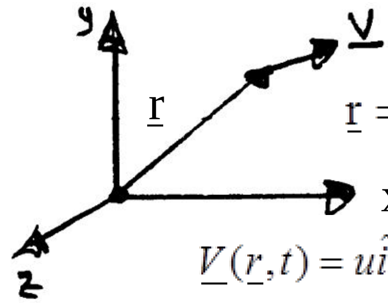
Chapter 4: Fluids Kinematics

4.1 Velocity and Description Methods

Primary dependent variable is fluid velocity vector $\underline{V} = \underline{V}(\underline{r})$; where $\underline{r}$ is the position vector

If $\underline{V}$ is known then pressure and forces can be determined using techniques to be discussed in subsequent chapters.



$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

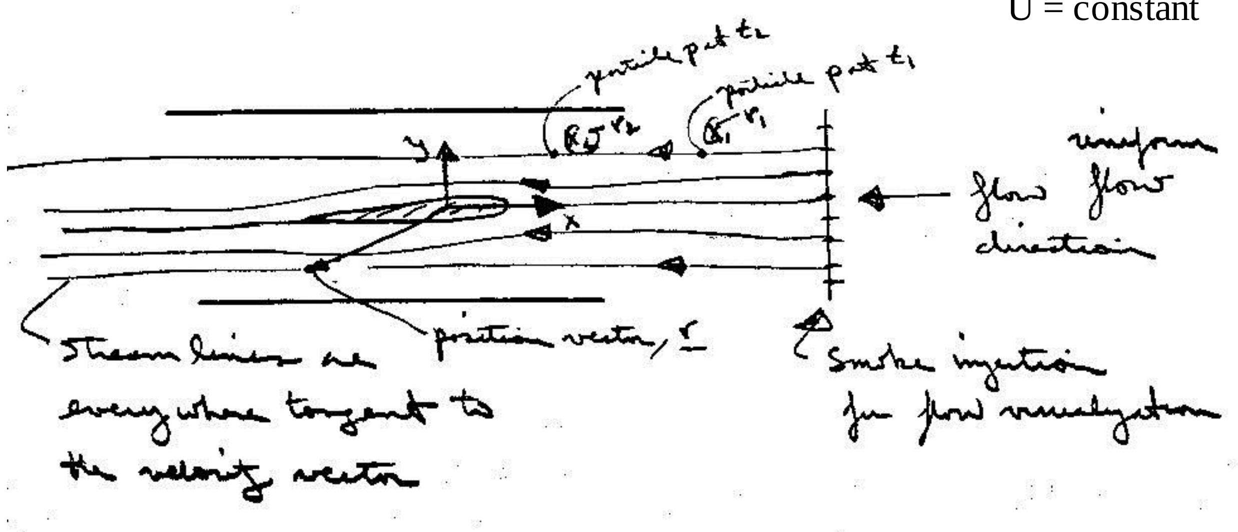
$$\underline{V}(\underline{r},t) = u\hat{i} + v\hat{j} + w\hat{k}$$

Consideration of the velocity field alone is referred to as flow field kinematics in distinction from flow field dynamics (force considerations).

Fluid mechanics and especially flow kinematics is a geometric subject and if one has a good understanding of the flow geometry then one knows a great deal about the solution to a fluid mechanics problem.

Consider a simple flow situation, such as an airfoil in a wind tunnel:

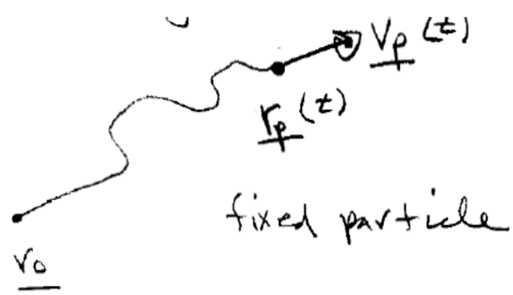
U = constant



Velocity: Lagrangian and Eulerian Viewpoints

There are two approaches to analyzing the velocity field: Lagrangian and Eulerian

