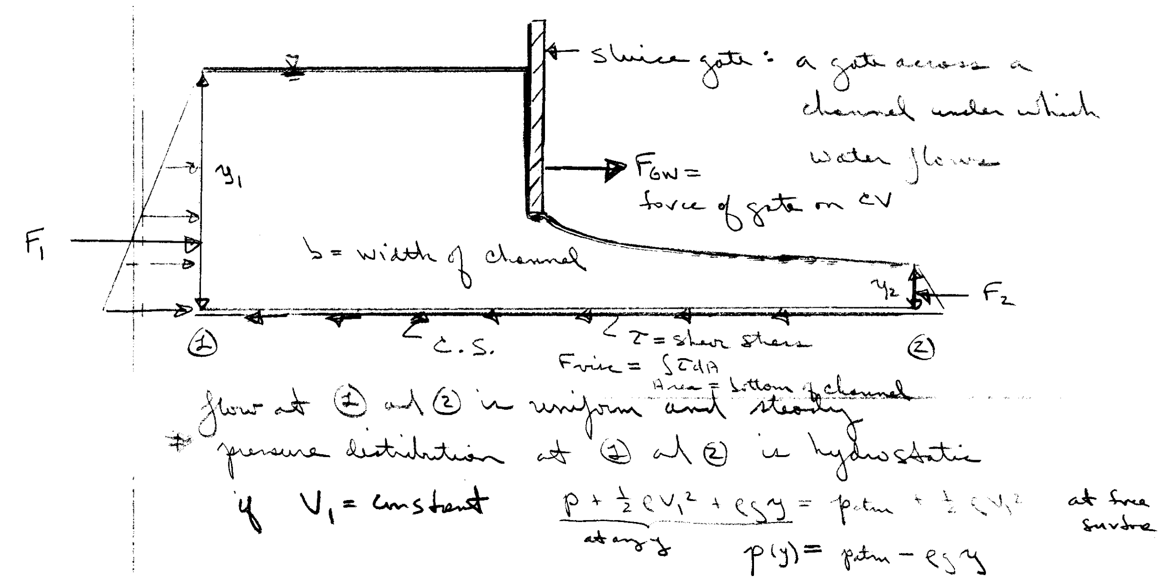
y-momentum: $\sum F_y = \sum \rho v \underline{V} \cdot \underline{A}$

$$p_2A_2\sin\theta + R_y - w_f - w_b = \rho V_{1y}(-V_1A_1) + \rho V_{2y}(V_2A_2)$$
$$= \rho Q(V_{2y} - V_{1y})$$

4. Problems involving Nonuniform Velocity Distribution
   See text pp. 215– 216

5. Force on a rectangular sluice gate

The force on the fluid due to the gate is calculated from the x-momentum equation:



$$\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$$

$$F_1 + F_{GW} - F_{visc} - F_2 = \rho V_1(-V_1 A_1) + \rho V_2(V_2 A_2)$$

$$F_{GW} = F_2 - F_1 + \rho Q(V_2 - V_1) + F_{visc}$$

(F_{visc}: usually can be neglected)

$$= \gamma \frac{y_2}{2} \cdot y_2 b - \gamma \frac{y_1}{2} \cdot y_1 b + \rho Q(V_2 - V_1)$$

$$F_{GW} = \frac{1}{2} b\gamma \left(y_2^2 - y_1^2\right) + \underbrace{\rho Q(V_2 - V_1)}_{\frac{\rho Q^2}{b}\left(\frac{1}{y_2} - \frac{1}{y_1}\right)}$$

$$V_1 = \frac{Q}{y_1 b}$$

$$V_2 = \frac{Q}{y_2 b}$$

<u>Moment of Momentum Equation</u>

See text pp. 221 – 229

5.3 Energy Equation

Derivation of the Energy Equation

The First Law of Thermodynamics

The difference between the heat added to a system and the work done by a system depends only on the initial and final states of the system; that is, depends only on the change in energy E: principle of conservation of energy

$$\Delta E = Q - W$$

ΔE = change in energy
Q = heat added to the system
W = work done by the system

$E = E_u + E_k + E_p$ = total energy of the system

- E_u: Internal energy due to molecular motion
- E_k: kinetic energy
- E_p: potential energy

The differential form of the first law of thermodynamics expresses the rate of change of E with respect to time

$$\frac{dE}{dt} = \dot{Q} - \dot{W}$$

- $\dot{Q}$: rate of heat transfer to system
- $\dot{W}$: rate of work being done by system

Energy Equation for Fluid Flow

The energy equation for fluid flow is derived from Reynolds transport theorem with

B_{system} = E = total energy of the system (extensive property)

β = E/mass = e = energy per unit mass (intensive property)
$= \hat{u} + e_k + e_p$

$$\frac{dE}{dt} = \frac{d}{dt}\int_{CV} \rho e d\mathcal{V} + \int_{CS} \rho e \underline{V} \cdot \underline{dA}$$

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{CV} \rho(\hat{u} + e_k + e_p) d\mathcal{V} + \int_{CS} \rho(\hat{u} + e_k + e_p)\underline{V} \cdot \underline{dA}$$

This can be put in a more useable form by noting the following:

$$e_k = \frac{\text{Total KE of mass with velocity V}}{\text{mass}} = \frac{\Delta M V^2/2}{\Delta M} = \frac{V^2}{2} \qquad V^2 = |\underline{V}|$$

$$e_p = \frac{E_p}{\Delta M} = \frac{\gamma \Delta\mathcal{V} z}{\rho \Delta\mathcal{V}} = gz \qquad \text{(for } E_p \text{ due to gravity only)}$$

