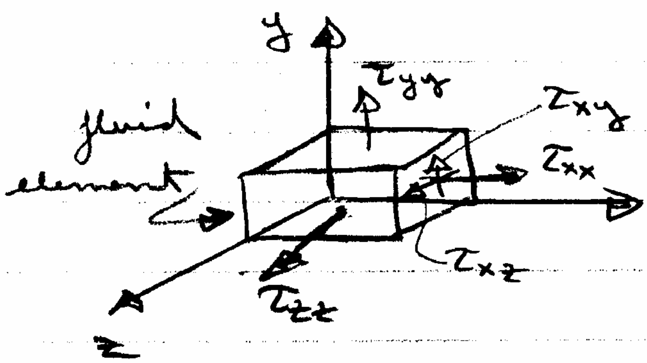
Chapter 3: Pressure and Fluid Statics

3.1 Pressure

For a static fluid, the only stress is the normal stress since by definition a fluid subjected to a shear stress must deform and undergo motion. Normal stresses are referred to as pressure p.

For the general case, the stress on a fluid element or at a point is a tensor



τ_{ij} = stress tensor

$$= \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

i = force
j = direction

For a static fluid,

$\tau_{ij} = 0 \quad i \neq j$ shear stresses = 0

$\tau_{ii} = -p = \tau_{xx} = \tau_{yy} = \tau_{zz} \quad i = j$ normal stresses = p

Also shows that p is isotropic, one value at a point which is independent of direction, a scalar.

Definition of Pressure:

$$p = \lim_{\delta A \to 0} \frac{\delta F}{\delta A} = \frac{dF}{dA} \quad N/m^2 = Pa\ (Pascal)$$

F = normal force acting over A

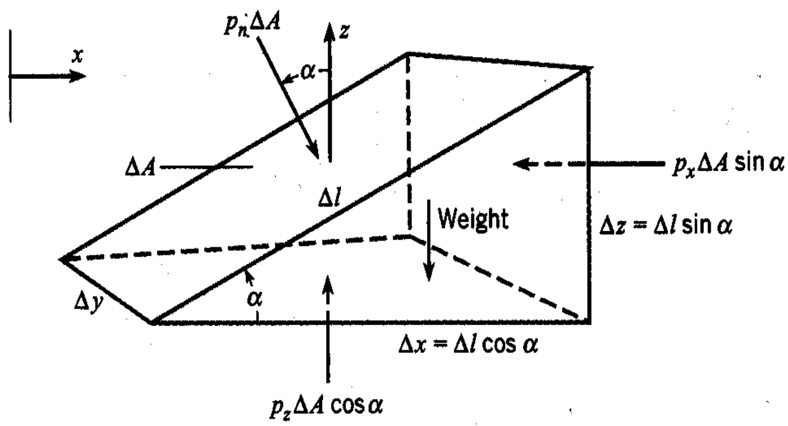
As already noted, p is a scalar, which can be easily demonstrated by considering the equilibrium of forces on a wedge-shaped fluid element

Geometry

$\Delta A = \Delta \ell \ \Delta y$

$\Delta x = \Delta \ell \cos\alpha$

$\Delta z = \Delta \ell \sin\alpha$



$\Sigma F_x = 0$

$p_n \Delta A \sin\alpha - p_x \Delta A \sin\alpha = 0$

$p_n = p_x$

$W = mg$

$= \rho \forall g$

$= \gamma \forall$

$\forall = ½\, \Delta x \Delta z \Delta y$

$\Sigma F_z = 0$

$-p_n \Delta A \cos\alpha + p_z \Delta A \cos\alpha - W = 0$

$$W = \frac{\gamma}{2}(\underbrace{\Delta\ell\cos\alpha}_{\Delta x})(\underbrace{\Delta\ell\sin\alpha}_{\Delta z})\Delta y$$

$$-p_n \Delta\ell \Delta y \cos\alpha + p_z\, \Delta\ell\Delta y \cos\alpha - \frac{\gamma}{2}\Delta\ell^2 \cos\alpha \sin\alpha \Delta y = 0$$