Lagrangian: keep track of individual fluids particles (i.e., solve $\underline{F} = M\underline{a}$ for each particle)



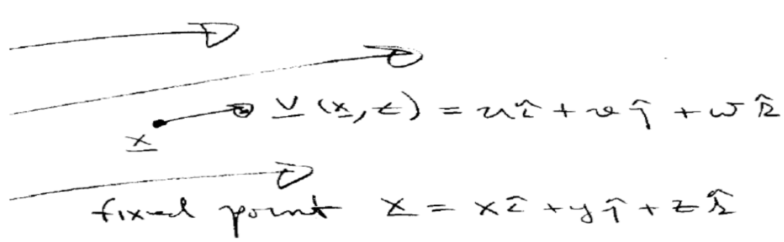
Say particle p is at position $\underline{r}_1(t_1)$ and at position $\underline{r}_2(t_2)$ then,

$$\underline{V_p} = \lim_{\Delta t \to 0} \frac{\underline{r_2} - \underline{r_1}}{t_2 - t_1}$$
$$= \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$
$$= u_p\hat{\imath} + v_p\hat{\jmath} + w_p\hat{k}$$

Of course the motion of one particle is insufficient to describe the flow field, so the motion of all particles must be considered simultaneously which would be a very difficult task. Also, spatial gradients are not given directly. Thus, the Lagrangian approach is only used in special circumstances.

Eulerian: focus attention on a fixed point in space

$$\underline{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$



In general,

$$\underline{V} = \underline{V}(\underline{x}, t) = \underbrace{u\hat{\imath} + v\hat{\jmath} + w\hat{k}}_{\text{velocity components}}$$

where,

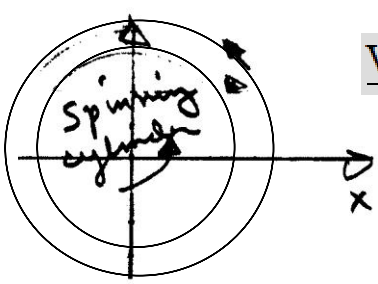
$$u = u(x, y, z, t),\ v = v(x, y, z, t),\ w = w(x, y, z, t)$$

This approach is by far the most useful since we are usually interested in the flow field in some region and not the history of individual particles.

However, must transform $\underline{F} = M\underline{a}$ from system to CV (recall Reynolds Transport Theorem (RTT) & CV analysis from thermodynamics)

Ex. Flow around a car

$\underline{V}$ can be expressed in any coordinate system; e.g., polar or spherical coordinates. Recall that such coordinates are called orthogonal curvilinear coordinates. The coordinate system is selected such that it is convenient for describing the problem at hand (boundary geometry or streamlines).



$$\underline{V} = v_r \hat{e}_r + v_\theta \hat{e}_\theta$$

$$x = r\cos\theta$$

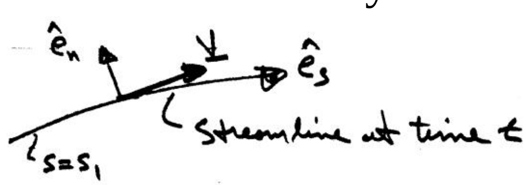
$$y = r\sin\theta$$

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Undoubtedly, the most convenient coordinate system is streamline coordinates:

$$\underline{V}(s,t) = v_s(s,t)\hat{e}_s(s,t)$$

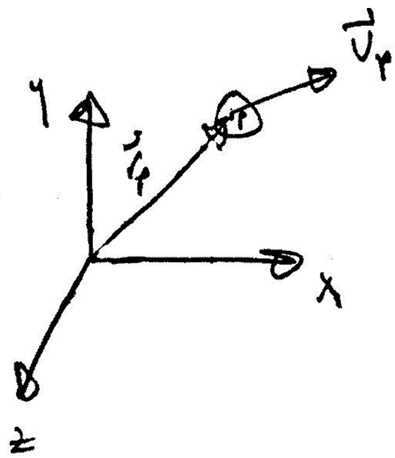


However, usually $\underline{V}$ not known a priori and even if known streamlines maybe difficult to generate/determine.