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{CV} \rho\left(\frac{V^2}{2} + gz + \hat{u}\right) d\mathcal{V} + \int_{Cs} \rho\left(\frac{V^2}{2} + gz + \hat{u}\right)\underline{V} \cdot \underline{dA}$$

rate of work done by system

rate of change of energy in CV

flux of energy out of CV (ie, across CS)

rate of heat transfer to sysem

Rate of Work Components: $\dot{W} = \dot{W}_s + \dot{W}_f$

For convenience of analysis, work is divided into shaft work W_s and flow work W_f

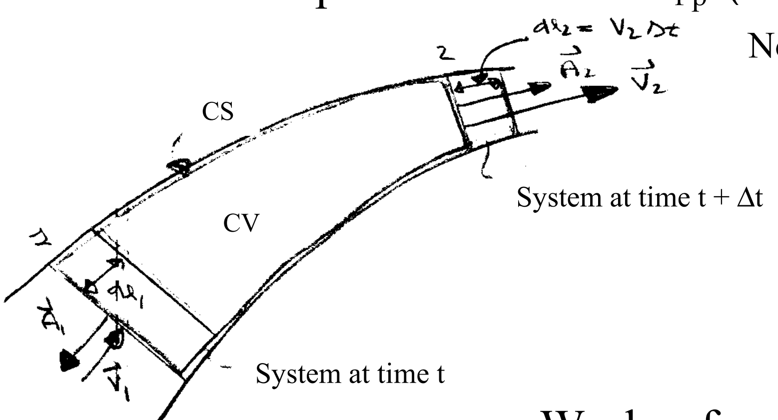
W_f = net work done on the surroundings as a result of normal and tangential stresses acting at the control surfaces

$= W_{f\,pressure} + W_{f\,shear}$

W_s = any other work transferred to the surroundings usually in the form of a shaft which either takes energy out of the system (turbine) or puts energy into the system (pump)

Flow work due to pressure forces $W_{f\,p}$ (for system)

Note: here $\underline{V}$ uniform over $\underline{A}$



Work = force × distance

at 2 $W_2 = p_2 A_2 \times V_2 \Delta t$ (on surroundings)

rate of work⇒ $\dot{W}_2 = p_2 A_2 V_2 = p_2 \underline{V}_2 \cdot \underline{A}_2$

at 1 $W_1 = -p_1 A_1 \times V_1 \Delta t$

$\dot{W}_1 = p_1 \underline{V}_1 \cdot \underline{A}_1$

neg. sign since pressure force on surrounding fluid acts in a direction opposite to the motion of the system boundary

In general,

$$\dot{W}_{fp} = p\underline{V} \cdot \underline{A}$$

for more than one control surface and $\underline{V}$ not necessarily uniform over $\underline{A}$:

$$\dot{W}_{fp} = \int_{CS} p\underline{V} \cdot \underline{dA} = \int_{CS} \rho\left(\frac{p}{\rho}\right)\underline{V} \cdot \underline{dA}$$

$$\dot{W}_f = \dot{W}_{fp} + \dot{W}_{fshear}$$

Basic form of energy equation

$$\dot{Q} - \dot{W}_s - \dot{W}_{fshear} - \int_{CS} \rho\left(\frac{p}{\rho}\right)\underline{V} \cdot \underline{dA} = \frac{d}{dt}\int_{CV} \rho\left(\frac{V^2}{2} + gz + \hat{u}\right)d\forall + \int_{CS} \rho\left(\frac{V^2}{2} + gz + \hat{u}\right)\underline{V} \cdot \underline{dA}$$

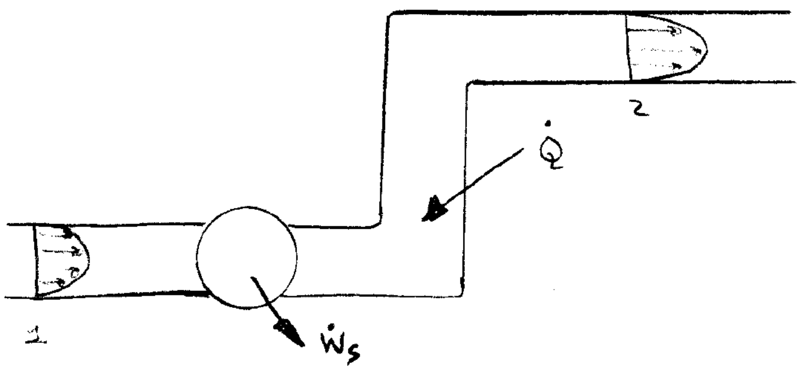
$$\dot{Q} - \dot{W}_s - \cancel{\dot{W}_{fshear}} = \frac{d}{dt}\int_{CV} \rho\left(\frac{V^2}{2} + gz + \hat{u}\right)d\forall + \int_{CS} \rho\left(\frac{V^2}{2} + gz + \underbrace{\hat{u} + \frac{p}{\rho}}_{h=\text{enthalpy}}\right)\underline{V} \cdot \underline{dA}$$

Usually this term can be eliminated by proper choice of CV, i.e. CS normal to flow lines. Also, at fixed boundaries the velocity is zero (no slip condition) and no shear stress flow work is done. Not included or discussed in text!