$\div \Delta\ell \Delta y \cos\alpha$

$$-p_n + p_z - \frac{\gamma}{2}\Delta\ell \sin\alpha = 0$$

$$-p_n + p_z - \frac{\gamma}{2}\Delta\ell\sin\alpha = 0$$

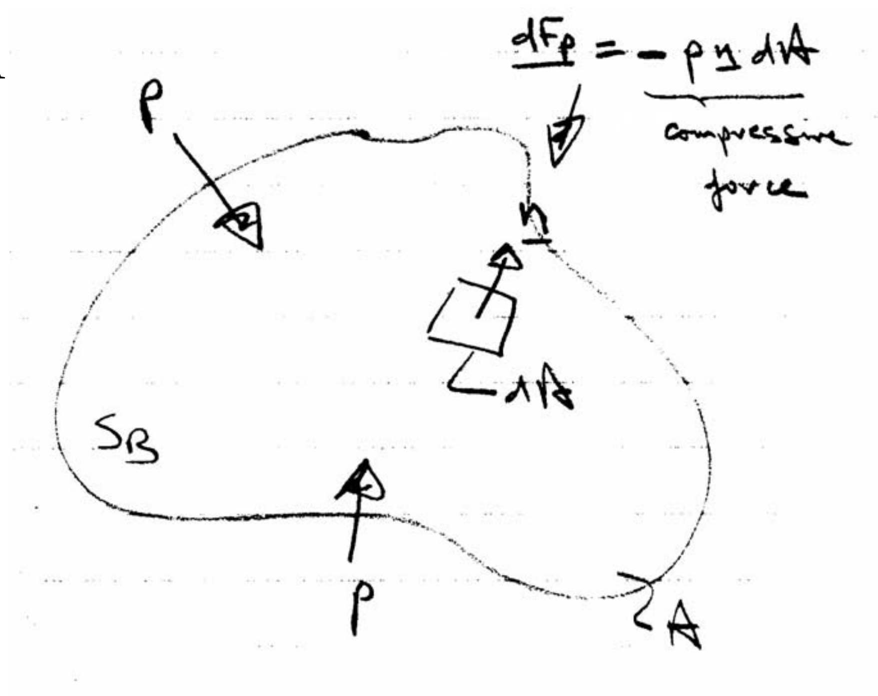
$$p_n = p_z \qquad \text{for } \Delta\ell \to 0$$

i.e., $p_n = p_x = p_y = p_z$

p is single valued at a point and independent of direction since α arbitrary and independent p_n of α

A body/surface in contact with a static fluid experiences a force due to p

$$\underline{F}_p = -\int_{S_B} p\underline{n}dA$$



Note: if p = constant, $\underline{F}_p = 0$ for a closed body

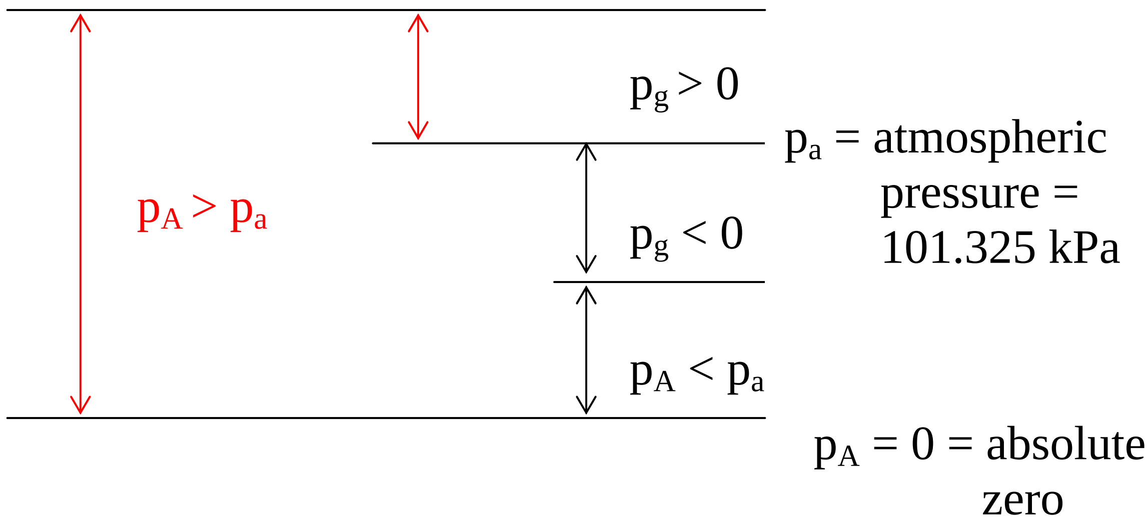
Scalar form of Green's Theorem:

$$\int_s f\,\underline{n}ds = \int_\forall \nabla f d\forall \qquad f = \text{constant} \Rightarrow \nabla f = 0$$

Pressure Transmission

Pascal's law: in a closed system, a pressure change produced at one point in the system is transmitted throughout the entire system.