4.2 Acceleration Field and Material Derivative

The acceleration of a fluid particle is the rate of change of its velocity.

In the Lagrangian approach the velocity of a fluid particle is a function of time only since we have described its motion in terms of its position vector.



$$\underline{r_p} = x_p(t)\hat{\imath} + y_p(t)\hat{\jmath} + z_p(t)\hat{k}$$

$$\underline{V_p} = \frac{d\underline{r_p}}{dt} = u_p\hat{\imath} + v_p\hat{\jmath} + w_p\hat{k}$$

$$\underline{a_p} = \frac{d\underline{V_p}}{dt} = \frac{d^2\underline{r_p}}{dt^2} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

$$a_x = \frac{du_p}{dt} \quad a_y = \frac{dv_p}{dt} \quad a_z = \frac{dw_p}{dt}$$

In the Eulerian approach the velocity is a function of both space and time such that,

$$\underline{V} = u(x, y, z, t)\hat{\imath} + v(x, y, z, t)\hat{\jmath} + w(x, y, z, t)\hat{k}$$

where (u, v, w) are velocity components in (x, y, z) directions, and $(x, y, z) = f(t)$ since we must follow the particle in evaluating $d\underline{V}/dt$.

$$\underline{a} = \frac{D\underline{V}}{Dt} = \frac{Du}{Dt}\hat{\imath} + \frac{Dv}{Dt}\hat{\jmath} + \frac{Dw}{Dt}\hat{k} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

where $\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$ are not arbitrary but assumed to follow a fluid particle, i.e. $\frac{\partial x_i}{\partial t} = u_i$

$$a_x = \underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{called substantial derivative } \frac{Du}{Dt}}$$

Similarly for a_y & a_z,

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

In vector notation this can be written concisely

$$\frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V}$$

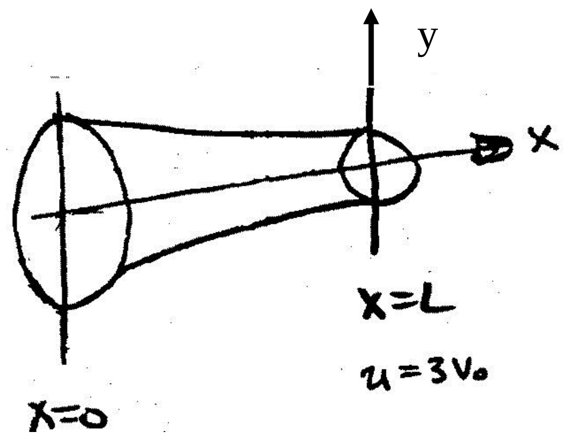
$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$ gradient operator

First term, $\frac{\partial \underline{V}}{\partial t}$, called local or temporal acceleration results from velocity changes with respect to time at a given point. Local acceleration results when the flow is unsteady.

Second term, $\underline{V} \cdot \nabla \underline{V}$, called convective acceleration because it is associated with spatial gradients of velocity in the flow field. Convective acceleration results when the flow is non-uniform, that is, if the velocity changes along a streamline.

The convective acceleration terms are nonlinear which causes mathematical difficulties in flow analysis; also, even in steady flow the convective acceleration can be large if spatial gradients of velocity are large.

Example: Flow through a converging nozzle can be approximated by a one dimensional velocity distribution $u = u(x)$. For the nozzle shown, assume that the velocity varies linearly from $u = V_o$ at the entrance to $u = 3V_o$ at the exit. Compute the acceleration $\frac{D\underline{V}}{Dt}$ as a function of x.



Evaluate $\frac{D\underline{V}}{Dt}$ at the entrance and exit if V_o = 10 ft/s and L =1 ft.

