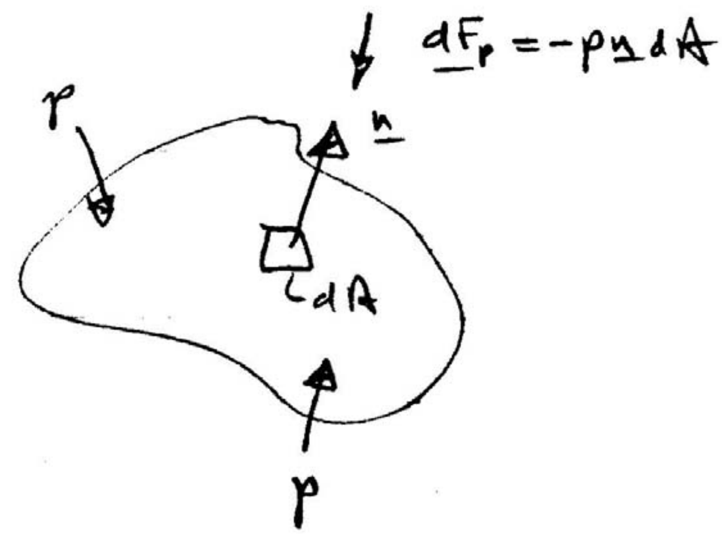
3.4 Hydrostatic Forces on Plane Surfaces

For a static fluid, the shear stress is zero and the only stress is the normal stress, i.e., pressure p. Recall that p is a scalar, which when in contact with a solid surface exerts a normal force towards the surface.

$$\underline{F}_p = -\int_A p\underline{n}dA$$



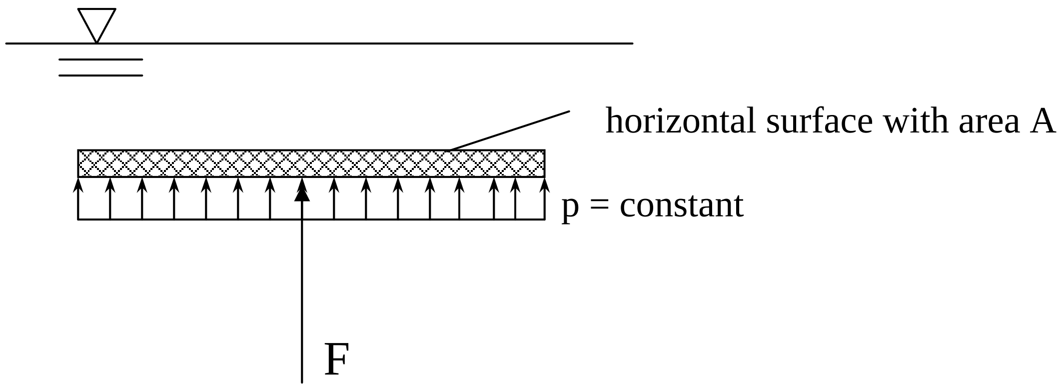
For a plane surface $\underline{n}$ = constant such that we can separately consider the magnitude and line of action of $\underline{F}_p$.

$$\left|\underline{F}_p\right| = F = \int_A pdA$$

Line of action is towards and normal to A through the center of pressure (x_{cp}, y_{cp}).

Unless otherwise stated, throughout the chapter assume p_{atm} acts at liquid surface. Also, we will use gage pressure so that p = 0 at the liquid surface.

Horizontal Surfaces



$$F = \int p dA = pA$$

Line of action is through centroid of A,
i.e., $(x_{cp}, y_{cp}) = (\bar{x}, \bar{y})$

Inclined Surfaces

$$\frac{dp}{dz} = -\gamma$$
$$\Delta p = -\gamma \Delta z$$

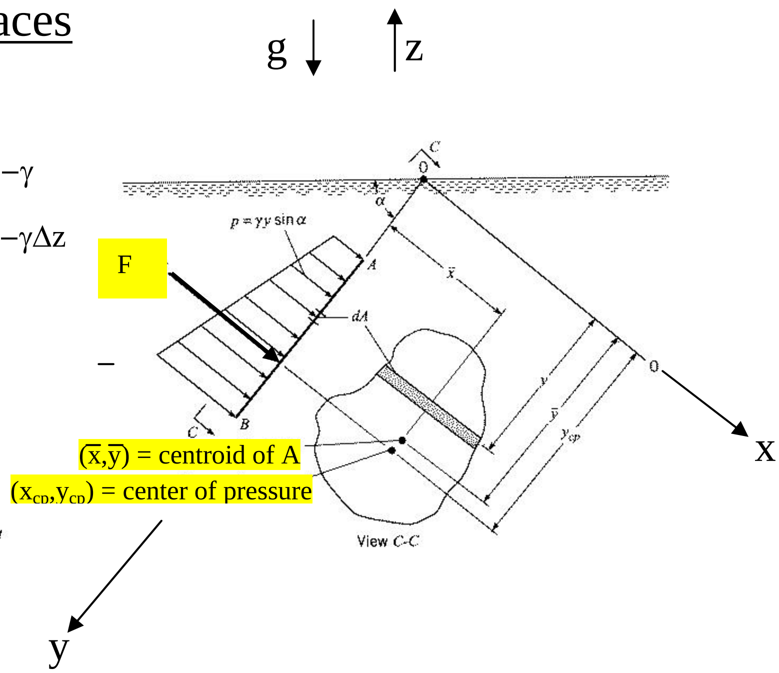


FIGURE 3.10
Distribution of hydrostatic pressure on a plane surface.

$$dF = pdA = \underbrace{\gamma y \sin\alpha}_{p}\, dA$$

γ and $\sin\alpha$ are constants

$$F = \int_A pdA = \gamma \sin\alpha \underbrace{\int_A ydA}_{\bar{y}A}$$

$$\bar{y} = \frac{1}{A}\int ydA$$

$$F = \underbrace{\gamma \sin\alpha\, \bar{y}}_{\bar{p}} A$$

1st moment of area

$\bar{p}$ = pressure at centroid of A

$$\boxed{F = \bar{p}A}$$

Magnitude of resultant hydrostatic force on plane surface is product of pressure at centroid of area and area of surface.