Simplified Forms of the Energy Equation

Energy Equation for Steady One-Dimensional Pipe Flow

Consider flow through the pipe system as shown



Energy Equation (steady flow)

$$\dot{Q} - \dot{W}_s = \int_{CS} \rho \left(\frac{V^2}{2} + gz + \frac{p}{\rho} + \hat{u} \right) \underline{V} \cdot \underline{dA}$$

$$\dot{Q} - \dot{W}_s + \int_{A_1} \left(\frac{p_1}{\rho} + gz_1 + \hat{u}_1 \right) \rho_1 V_1 A_1 + \int_{A_1} \frac{\rho_1 V_1^3}{2} dA_1$$

$$= \int_{A_2} \left(\frac{p_2}{\rho} + gz_2 + \hat{u}_2 \right) \rho_2 V_2 A_2 + \int_{A_2} \frac{\rho_2 V_2^3}{2} dA_2$$

*Although the velocity varies across the flow sections the streamlines are assumed to be straight and parallel; consequently, there is no acceleration normal to the streamlines and the pressure is hydrostatically distributed, i.e., $p/\rho + gz =$ constant.

*Furthermore, the internal energy u can be considered as constant across the flow sections, i.e. T = constant. These quantities can then be taken outside the integral sign to yield

$$\dot{Q} - \dot{W}_s + \left(\frac{p_1}{\rho} + gz_1 + \hat{u}_1\right)\rho\int_{A_1} V_1 dA_1 + \rho\int_{A_1} \frac{V_1^3}{2} dA_1$$

$$= \left(\frac{p_2}{\rho} + gz_2 + \hat{u}_2\right)\rho\int_{A_2} V_2 dA_2 + \rho\int_{A_2} \frac{V_2^3}{2} dA_2$$

Recall that $Q = \int \underline{V} \cdot \underline{dA} = \overline{V}A$

So that $\rho\int \underline{V} \cdot \underline{dA} = \rho\overline{V}A = \dot{m}$ mass flow rate

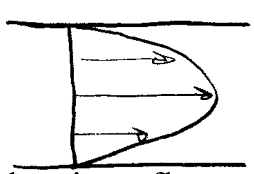
Define: $$\underbrace{\rho\int_A \frac{V^3}{2} dA}_{\text{K.E. flux}} = \alpha\underbrace{\frac{\rho\overline{V}^3 A}{2}}_{\text{K.E. flux for } V=\overline{V}=\text{constant across pipe}} = \alpha\frac{\overline{V}^2}{2}\dot{m}$$

i.e., $$\alpha = \frac{1}{A}\int_A \left(\frac{V}{\overline{V}}\right)^3 dA$$ = kinetic energy correction factor

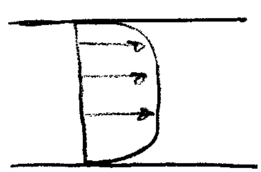
$$\dot{Q} - \dot{W} + \left(\frac{p_1}{\rho} + gz_1 + \hat{u}_1 + \alpha_1\frac{\overline{V}_1^2}{2}\right)\dot{m} = \left(\frac{p_2}{\rho} + gz_2 + \hat{u}_2 + \alpha_2\frac{\overline{V}_2^2}{2}\right)\dot{m}$$

$$\frac{1}{\dot{m}}\left(\dot{Q} - \dot{W}\right) + \frac{p_1}{\rho} + gz_1 + \hat{u}_1 + \alpha_1\frac{\overline{V}_1^2}{2} = \frac{p_2}{\rho} + gz_2 + \hat{u}_2 + \alpha_2\frac{\overline{V}_2^2}{2}$$

Nnote that: $\alpha = 1$ if V is constant across the flow section
$\alpha > 1$ if V is nonuniform



laminar flow $\alpha = 2$



turbulent flow $\alpha = 1.05$ ~ 1 may be used

Shaft Work

Shaft work is usually the result of a turbine or a pump in the flow system. When a fluid passes through a turbine, the fluid is doing shaft work on the surroundings; on the other hand, a pump does work on the fluid

$\dot{W}_s = \dot{W}_t - \dot{W}_p$ where $\dot{W}_t$ and $\dot{W}_p$ are magnitudes of power $\left(\frac{\text{work}}{\text{time}}\right)$

Using this result in the energy equation and deviding by g results in

$$\underbrace{\frac{\dot{W}_p}{\dot{m}g} + \frac{p_1}{\gamma} + z_1 + \alpha_1 \frac{\overline{V}_1^2}{2} = \frac{\dot{W}_t}{\dot{m}g} + \frac{p_2}{\gamma} + z_2 + \alpha_2 \frac{\overline{V}_2^2}{2}}_{\text{mechanical part}} + \underbrace{\frac{\hat{u}_2 - \hat{u}_1}{g} - \frac{\dot{Q}}{\dot{m}g}}_{\text{thermal part}}$$

Note: each term has dimensions of length

Define the following:

$$h_p = \frac{\dot{W}_p}{\dot{m}g} = \frac{\dot{W}_p}{\rho Q g} = \frac{\dot{W}_p}{\gamma Q}$$

$$h_t = \frac{\dot{W}_t}{\dot{m}g}$$

$$h_L = \frac{\hat{u}_2 - \hat{u}_1}{g} - \frac{\dot{Q}}{\dot{m}g} = head\ loss$$

Head Loss

In a general fluid system a certain amount of mechanical energy is converted to thermal energy due to viscous action. This effect results in an increase in the fluid internal energy. Also, some heat will be generated through energy dissipation and be lost (i.e. $-\dot{Q}$). Therefore the term

from 2^{nd} law

$$h_L = \frac{\hat{u}_2 - \hat{u}_1}{g} - \frac{\dot{Q}}{g\dot{m}} > 0$$

represents a loss in mechanical energy due to viscous stresses

Note that adding $\dot{Q}$ to system will not make $h_L = 0$ since this also increases Δu. It can be shown from 2^{nd} law of thermodynamics that $h_L > 0$.