$u = V_o$

$u(x) = mx + b$
$u(0) = b = V_o$
$m = \frac{\Delta u}{\Delta x} = \frac{3V_o - V_o}{L} = \frac{2V_o}{L}$

We have $\underline{V} = u(x)\hat{i}$, $\frac{Du}{Dt} = u\frac{\partial u}{\partial x} = a_x$

Assume linear variation between inlet and exit

$$u(x) = \frac{2V_o}{L}(x) + V_o = V_o\left(\frac{2x}{L} + 1\right)$$

$$\frac{\partial u}{\partial x} = \frac{2V_0}{L} \Rightarrow \boxed{a_x = \frac{2V_o^2}{L}\left(\frac{2x}{L} + 1\right)}$$

@ x = 0 $\quad a_x = 200 \text{ ft/s}^2$

@ x = L $\quad a_x = 600 \text{ ft/s}^2$

Additional considerations: Separation, Vortices, Turbulence, and Flow Classification

We will take this opportunity and expand on the material provided in the text to give a general discussion of fluid flow classifications and terminology.

1. One-, Two-, and Three-dimensional Flow

   1D: $\underline{V} = u(y)\hat{i}$

   2D: $\underline{V} = u(x,y)\hat{i} + v(x,y)\hat{j}$

   3D: $\underline{V} = \underline{V}(\underline{x}) = u(x,y,z)\hat{i} + v(x,y,z)\hat{j} + w(x,y,z)\hat{k}$

2. Steady vs. Unsteady Flow

   $\underline{V} = \underline{V}(\underline{x},t)$ unsteady flow

   $\underline{V} = \underline{V}(\underline{x})$ steady flow

3. Incompressible and Compressible Flow

   $\frac{D\rho}{Dt} = 0 \Rightarrow$ incompressible flow

   $Ma = \frac{V}{c}$

   V: representative velocity

   c: speed of sound in fluid

$Ma < .3$	incompressible
$Ma > .3$	compressible
$Ma = 1$	sonic (commercial aircraft Ma~.8)
$Ma > 1$	supersonic

Ma is the most important nondimensional parameter for compressible flow (Chapter 7 Dimensional Analysis)

4. Viscous and Inviscid Flows

Inviscid flow: ($\mu = 0$)	neglect μ, which simplifies analysis but must decide when this is a good approximation (D' Alembert paradox body in steady motion $C_D = 0$!)
Viscous flow: ($\mu \neq 0$)	retain μ, i.e., "Real-Flow Theory" more complex analysis, but often no choice

5. Rotational vs. Irrotational Flow

$\underline{\Omega} = \nabla \times \underline{V} \neq 0$ rotational flow

$\underline{\Omega} = 0$ irrotational flow

Generation of vorticity usually is the result of viscosity $\therefore$ viscous flows are always rotational, whereas inviscid flows are usually irrotational. Inviscid, irrotational, incompressible flow is referred to as ideal-flow theory.

6. Laminar vs. Turbulent Viscous Flows
   Laminar flow = smooth orderly motion composed of thin sheets (i.e., laminas) gliding smoothly over each other

   Turbulent flow = disorderly high frequency fluctuations superimposed on main motion. Fluctuations are visible as eddies which continuously mix, i.e., combine and disintegrate (average size is referred to as the scale of turbulence).

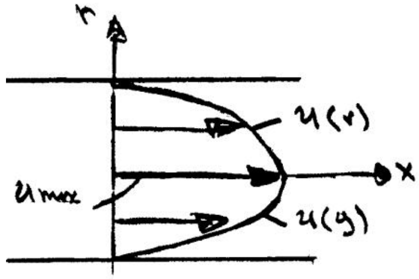
Reynolds decomposition

$$\underbrace{u = \bar{u}}_{\text{mean motion}} + \underbrace{u'(t)}_{\text{turbulent fluctuation}}$$

usually $u' \sim (.01\text{-}.1)\bar{u}$, but influence is as if μ increased by 100-10,000 or more.