Center of Pressure

Center of pressure is in general below centroid since pressure increases with depth. Center of pressure is determined by equating the moments of the resultant and distributed forces about any arbitrary axis.

Determine y_{cp} by taking moments about horizontal axis 0-0

$$y_{cp}F = \int_A y\,dF$$

$$\int_A y\,p\,dA$$

$$\int_A y(\gamma y \sin\alpha)dA$$

$$= \gamma \sin\alpha \underbrace{\int_A y^2 dA}$$

I_o = 2nd moment of area about 0-0
= moment of inertia

transfer equation: $I_o = \bar{y}^2 A + \bar{I}$

$\bar{I}$ = moment of inertia with respect to horizontal centroidal axis

$$y_{cp}F = \gamma \sin\alpha(\bar{y}^2 A + \bar{I})$$

$$y_{cp}(\bar{p}A) = \gamma \sin\alpha(\bar{y}^2 A + \bar{I})$$

$$y_{cp}\gamma \sin\alpha\, \bar{y}A = \gamma \sin\alpha(\bar{y}^2 A + \bar{I})$$

$$y_{cp}\bar{y}A = \bar{y}^2 A + \bar{I}$$

$$y_{cp} = \bar{y} + \frac{\bar{I}}{\bar{y}A}$$

y_{cp} is below centroid by $\bar{I}/\bar{y}A$

$y_{cp} \rightarrow \bar{y}$ for large $\bar{y}$

For $p_o \neq 0$, y must be measured from an equivalent free surface located p_o/γ above $\bar{y}$.

Determine x_{cp} by taking moment about y axis

$$x_{cp}F = \int_A x\,dF$$
$$= \int_A xp\,dA$$

$$x_{cp}(\gamma\bar{y}\sin\alpha A) = \int_A x(\gamma y\sin\alpha)dA$$

$$x_{cp}\bar{y}A = \underbrace{\int_A xy\,dA}$$

I_{xy} = product of inertia

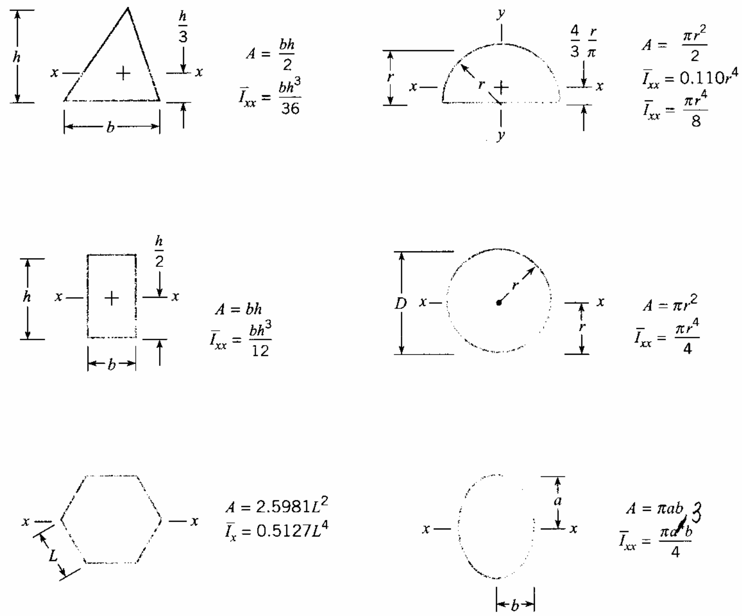
$$= \bar{I}_{xy} + \bar{x}\bar{y}A \quad \text{transfer equation}$$

$$x_{cp}\bar{y}A = \bar{I}_{xy} + \bar{x}\bar{y}A$$

$$\boxed{x_{cp} = \frac{\bar{I}_{xy}}{\bar{y}A} + \bar{x}}$$

For plane surfaces with symmetry about an axis normal to 0-0, $\bar{I}_{xy} = 0$ and $x_{cp} = \bar{x}$.

FIGURE A.1
Centroids and moments of inertia of plane areas



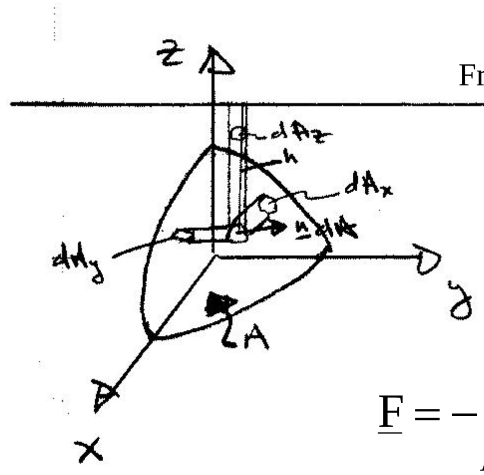
Volume and Area Formulas:

$$A_{\text{circle}} = \pi r^2 = \pi D^2/4$$

$$A_{\text{sphere surface}} = \pi D^2$$

$$\forall_{\text{sphere}} = \frac{1}{6}\pi D^3$$

3.5 Hydrostatic Forces on Curved Surfaces



$$\underline{F} = -\int_A p\underline{n}dA$$

$p = \gamma h$

h = distance below free surface

Horizontal Components (x and y components)

$$F_x = \underline{F}\cdot\hat{i} = -\int_A p\underbrace{\underline{n}\cdot\hat{i}dA}$$

$$= -\int_{A_x} p dA_x$$

dA_x = projection of $\underline{n}dA$ onto plane $\perp$ to x-direction

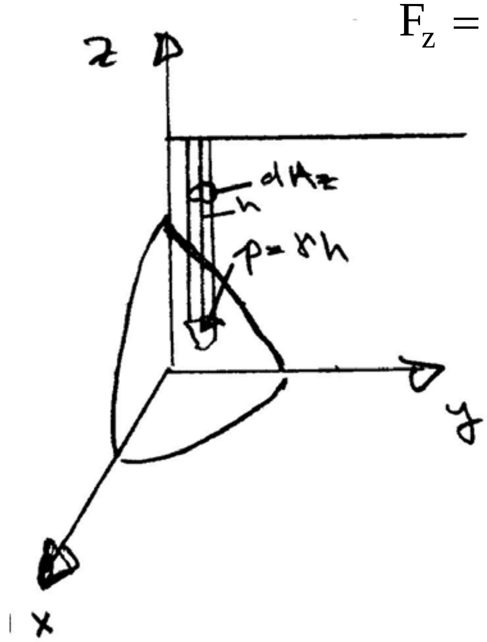
$$F_y = \underline{F}\cdot\hat{j} = -\int_{A_y} p dA_y \qquad dA_y = \underline{n}\cdot\hat{j}dA$$

= projection $\underline{n}dA$ onto plane $\perp$ to y-direction

Therefore, the horizontal components can be determined by some methods developed for submerged plane surfaces.

The horizontal component of force acting on a curved surface is equal to the force acting on a vertical projection of that surface including both magnitude and line of action.

Vertical Components



$$F_z = \underline{F}\cdot\hat{k} = -\int_A p\underline{n}\cdot\hat{k}dA$$

$$= -\int_{A_z} p dA_z \qquad p = \gamma h$$

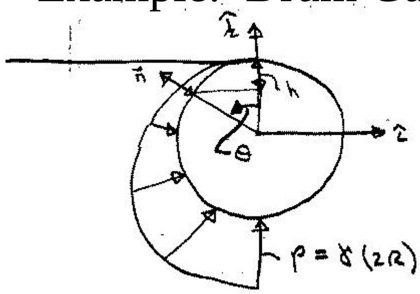
h=distance below free surface

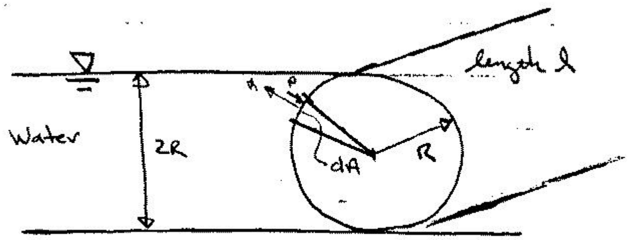
$$= \gamma \int_{A_z} h dA_z = \gamma V$$

= weight of fluid above surface A

The vertical component of force acting on a curved surface is equal to the net weight of the column of fluid above the curved surface with line of action through the centroid of that fluid volume.

Example: Drum Gate





Pressure Diagram

$p = \gamma h = \gamma R(1\text{-}\cos\theta)$

$\underline{n} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{k}$

$dA = \ell\ \ R d\theta$

$$\underline{F} = -\int_0^{\pi} \underbrace{\gamma R(1-\cos\theta)}_{p}\underbrace{(-\sin\theta\,\hat{i} + \cos\theta\,\hat{k})}_{\underline{n}}\underbrace{\ell R d\theta}_{dA}$$

$$\underline{F}\cdot\hat{i} = F_x = +\gamma\,\ell\,R^2\int_0^{\pi}(1-\cos\theta)\sin\theta d\theta$$

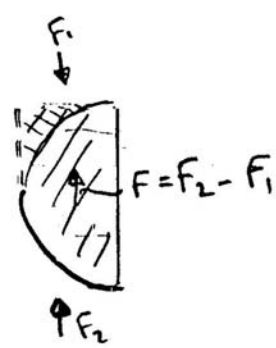
$$= \gamma\,\ell\,R^2\left[-\cos\theta + \frac{1}{4}\cos 2\theta\right|_0^{\pi} = 2\gamma\ell R^2$$

$= (\gamma \underset{\bar{p}}{R})(\underset{A}{2R\,\ell}) \Rightarrow$ <u>same force as that on projection of area onto vertical plane</u>

$$F_z = -\gamma\,\ell\,R^2\int_0^{\pi}(1-\cos\theta)\cos\theta d\theta$$

$$= -\gamma\,\ell\,R^2\left[\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right|_0^{\pi}$$