Drop — over $\overline{V}$ and understand that V in energy equation refers to average velocity.

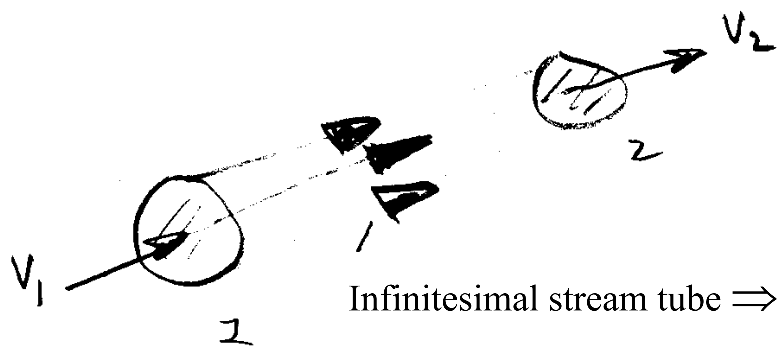
Using the above definitions in the energy equation results in (steady 1-D incompressible flow)

$$\underbrace{\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_L}$$

form of energy equation used for this course!

Comparison of Energy Equation and Bernoulli Equation

Apply energy equation to a stream tube without any shaft work



Infinitesimal stream tube ⇒ $\alpha_1 = \alpha_2 = 1$

Energy eq : $$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$

- If $h_L = 0$ (i.e., $\mu = 0$) we get Bernoulli equation and conservation of mechanical energy along a streamline

- Therefore, energy equation for steady 1-D pipe flow can be interpreted as a modified Bernoulli equation to include viscous effects (h_L) and shaft work (h_p or h_t)

Summary of the Energy Equation

The energy equation is derived from RTT with

B = E = total energy of the system

β = e = E/M = energy per unit mass

$$= \hat{u} + \frac{1}{2}V^2 + gz$$

internal KE PE

$$\frac{dE}{dt} = \frac{d}{dt}\int_{CV} \rho e d\forall + \int_{CS} \rho e \underline{V}\cdot\underline{dA} = \dot{Q} - \dot{W}$$

from 1st Law of Thermodynamics

$\dot{Q}$: heat add; $\dot{W}$: work done

$$\dot{W} = \dot{W}_s + \dot{W}_p + \dot{W}_v$$

$\dot{W}_s$: shaft work done on or by system (pump or turbine)

$\dot{W}_p$: pressure work done on CS

$\dot{W}_v$: Viscous stress work on CS (Neglected in text presentation)

$$\dot{W}_p = \int_{CV} p\underline{V}\cdot\underline{dA} = \int_{CS} \rho(p/\rho)\underline{V}\cdot\underline{dA}$$

$$\dot{W}_s = \dot{W}_t - \dot{W}_p$$

$$\dot{Q} - \dot{W}_t + \dot{W}_p = \frac{d}{dt}\int_{CV} \rho e d\forall + \int_{CS} \rho(e + p/e)\underline{V}\cdot\underline{dA}$$

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

For steady 1-D pipe flow (one inlet and one outlet):

1) Streamlines are straight and parallel

$\Rightarrow p/\rho + gz =$ constant across CS

2) T = constant ⇒ u = constant across CS

3) define $\alpha = \frac{1}{A}\int_{CS}\left(\frac{V}{\overline{\overline{V}}}\right)^3 dA$ = KE correction factor

$$\Rightarrow \quad \frac{\rho}{2}\int V^3 dA = \alpha\frac{\rho\overline{V}^3}{2}A = \alpha\frac{\overline{V}^2}{2}\dot{m}$$

$$\overbrace{\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2}^{\text{mechanical energy}}+h_t+h_L$$

Thermal energy (→ h_L)

$h_p = \dot{W}_p/\dot{m}g$

$h_t = \dot{W}_t/\dot{m}g$

$h_L = \frac{\hat{u}_2-\hat{u}_1}{g} - \frac{\dot{Q}}{\dot{m}g}$ = head loss

> 0 represents loss in mechanical energy due to viscosity

<u>Note: each term has</u> units of length

V is average velocity (vector dropped) and corrected by α

Concept of Hydraulic and Energy Grade Lines

$$\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2+h_t+h_L$$

Define $\text{HGL}=\dfrac{p}{\gamma}+z$

$\text{EGL}=\dfrac{p}{\gamma}+z+\alpha\dfrac{V^2}{2g}$

} point-by-point application is graphically displayed

HGL corresponds to pressure tap measurement + z
EGL corresponds to stagnation tube measurement + z

$EGL_1 = EGL_2 + h_L$
for $h_p = h_t = 0$

EGL = HGL if V = 0

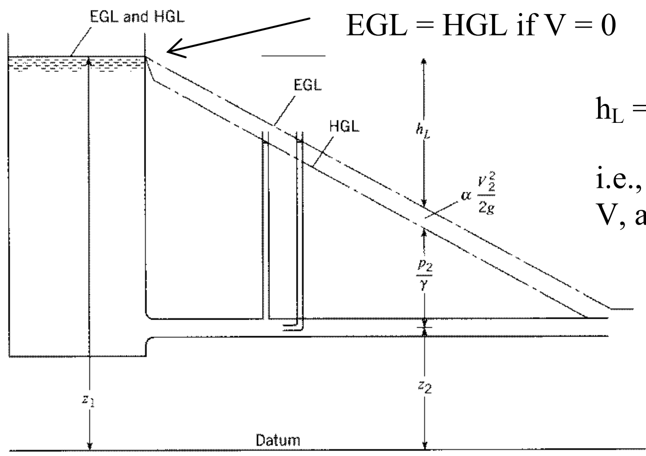


FIGURE 7.4
EGL and HGL in a straight pipe.