Example: Pipe Flow (Chapter 8 = Flow in Conduits)

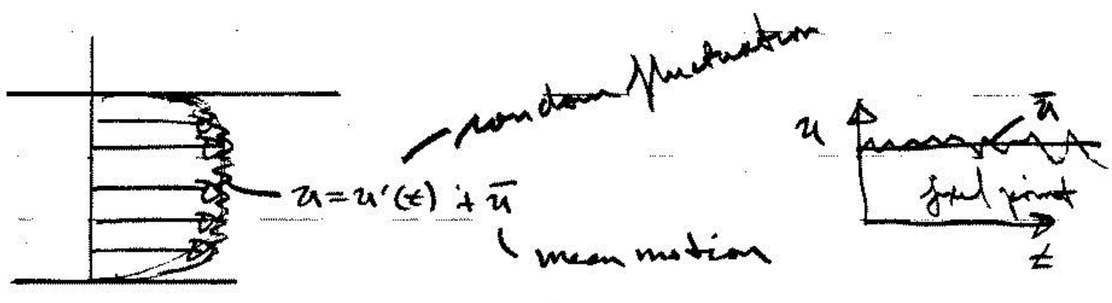
Laminar flow:



$$u(r) = \frac{R^2 - r^2}{4\mu}\left(-\frac{dp}{dx}\right)$$

u(y),velocity profile in a paraboloid

Turbulent flow: fuller profile due to turbulent mixing extremely complex fluid motion that defies closed form analysis.



Turbulent flow is the most important area of motion fluid dynamics research.

The most important nondimensional number for describing fluid motion is the Reynolds number (Chapter 8)

$$Re = \frac{VD\rho}{\mu} = \frac{VD}{\nu}$$

V = characteristic velocity

D = characteristic length

For pipe flow

$V = \overline{V}$ = average velocity

D = pipe diameter

Re < 2300	laminar flow
Re > 2300	turbulent flow

Also depends on roughness, free-stream turbulence, etc.

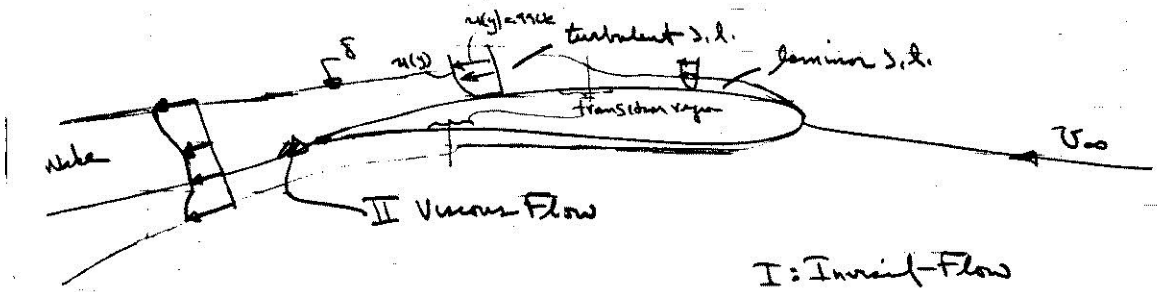
7. Internal vs. External Flows

   Internal flows = completely wall bounded;
   Usually requires viscous analysis, except near entrance (Chapter 8)

   External flows = unbounded; i.e., at some distance from body or wall flow is uniform (Chapter 9, Surface Resistance)

   External Flow exhibits flow-field regions such that both inviscid and viscous analysis can be used depending on the body shape and Re.

Flow Field Regions (high Re flows)



$$Re = \frac{Vc}{\nu} = \frac{\text{inertia force}}{\text{viscous force}}$$

Important features:

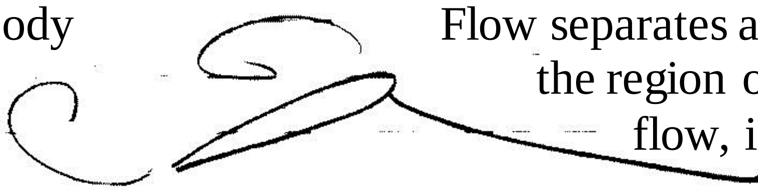
1) low Re viscous effects important throughout entire fluid domain: creeping motion
2) high Re flow about streamlined body viscous effects confined to narrow region: boundary layer and wake
3) high Re flow about bluff bodies: in regions of adverse pressure gradient flow is susceptible to separation and viscous-inviscid interaction is important