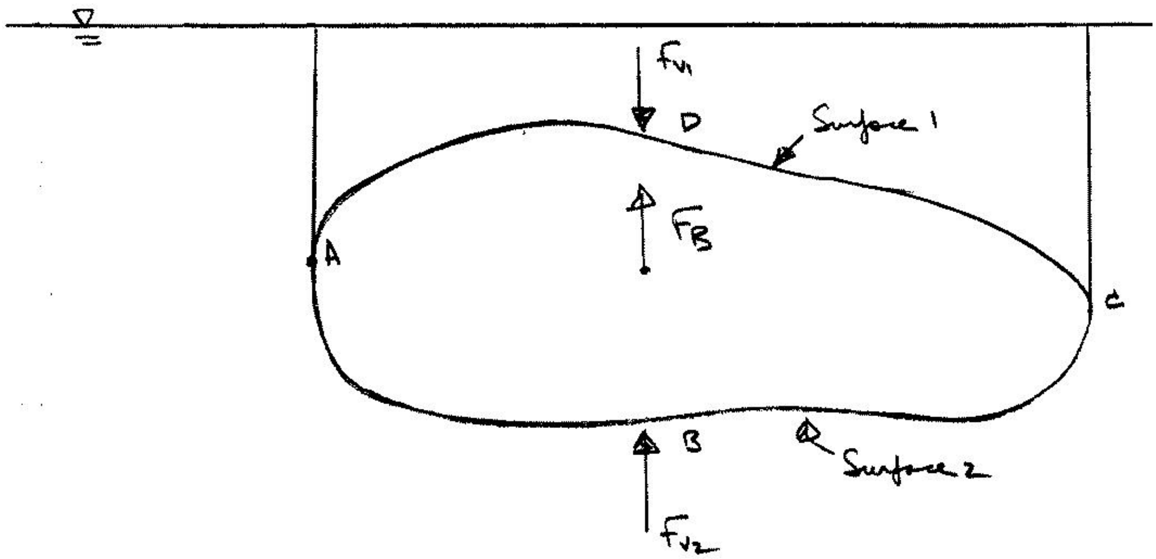
$$= \gamma\,\ell\,R^2\frac{\pi}{2} = \gamma\,\ell\left(\frac{\pi R^2}{2}\right) = \gamma \cancel{V}$$



$\Rightarrow$ net weight of water above surface

3.6 Buoyancy

Archimedes Principle



$F_B = F_{v2} - F_{v1}$

= fluid weight above Surface 2 (ABC)
– fluid weight above Surface 1 (ADC)

= fluid weight equivalent to body volume $\cancel{V}$

$F_B = \rho g \cancel{V}$ $\qquad$ $\cancel{V}$ = submerged volume

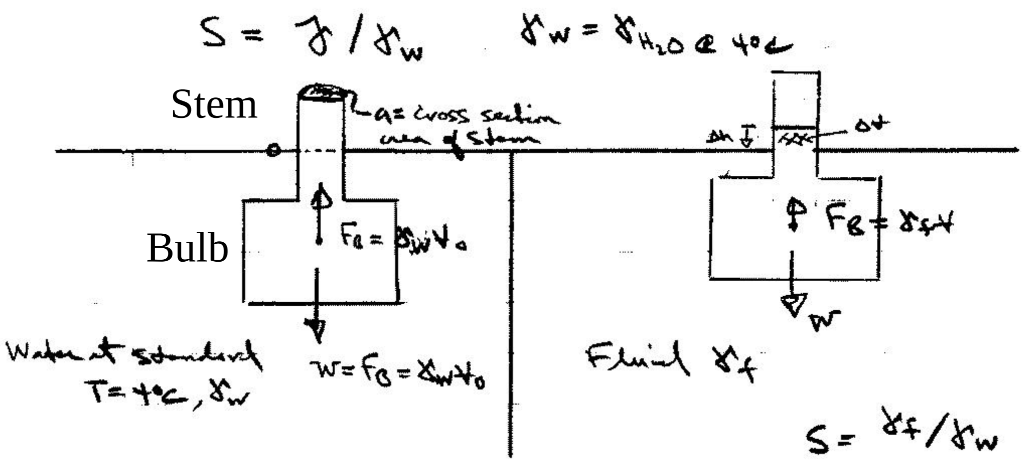
Line of action is through centroid of $\cancel{V}$ = center of buoyancy

Net Horizontal forces are zero since

$$F_{BAD} = F_{BCD}$$

Hydrometry

A hydrometer uses the buoyancy principle to determine specific weights of liquids.



$$W = mg = \gamma_f \cancel{V} = S\gamma_w \cancel{V}$$

$$W = \gamma_w \cancel{V}_o = S\gamma_w(\cancel{V}_o - \Delta\cancel{V}) = \underbrace{S\gamma_w}_{\gamma_f}\underbrace{(\cancel{V}_o - a\Delta h)}_{\cancel{V}}$$

a = cross section area stem

$$\cancel{V}_o/S = \cancel{V}_o - a\Delta h \qquad a\Delta h = \cancel{V}_o - \cancel{V}_o/S$$

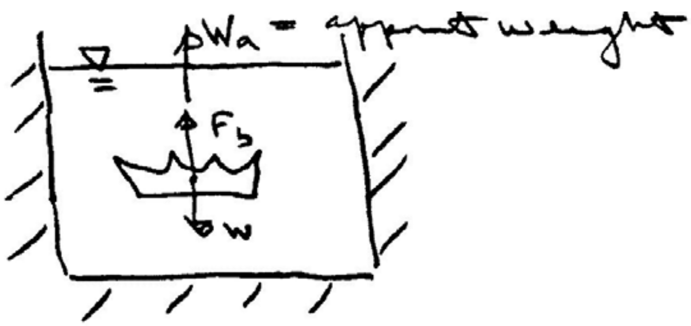
$$\Delta h = \frac{\cancel{V}_o}{a}\cdot\left(1-\frac{1}{S}\right) = \Delta h(S)$$

$$\Delta h = \frac{\cancel{V}_o}{a}\cdot\frac{S-1}{S}$$ calibrate scale using fluids of known S

$$S = \frac{\cancel{V}_o}{\cancel{V}_0 - a\Delta h}$$

Example (apparent weight)