Absolute Pressure, Gage Pressure, and Vacuum



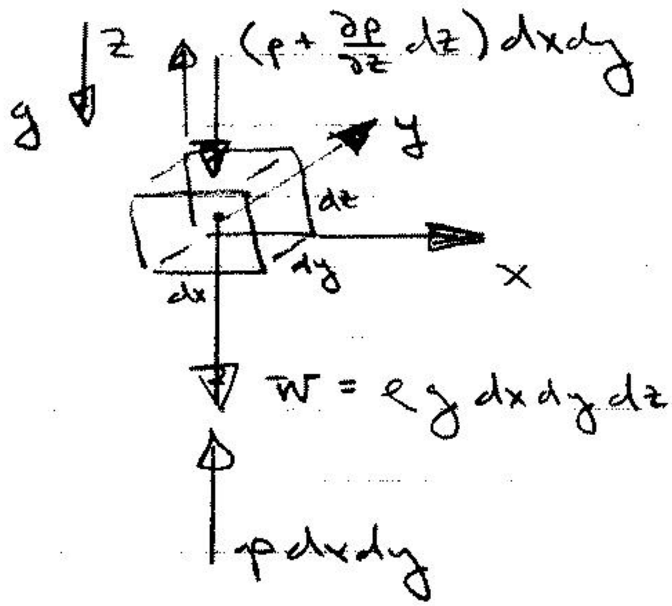
For $p_A > p_a$, $p_g = p_A - p_a$ = gage pressure

For $p_A < p_a$, $p_{vac} = -p_g = p_a - p_A$ = vacuum pressure

3.2 Pressure Variation with Elevation

Basic Differential Equation

For a static fluid, pressure varies only with elevation within the fluid. This can be shown by consideration of equilibrium of forces on a fluid element



1[st] order Taylor series estimate for pressure variation over dz

Newton's law (momentum principle) applied to a static fluid

$\Sigma \underline{F} = m\underline{a} = 0$ for a static fluid

i.e., $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$

$\Sigma F_z = 0$

$$p\,dxdy - \left(p + \frac{\partial p}{\partial z}dz\right)dxdy - \rho g\,dxdydz = 0$$

$$\frac{\partial p}{\partial z} = -\rho g = -\gamma$$

Basic equation for pressure variation with elevation

$$\sum F_y = 0 \qquad \sum F_x = 0$$

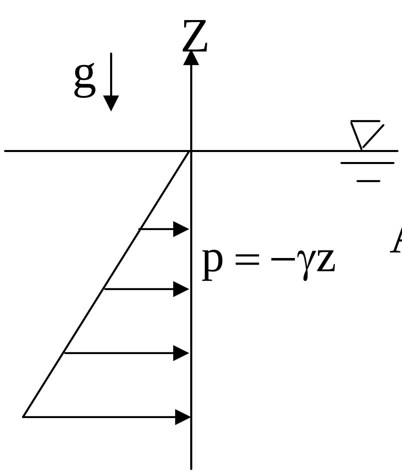
$$p\,dx\,dz - \left(p + \frac{\partial p}{\partial y}dy\right)dx\,dz = 0 \qquad p\,dy\,dz - \left(p + \frac{\partial p}{\partial x}dx\right)dy\,dz = 0$$

$$\frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial x} = 0$$

For a static fluid, the pressure only varies with elevation z and is constant in horizontal xy planes.

The basic equation for pressure variation with elevation can be integrated depending on whether ρ = constant or ρ = ρ(z), i.e., whether the fluid is incompressible (liquid or low-speed gas) or compressible (high-speed gas) since g ~ constant

<u>Pressure Variation for a Uniform-Density Fluid</u>



$$\frac{\partial p}{\partial z} = -\rho g = -\gamma \qquad \rho = \text{constant for liquid}$$

$$\Delta p = -\gamma \Delta z$$

$$p_2 - p_1 = -\gamma\left(z_2 - z_1\right)$$

Alternate forms:

$$p_1 + \gamma z_1 = p_2 + \gamma z_2 = \text{constant}$$

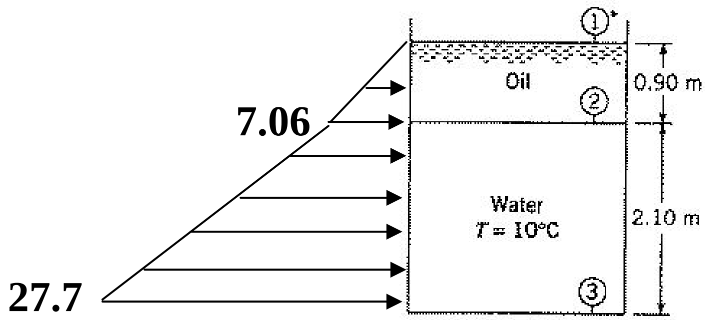
$$p + \gamma z = \text{constant} \qquad \text{piezometric pressure}$$