8. Separated vs. Unseparated Flow

Streamlined body

Flow remains attached w/o separation

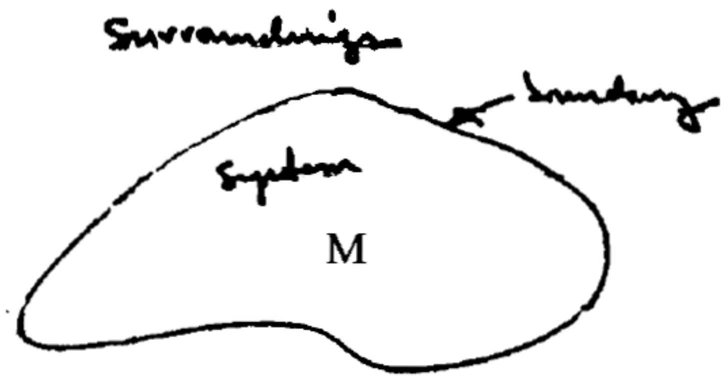
Bluff body



Flow separates and creates the region of reverse flow, i.e. separation

4.3 Basic Control-Volume Approach and RTT

Laws of mechanics are written for a system, i.e., a fixed amount of matter.



1. Conservation of mass: $\frac{dM}{dt} = 0$
2. Conservation of momentum: $\underline{F} = M\underline{a} = \frac{d(M\underline{V})}{dt}$
3. Conservation of energy: $\frac{dE}{dt} = \dot{Q} - \dot{W}$

ΔE=heat added – work done

Also

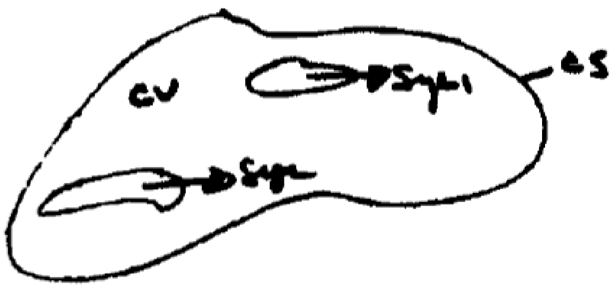
Conservation of angular momentum: $\frac{d\underline{H}_G}{dt} = \underline{M}_G$

Second Law of Thermodynamics: $\frac{dS}{dt} = \frac{\delta\dot{Q}}{T} + \dot{\sigma}$

$\dot{\sigma}$, entropy production due to system irreversibilities

$\dot{\sigma} \geq 0$

In fluid mechanics we are usually interested in a region of space, i.e, control volume and not particular systems. Therefore, we need to transform GDE's from a system to a control volume, which is accomplished through the use of RTT (actually derived in thermodynamics for CV forms of continuity and 1st and 2nd laws, but not in general form or referred to as RTT).



Note GDE's are of form:

$$\frac{d}{dt}\underbrace{\left(M, M\underline{V}, E\right)}_{\text{system extensive properties } B_{sys} \text{ depend on mass}} = RHS$$

i.e., involve $\frac{dB_{sys}}{dt}$ which needs to be related to changes in CV. Recall, definition of corresponding system intensive properties

$$\beta = (1, \underline{V}, e) \qquad \text{independent of mass}$$

where

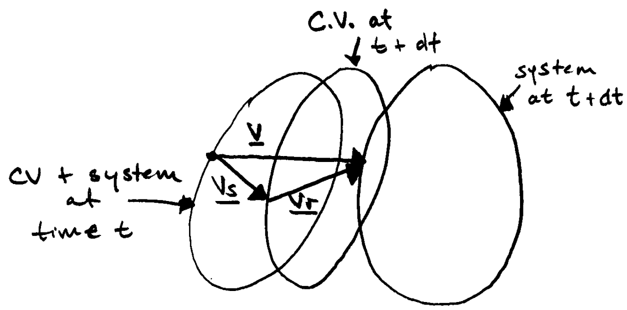
$$B = \int \beta dm = \int \beta \rho d\forall$$

i.e., $\beta = \frac{dB}{dm}$

Reynolds Transport Theorem (RTT)

Need relationship between $\frac{d}{dt}\left(B_{sys}\right)$ and changes in

$B_{cv} = \int\limits_{CV} \beta \, dm = \int\limits_{CV} \beta\rho \, d\forall$.