King Hero ordered a new crown to be made from pure gold. When he received the crown he suspected that other metals had been used in its construction. Archimedes discovered that the crown required a force of 4.7# to suspend it when immersed in water, and that it displaced 18.9 in^3 of water. He concluded that the crown was not pure gold. Do you agree?



$$\sum F_{vert} = 0 = W_a + F_b - W = 0 \Rightarrow W_a = W - F_b = (\gamma_c - \gamma_w)\cancel{V}$$

$$W = \gamma_c \cancel{V}, \quad F_b = \gamma_w \cancel{V}$$

$$\text{or} \quad \gamma_c = \frac{W_a}{\cancel{V}} + \gamma_w = \frac{W_a + \gamma_w \cancel{V}}{\cancel{V}}$$

$$\gamma_c = \frac{4.7 + 62.4 \times 18.9/1728}{18.9/1728} = 492.1 = \rho_c g$$

$$\Rightarrow \quad \rho_c = 15.3 \text{ slugs/ft}^3$$

$\sim \rho_{steel}$ and since gold is heavier than steel the crown can not be pure gold

3.7 Stability of Immersed and Floating Bodies

Here we'll consider transverse stability. In actual applications both transverse and longitudinal stability are important.

Immersed Bodies

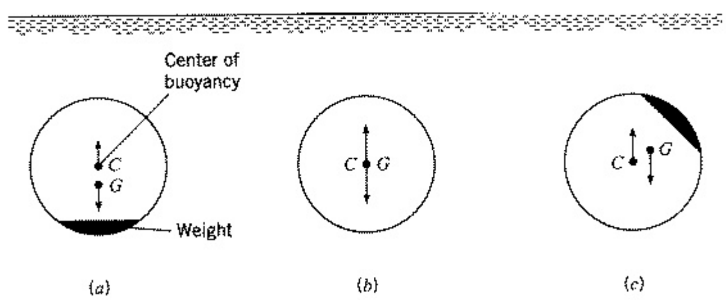


FIGURE 3.15
Conditions of stability for immersed bodies. (a) Stable. (b) Neutral. (c) Unstable.

Static equilibrium requires: $\sum F_v = 0$ and $\sum M = 0$

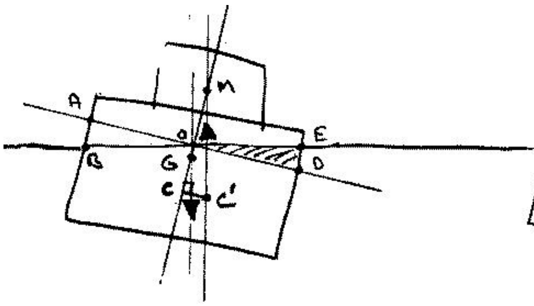
$\sum M = 0$ requires that the centers of gravity and buoyancy coincide, i.e., C = G and body is neutrally stable

If C is above G, then the body is stable (righting moment when heeled)

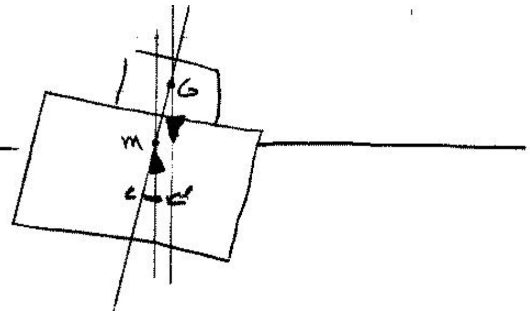
If G is above C, then the body is unstable (heeling moment when heeled)

Floating Bodies

For a floating body the situation is slightly more complicated since the center of buoyancy will generally shift when the body is rotated depending upon the shape of the body and the position in which it is floating.



Positive $\overline{GM}$



Negative $\overline{GM}$

The center of buoyancy (centroid of the displaced volume) shifts laterally to the right for the case shown because part of the original buoyant volume AOB is transferred to a new buoyant volume EOD.

The point of intersection of the lines of action of the buoyant force before and after heel is called the metacenter M and the distance GM is called the metacentric height. If GM is positive, that is, if M is above G, then the ship is stable; however, if GM is negative, the ship is unstable.

Floating Bodies

α = small heel angle

$\bar{x} = CC'$ = lateral displacement of C

C = center of buoyancy i.e., centroid of displaced volume $\cancel{V}$

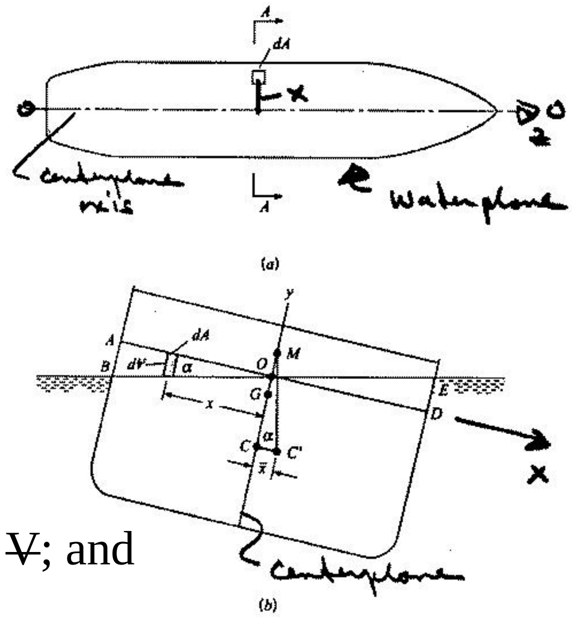


Fig. 3.17

Solve for GM: find $\bar{x}$ using
(1) basic definition for centroid of $\cancel{V}$; and
(2) trigonometry