$h_L = f\dfrac{L}{D}\dfrac{V^2}{2g}$

i.e., linear variation in L for D, V, and f constant

f = friction factor
f = f(Re)

pressure tap: $\dfrac{p_2}{\gamma}=h$

stagnation tube: $\dfrac{p_2}{\gamma}+\alpha\dfrac{V_2^2}{2g}=h$

} h = height of fluid in tap/tube

$EGL_1 + h_p = EGL_2 + h_t + h_L$

$EGL_2 = EGL_1 + \underbrace{h_p - h_t}_{\text{abrupt change due to } h_p \text{ or } h_t} - h_L$

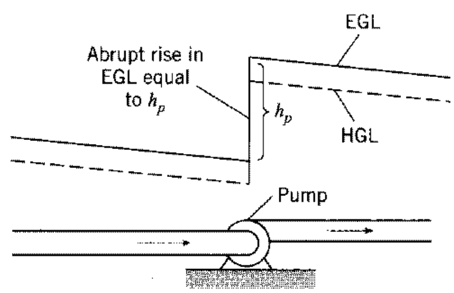
$h_L = f\dfrac{L}{D}\dfrac{V^2}{2g}$

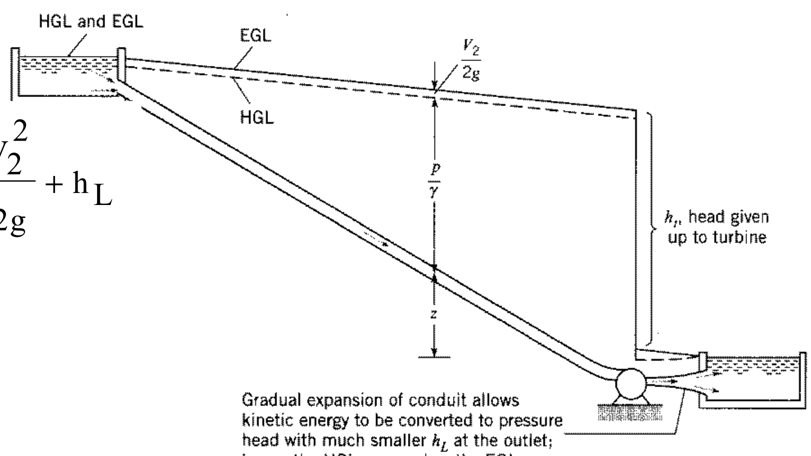
Helpful hints for drawing HGL and EGL

1. $EGL = HGL + \alpha V^2/2g = HGL$ for $V = 0$

2.&3. $h_L = f\frac{L}{D}\frac{V^2}{2g}$ in pipe means EGL and HGL will slope downward, except for abrupt changes due to h_t or h_p

FIGURE 7.5
Rise in EGL and HGL due to pump.





$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L$$

$HGL_2 = EGL_1 - h_L$

$h_L = \frac{V^2}{2g}$ for abrupt expansion

Drop in EGL and HGL due to turbine.

4. $p = 0 \Rightarrow \text{HGL} = z$

5. for $h_L = f\frac{L}{D}\frac{V^2}{2g} = \text{constant} \times L$ ← i.e., linearly increased for increasing L with slope $\frac{f}{D}\frac{V^2}{2g}$

EGL/HGL slope downward

6. for change in D ⇒ change in V

i.e.

$$\left.\begin{aligned} V_1A_1 &= V_2A_2 \\ V_1\frac{\pi D_1^2}{4} &= V_2\frac{\pi D_2^2}{4} \\ V_1D_1^2 &= V_1D_2^2 \end{aligned}\right\} \Rightarrow$$