Moving deforming CV:

$\underline{V}_r = \underline{V} - \underline{V}_s$

$\underline{V}$ = fluid velocity
$\underline{V}_s$ = CS defining CV velocity
$\underline{V}_r$ = relative velocity

all in same coordinate system

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \frac{(B_{cv} + \Delta B)_{t+\Delta t} - (B_{cv} + \Delta B)_t}{\Delta t}$$

$$= \underbrace{\lim_{\Delta t \to 0} \frac{B_{cv_{t+\Delta t}} - B_{cv_t}}{\Delta t}}_{1} + \underbrace{\lim_{\Delta t \to 0} \frac{\Delta B_{t+\Delta t} - \Delta B_t}{\Delta t}}_{2}$$

1 = time rate of change of B in CV $= \frac{dB_{cv}}{dt} = \frac{d}{dt}\int\limits_{CV} \beta\rho \, d\forall$

2 = net outflux of B from CV across CS =

$$\int\limits_{CS} \beta\rho \underline{V}_R \cdot \underline{n} \; DA$$

$$\frac{dB_{SYS}}{dt} = \frac{d}{dt}\int\limits_{CV} \beta\rho \, d\forall + \int\limits_{CS} \beta\rho \underline{V}_R \cdot \underline{n} \; dA$$

General form RTT for moving deforming control volume

Special Cases:

1) Non-deforming CV moving at constant velocity

$$\frac{dB_{SYS}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\beta\rho)\, d\forall + \int_{CS} \beta\rho \underline{V}_R \cdot \underline{n}\ dA$$

2) Fixed CV

$$\frac{dB_{SYS}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\beta\rho)\, d\forall + \int_{CS} \beta\rho \underline{V} \cdot \underline{n}\ dA$$

Gauss's Theorem: $\int_{CV} \nabla \cdot \underline{b}\ d\forall = \int_{CS} \underline{b} \cdot \underline{n}\ dA$

$$\frac{dB_{SYS}}{dt} = \int_{CV} \left[\frac{\partial}{\partial t}(\beta\rho) + \nabla \cdot (\beta\rho \underline{V}) \right] d\forall$$

Since CV fixed and arbitrary $\lim_{d\forall \to 0}$ gives differential eq.

3) Steady Flow: $\frac{\partial}{\partial t} = 0$

4) Uniform flow across discrete CS (steady or unsteady)

$$\int_{CS} \beta\rho \underline{V} \cdot \underline{n}\ dA = \sum_{CS} \beta\rho \underline{V} \cdot \underline{n}\ dA \quad \text{(- inlet, + outlet)}$$

Continuity Equation:

B = M = mass of system
$\beta = 1$

$\frac{dM}{dt} = 0$ by definition, system = fixed amount of mass

Integral Form:

$$\frac{dM}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho \underline{V}_R \cdot \underline{n} \, dA$$

$$-\frac{d}{dt}\int_{CV} \rho \, d\forall = \int_{CS} \rho \underline{V}_R \cdot \underline{n} \, dA$$

Rate of decrease of mass in CV = net rate of mass outflow across CS

Note simplifications for non-deforming CV, fixed CV, steady flow, and uniform flow across discrete CS

Simplifications:

1. Steady flow: $-\frac{d}{dt}\int_{CV} \rho \, d\forall = 0$

2. $\underline{V}$ = constant over discrete $\underline{dA}$ (flow sections):
$$\int_{CS} \rho \underline{V} \cdot \underline{dA} = \sum_{CS} \rho \underline{V} \cdot \underline{A}$$

3. Incompressible fluid (ρ = constant)

$$\int_{CS} \underline{V} \cdot \underline{dA} = -\frac{d}{dt}\int_{CV} d\forall \qquad \text{conservation of volume}$$

4. Steady One-Dimensional Flow in a Conduit:

$$\sum_{CS} \rho \underline{V} \cdot \underline{A} = 0$$

$$-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0$$

for ρ = constant $\quad Q_1 = Q_2$