(1) Basic definition of centroid of volume $\cancel{V}$

$$\bar{x}\cancel{V} = \int x\,d\cancel{V} = \sum x_i \Delta\cancel{V}_i \qquad \text{moment about centerplane}$$

$$\bar{x}\cancel{V} = \underbrace{\text{moment } V \text{ before heel}}_{\substack{= 0 \text{ due to symmetry of} \\ \text{original } V \text{ about y axis} \\ \text{i.e., ship centerplane}}} - \text{moment of } \cancel{V}_{AOB} + \text{moment of } \cancel{V}_{EOD}$$

$$\bar{x}\cancel{V} = -\int_{AOB} (-x)\,d\cancel{V} + \int_{EOD} x\,d\cancel{V}$$

$\tan\alpha = y/x$

$d\cancel{V} = y\,dA = x \tan\alpha\, dA$

$$\bar{x}\cancel{V} = \int_{AOB} x^2 \tan\alpha\, dA + \int_{EOD} x^2 \tan\alpha\, dA$$

$$\bar{x}\forall = \tan\alpha \underbrace{\int x^2 dA}_{\text{moment of inertia of ship waterplane about z axis O-O; i.e., } I_{OO}}$$

ship waterplane area

I_{OO} = moment of inertia of waterplane area about centerplane axis

(2) Trigonometry

$$\bar{x}\forall = \tan\alpha I_{OO}$$

$$CC' = \bar{x} = \frac{\tan\alpha I_{OO}}{\forall} = CM \tan\alpha$$

$$CM = I_{OO} / \forall$$

$$GM = CM - CG$$

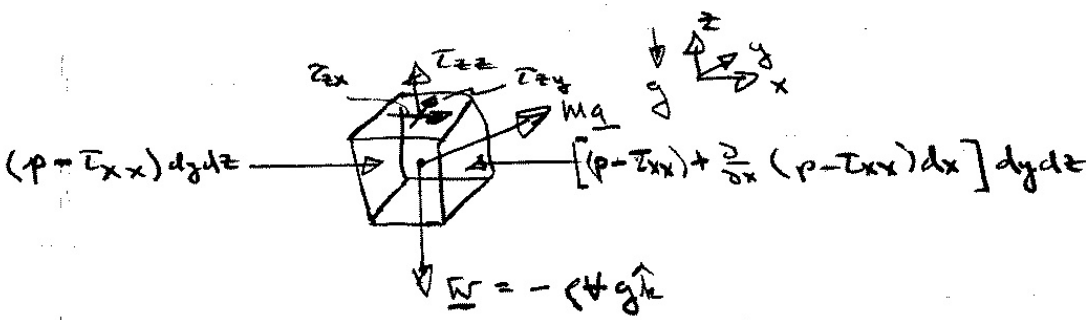
$$GM = \frac{I_{OO}}{\forall} - CG$$

GM > 0 Stable

GM < 0 Unstable

3.8 Fluids in Rigid-Body Motion

For fluids in motion, the pressure variation is no longer hydrostatic and is determined from application of Newton's 2nd Law to a fluid element.



τ_{ij} = viscous stresses
p = pressure
M$\underline{a}$ = inertia force
$\underline{W}$ = weight (body force)

net surface force in X direction

$$X_{net} = \left(\underset{\text{pressure}}{-\frac{\partial p}{\partial x}} + \underbrace{\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}_{\text{viscous}} \right) \cancel{V}$$

Newton's 2nd Law

$$M\underline{a} = \sum \underline{F} = \underline{F}_B + \underline{F}_S$$

per unit ($\div\, \cancel{V}$) volume $\quad \rho \underline{a} = \underline{f}_b + \underline{f}_s$

$$\underline{a} = \frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V}$$

$$\underline{f}_s = \text{body force} = -\rho g \hat{k}$$

$$\underline{f}_s = \text{surface force} = \underline{f}_p + \underline{f}_v$$

$$\underline{f}_p = \text{surface force due to p} = -\nabla p$$

$$\underline{f}_v = \text{surface force due to viscous stresses } \tau_{ij}$$

$$\rho \frac{D\underline{V}}{Dt} = \underline{f}_b + \underline{f}_p + \cancel{\underline{f}_v}$$

Neglected in this chapter and included later in Section 6.4 when deriving complete Navier-Stokes equations

$$\rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} - \nabla p$$

inertia force = body force due to gravity + surface force due to pressure gradients

x: $$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x}$$

Note: for $\underline{V} = 0$

$$\nabla p = -\rho g \hat{k}$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -\rho g = -\gamma$$

y: $$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}$$

$$\rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = -\frac{\partial p}{\partial y}$$

z: $$\rho \frac{Dw}{Dt} = -\rho g - \frac{\partial p}{\partial z} = -\frac{\partial}{\partial z}(p + \gamma z)$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z}(p + \gamma z)$$

or $\rho \underline{a} = -\nabla(p + \gamma z)$ Euler's equation for inviscid flow