change in distance between HGL & EGL and slope change due to change in h_L

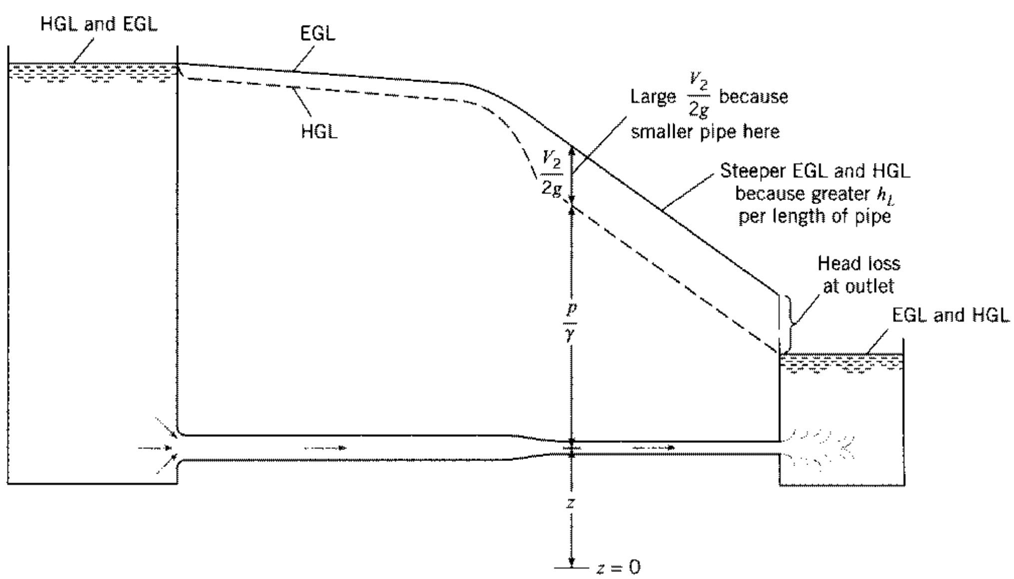


FIGURE 7.8
Change in EGL and HGL due to change in diameter of pipe.

7. If HGL < z then $p/\gamma < 0$ i.e., cavitation possible

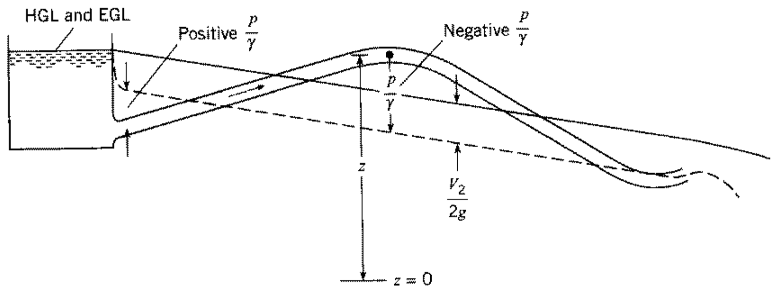


FIGURE 7.9
Subatmospheric pressure when pipe is above HGL.

condition for cavitation:

$$p = p_{va} = 2000 \frac{N}{m^2}$$

gage pressure $p_{va,g} = p_A - p_{atm} \approx -p_{atm} = -100{,}000 \frac{N}{m^2}$

$$\frac{p_{va,g}}{\gamma} \approx -10\text{m}$$

(γ = 9810 N/m^3)

4.15 METHOD OF SOLUTION OF FLOW PROBLEMS

For the solutions of problems of liquid flow there are two fundamental equations, the equation of continuity (3.10) and the energy equation in one of the forms from Eqs. (4.5) to (4.10). The following procedure may be employed:

1. Choose a datum plane through any convenient point.
2. Note at what sections the velocity is known or is to be assumed. If at any point the section area is great compared with its value elsewhere, the velocity head is so small that it may be disregarded.
3. Note at what points the pressure is known or is to be assumed. In a body of liquid at rest with a free surface the pressure is known at every point within the body. The pressure in a jet is the same as that of the medium surrounding the jet.
4. Note whether or not there is any point where all three terms, pressure, elevation, and velocity, are known.
5. Note whether or not there is any point where there is only one unknown quantity.

It is generally possible to write an energy equation that will fulfill conditions 4 and 5. If there are two unknowns in the equation, then the continuity equation must be used also. The application of these principles is shown in the following illustrative examples.

Illustrative Example 4.7 A pipeline with a pump leads to a nozzle as shown in the accompanying figure. Find the flow rate when the pump develops a head of 80 ft. Assume that the head loss in the 6-in-diameter pipe may be expressed by $h_L = 5V_6^2/2g$, while the head loss in the 4-in-diameter pipe is $h_L = 12V_4^2/2g$. Sketch the energy line and hydraulic grade line, and find the pressure head at the suction side of the pump.

Select the datum as the elevation of the water surface in the reservoir. Note from continuity that

$$V_6 = (\tfrac{3}{6})^2 V_3 = 0.25V_3 \qquad \text{and} \qquad V_4 = (\tfrac{3}{4})^2 V_3 = 0.563V_3$$

where V_3 is the jet velocity. Writing an energy equation from the surface of the reservoir to the jet,

$$\left(z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g}\right) - h_{L_6} + h_p - h_{L_4} = z_3 + \frac{p_3}{\gamma} + \frac{V_3^2}{2g}$$

$$0 + 0 + 0 - 5\frac{V_6^2}{2g} + 80 - 12\frac{V_4^2}{2g} = 10 + 0 + \frac{V_3^2}{2g}$$

Express all velocities in terms of V_3:

$$-\frac{5(0.25V_3)^2}{2g} + 80 - 12\frac{(0.563V_3)^2}{2g} = 10 + \frac{V_3^2}{2g}$$

$$V_3 = 29.7 \text{ fps}$$

$$Q = A_3V_3 = \frac{\pi}{4}\left(\frac{3}{12}\right)^2 29.7 = 1.45 \text{ cfs}$$