$\nabla \cdot \underline{V} = 0$ Continuity equation for incompressible flow

4 equations in four unknowns $\underline{V}$ and p

Examples of Pressure Variation From Acceleration

Uniform Linear Acceleration:

$$\rho \underline{a} = -\rho g \hat{k} - \nabla p$$

$$\nabla p = -\rho\left(\underline{a} + g\hat{k}\right) = \rho\left(\underline{g} - \underline{a}\right) \qquad \underline{g} = -g\hat{k}$$

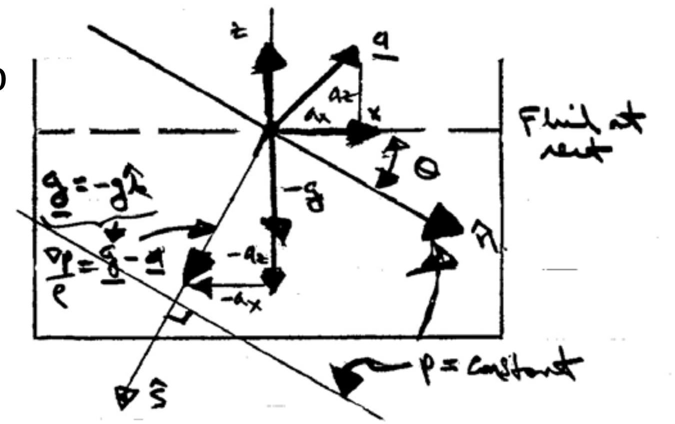
$$\nabla p = -\rho\left[a_x \hat{i} + (g + a_z)\hat{k}\right] \qquad \underline{a} = a_x\hat{i} + a_z\hat{k}$$

$$\frac{\partial p}{\partial x} = -\rho a_x \qquad \frac{\partial p}{\partial z} = -\rho\left(g + a_z\right)$$

$\hat{s}$ = unit vector in direction of ∇p

$= \nabla p \,/\, |\nabla p|$

$$= \frac{-\left[a_x\hat{i} + (g + a_z)\hat{k}\right]}{\left[a_x^2 + (g + a_z)^2\right]^{1/2}}$$



$\hat{n}$ = unit vector in direction of $\underbrace{\text{p = constant}}$

$= \hat{s} \times \hat{j}$ ijkijk

$$= \frac{-a_x\hat{k} + (g + a_z)\hat{i}}{\left[a_x^2 + (g + a_z)^2\right]^{1/2}}$$

$\perp$ to ∇p
by definition lines of constant p are normal to ∇p

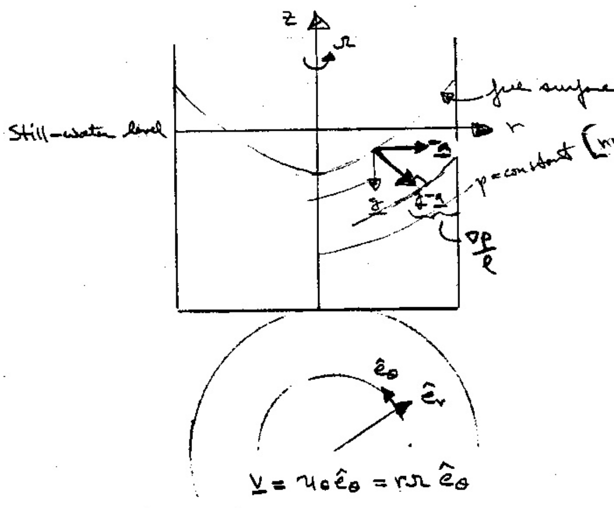
$\theta = \tan^{-1} a_x / (g + a_z)$ = angle between $\hat{n}$ and x

$$\frac{dp}{ds} = \nabla p \cdot \hat{s} = \rho\left[\underbrace{a_x^2 + \left(g + a_z\right)^2}_{G}\right]^{1/2} \quad > \quad \rho g$$

$p = \rho G s + \text{constant} \Rightarrow p_{gage} = \rho G s$

Rigid Body Rotation:

Consider a cylindrical tank of liquid rotating at a constant rate $\underline{\Omega} = \Omega\hat{k}$ $\qquad \Omega = \omega$ in text



$$\underline{a} = \underline{\Omega} \times \left(\underline{\Omega} \times r_o\right)$$

centripetal acceleration

$$= -r\Omega^2 \hat{e}_r$$

$$= -\frac{V^2}{r}\hat{e}_r$$

$$\nabla p = \rho(\underline{g} - \underline{a})$$

$$= -\rho g\hat{k} + \rho r\Omega^2 \hat{e}_r$$

$$\boxed{\nabla = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z}$$

grad in cylindrical coordinates

i.e., $\dfrac{\partial p}{\partial r} = \rho r\Omega^2 \qquad \underbrace{\dfrac{\partial p}{\partial z} = -\rho g} \qquad \dfrac{\partial p}{\partial \theta} = 0$

along path of $\underline{a} = 0$ pressure distribution is hydrostatic

and $p = \overbrace{\dfrac{\rho}{2} r^2 \Omega^2}^{C\,(r)} + f(z) + c$

$p_z = -\rho g$
$p = -\rho g z + C(r) + c$

$$p = \frac{\rho}{2} r^2 \Omega^2 - \rho g z + \text{constant}$$

$$\boxed{\frac{p}{\gamma} + z - \frac{V^2}{2g} = \text{constant} \qquad V = r\Omega}$$