Head loss in suction pipe:

$$h_L = 5\frac{V_6^2}{2g} = \frac{5(0.25V_3)^2}{2g} = \frac{0.312V_3^2}{2g}$$

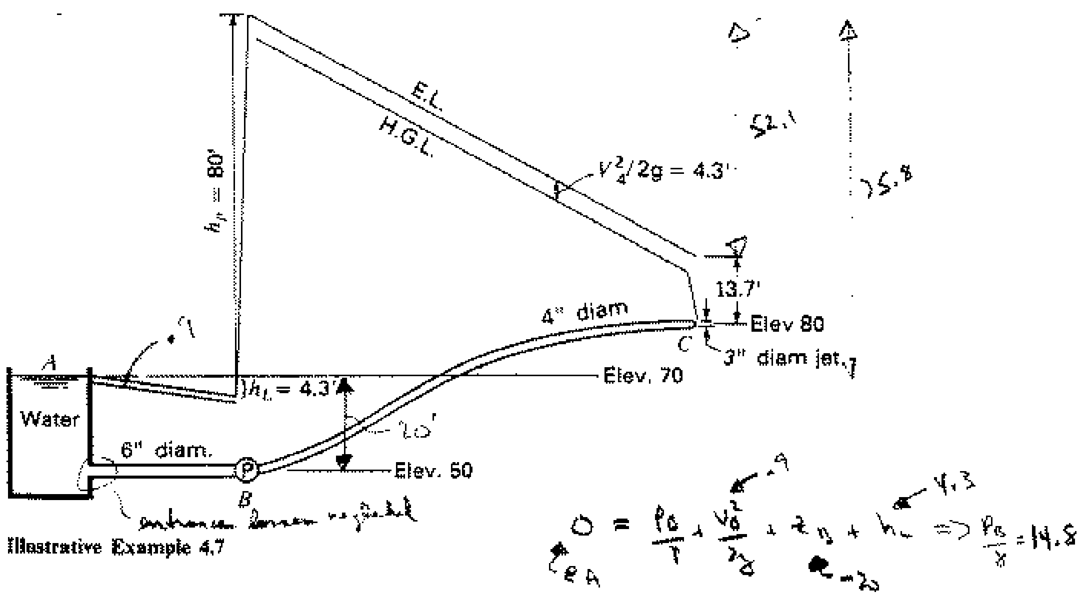
$$= 4.3 \text{ ft}$$

Head loss in discharge pipe:

$$h_L = 12\frac{V_4^2}{2g} = \frac{12(0.563V_3)^2}{2g} = 52.1 \text{ ft}$$

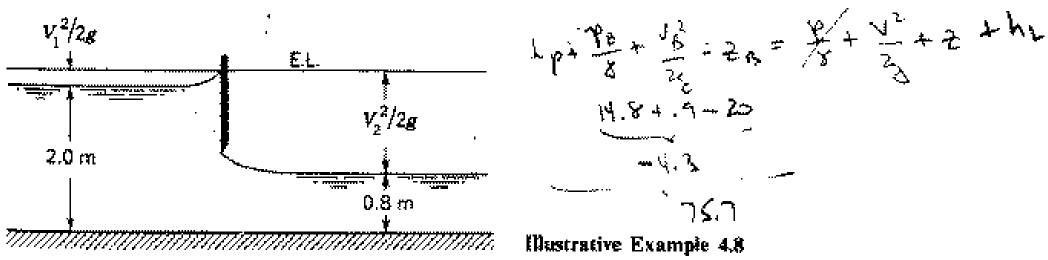
$$\frac{V_3^2}{2g} = 13.7 \text{ ft} \qquad \frac{V_4^2}{2g} = 4.3 \text{ ft} \qquad \frac{V_6^2}{2g} = 0.86 \text{ ft} \approx 0.9 \text{ ft}$$

The energy line and hydraulic grade line are drawn on the figure to scale. Inspection of the figure shows that the pressure head on the suction side of the pump is $p_B/\gamma = 14.8$ ft. Likewise, the pressure head at any point in the pipe may be found if the figure is to scale.



Illustrative Example 4.7

Illustrative Example 4.8 Given the two-dimensional flow as shown in the accompanying figure. Determine the flow rate. Assume no head loss.



Illustrative Example 4.8

Application of the Energy, Momentum, and Continuity Equations in Combination

In general, when solving fluid mechanics problems, one should use all available equations in order to derive as much information as possible about the flow. For example, consistent with the approximation of the energy equation we can also apply the momentum and continuity equations

Energy:

$$\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2+h_t+h_L$$

Momentum:

$$\sum F_s=\rho V_2^2 A_2-\rho V_1^2 A_1=\rho Q(V_2-V_1)$$

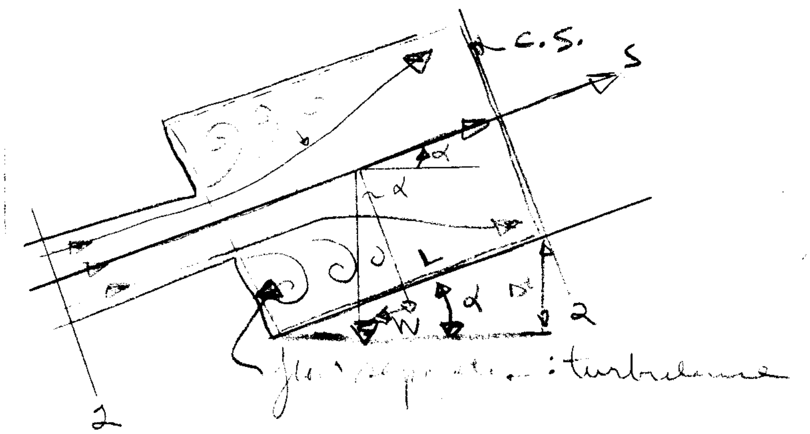
Continuity:

$$A_1V_1 = A_2V_2 = Q = \text{constant}$$

(Momentum and continuity: one inlet and one outlet, ρ = constant)

Abrupt Expansion

Consider the flow from a small pipe to a larger pipe. Would like to know $h_L = h_L(V_1, V_2)$. Analytic solution to exact problem is extremely difficult due to the occurrence of flow separations and turbulence. However, if the assumption is made that the pressure in the separation region remains approximately constant and at the value at the point of separation, i.e, p_1, an approximate solution for h_L is possible:



Apply Energy Eq from 1-2 ($\alpha_1 = \alpha_2 = 1$)

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L$$

Momentum eq. For CV shown (shear stress neglected)

$$\sum F_s = p_1 A_2 - p_2 A_2 - \underbrace{W \sin\alpha} = \sum \rho u \underline{V} \cdot \underline{A}$$

$$= \rho V_1(-V_1 A_1) + \rho V_2 (V_2 A_2)$$

$$= \rho V_2^2 A_2 - \rho V_1^2 A_1$$

$$W \sin\alpha = \underbrace{\gamma A_2 L}_{W} \underbrace{\frac{\Delta z}{L}}_{\sin\alpha}$$

next divide momentum equation by γA_2

$$\div \gamma A_2 \qquad \underbrace{\frac{p_1}{\gamma} - \frac{p_2}{\gamma} - (z_1 - z_2)}_{\text{from energy equation}} = \frac{V_2^2}{g} - \frac{V_1^2}{g}\frac{A_1}{A_2} = \frac{V_1^2}{g}\frac{A_1}{A_2}\left(\frac{A_1}{A_2} - 1\right)$$

$\Downarrow$

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} + h_L = \frac{V_2^2}{g} - \frac{V_1^2}{g}\frac{A_1}{A_2}$$

$$h_L = \frac{V_2^2}{2g} + \frac{V_1^2}{2g}\left(1 - \frac{2A_1}{A_2}\right)$$

$$h_L = \frac{1}{2g}\left[V_2^2 + V_1^2 \underbrace{- 2V_1^2\frac{A_1}{A_2}}_{-2V_1V_2}\right]$$

continuity eq.

$$V_1A_1 = V_2A_2$$

$$\frac{A_1}{A_2} = \frac{V_2}{V_1}$$

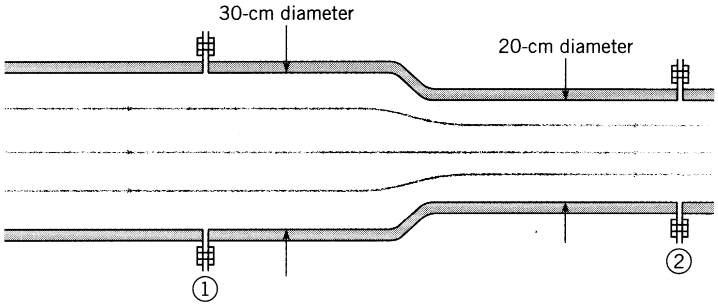
$$\boxed{h_L = \frac{1}{2g}\left[V_2 - V_1\right]^2}$$

If $V_2 \ll V_1$,

$$\boxed{h_L = \frac{1}{2g}V_1^2}$$

Forces on Transitions

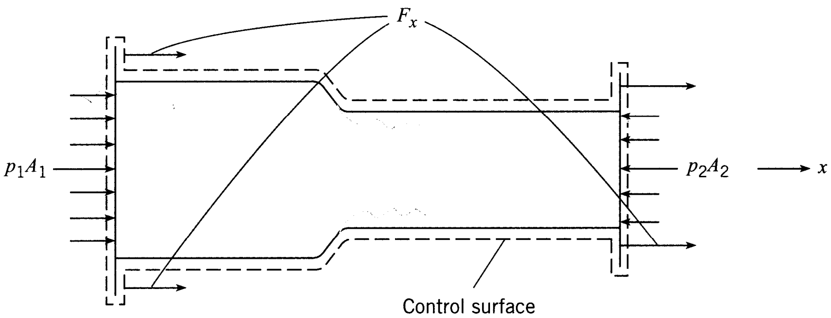
Example 7-6



$Q = .707\ m^3/s$

$\text{head loss} = .1\dfrac{V_2^2}{2g}$

(empirical equation)



Fluid = water
$p_1 = 250$ kPa
$D_1 = 30$ cm
$D_2 = 20$ cm
$F_x = ?$

First apply momentum theorem

$$\sum F_x = \sum \rho u \underline{V} \cdot \underline{A}$$

$$F_x + p_1A_1 - p_2A_2 = \rho V_1(-V_1A_1) + \rho V_2(V_2A_2)$$

$$F_x = \rho Q(V_2 - V_1) - p_1A_1 + p_2A_2$$

↖ force required to hold transition in place

The only unknown in this equation is p_2, which can be obtained from the energy equation.

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+h_L$$ note: $z_1 = z_2$ and $\alpha = 1$

$$p_2 = p_1-\gamma\left[\frac{V_2^2}{2g}-\frac{V_1^2}{2g}+h_L\right]$$ drop in pressure

$$\Rightarrow F_x = \rho Q(V_2-V_1)+\underbrace{A_2\left[p_1-\gamma\left(\frac{V_2^2}{2g}-\frac{V_1^2}{2g}+h_L\right)\right]}_{p_2}-p_1A_1$$

(note: if $p_2 = 0$ same as nozzle)

In this equation,

$V_1 = Q/A_1 = 10$ m/s
$V_2 = Q/A_2 = 22.5$ m/s

$$h_L = .1\frac{V_2^2}{2g} = 2.58\text{m}$$

continuity $A_1V_1 = A_2V_2$

$$V_2 = \frac{A_1}{A_2}V_1$$

i.e. $V_2 > V_1$

$F_x = -8.15$ kN is negative x direction to hold transition in place