The constant is determined by specifying the pressure at one point; say, $p = p_o$ at $(r, z) = (0, 0)$

$$p = p_o - \rho g z + \frac{1}{2} r^2 \Omega^2$$

Note: pressure is linear in z and parabolic in r

Curves of constant pressure are given by

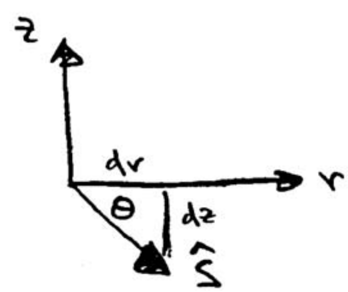
$$z = \frac{p_1 - p_o}{\rho g} + \frac{r^2 \Omega^2}{2g} = a + br^2$$

which are paraboloids of revolution, concave upward, with their minimum point on the axis of rotation

Free surface is found by requiring volume of liquid to be constant (before and after rotation)

The unit vector in the direction of ∇p is

$$\hat{s} = \frac{-\rho g \hat{k} + \rho r \Omega^2 \hat{e}_r}{\left[(\rho g)^2 + \left(\rho r \Omega^2 \right)^2 \right]^{1/2}}$$



$$\underbrace{\tan\theta = \frac{dz}{dr} = -\frac{g}{r\Omega^2}}$$ slope of $\hat{s}$

i.e., $r = C_1 \exp\left(-\frac{\Omega^2 z}{g} \right)$ equation of ∇p surfaces

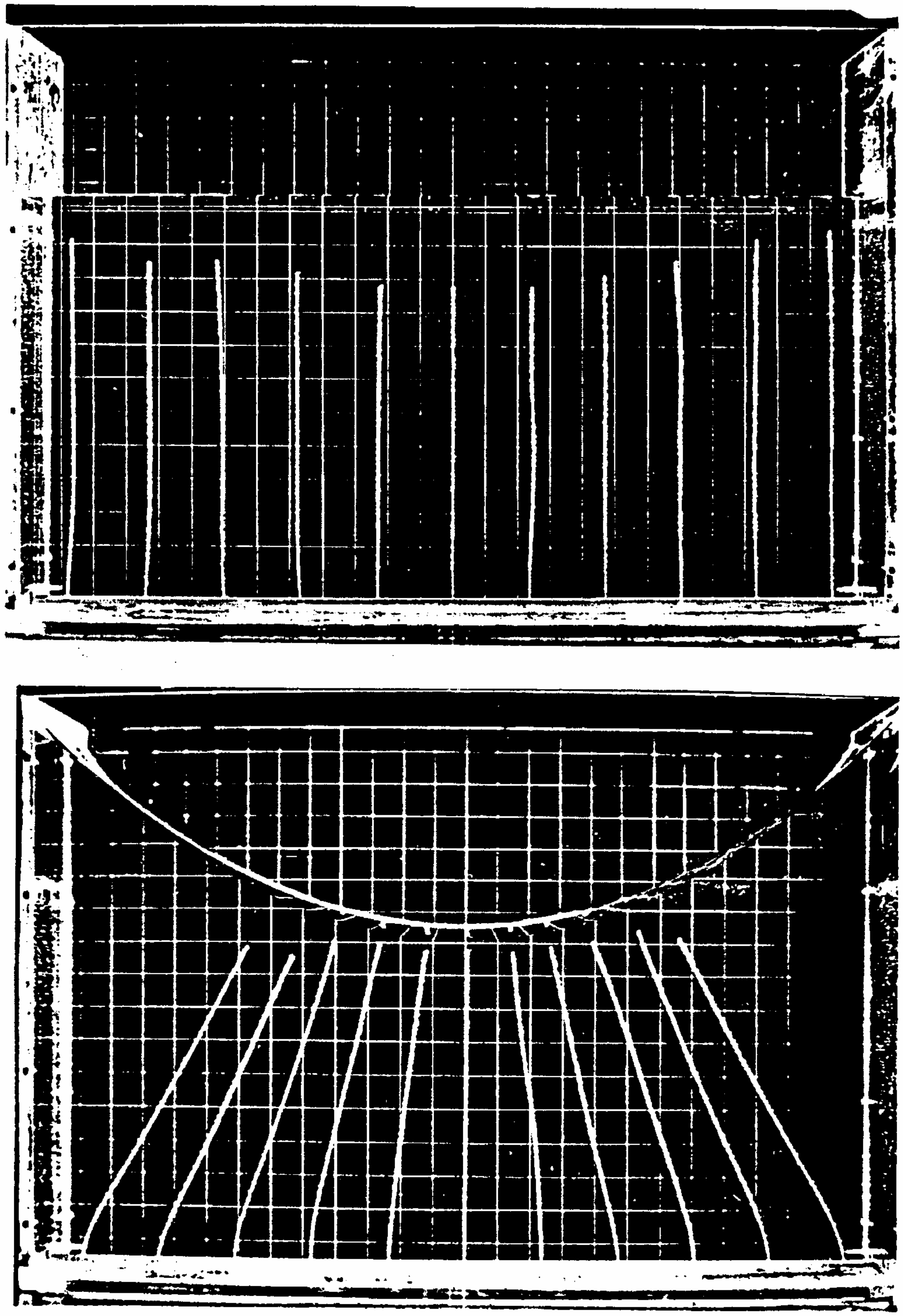


Fig. 2.23 Experimental demonstration with buoyant streamers of the fluid force field in rigid-body rotation: (top) fluid at rest (streamers hang vertically upward); (bottom) rigid-body rotation (streamers are aligned with the direction of maximum pressure gradient). (*From Ref. 5. Courtesy of R. Ian Fletcher.*)