$$p\left(z = 0\right) = 0 \qquad \text{gage}$$

i.e., $p = -\gamma z$ increase linearly with depth ↓
decrease linearly with height ↑

$$\frac{p}{\gamma} + z = \text{constant} \qquad \text{piezometric head}$$

EXAMPLE 3.4 Oil with a specific gravity of 0.80 forms a layer 0.90 m deep in an open tank that is otherwise filled with water. The total depth of water and oil is 3 m. What is the gage pressure at the bottom of the tank?



$$p + \gamma z = \text{constant}$$
$$p_1 + \gamma z_1 = p_2 + \gamma z_2$$
$$p_2 = p_1 + \gamma (z_1 - z_2)$$
$$p_1 = p_{atm} = 0$$
$$p_2 = \gamma_{oil} \Delta z = .8 \times 9810 \times .9 = 7.06 \text{kPa}$$
$$p_3 = p_2 + \gamma_{water} (z_2 - z_3)$$
$$= 7060 + 9810 \times 2.1$$
$$= 27.7 \text{kPa}$$

Solution First determine the pressure at the oil–water interface, staying within the oil, and then calculate the pressure at the bottom.

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2$$

where p_1 is the pressure at free surface of oil, z_1 is the elevation of free surface of oil, p_2 is the pressure at interface between oil and water, and z_2 is the elevation at interface between oil and water. For this example, $p_1 = 0$, $\gamma = 0.80 \times 9810 \text{ N/m}^3$, $z_1 = 3$ m, and $z_2 = 2.10$ m. Therefore,

$$p_2 = 0.90 \text{ m} \times 0.80 \times 9810 \text{ N/m}^3 = 7.06 \text{ kPa gage}$$

Now obtain p_3 from

$$\frac{p_2}{\gamma} + z_2 = \frac{p_3}{\gamma} + z_3$$

where p_2 has already been calculated and $\gamma = 9810 \text{ N/m}^3$.

$$p_3 = 9810\left(\frac{7060}{9810} + 2.10\right) = 27.7 \text{ kPa gage}$$

Pressure Variation for Compressible Fluids:

Basic equation for pressure variation with elevation

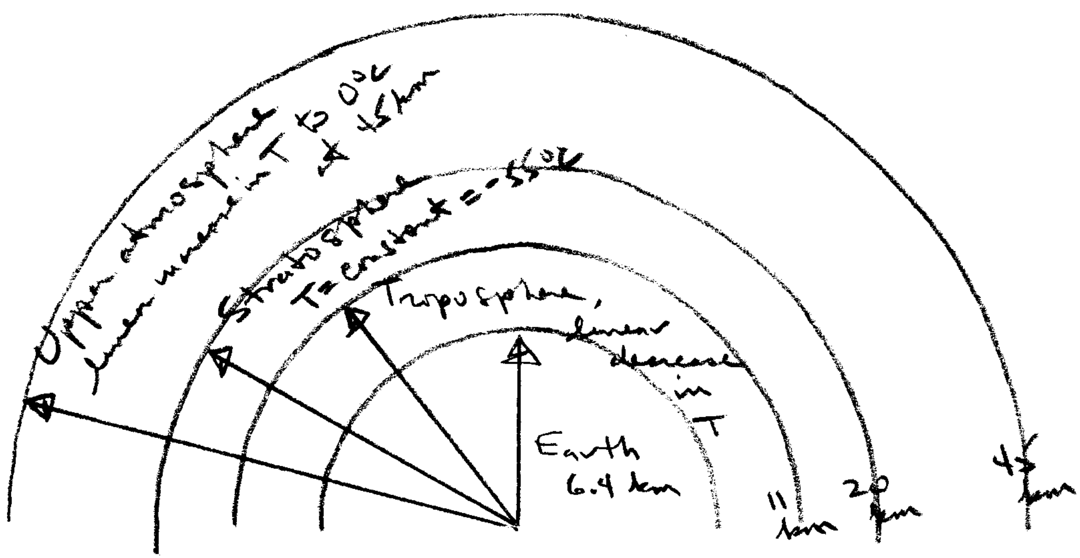
$$\frac{dp}{dz} = -\gamma = -\gamma(p,z) = \rho g$$

Pressure variation equation can be integrated for $\gamma(p,z)$ known. For example, here we solve for the pressure in the atmosphere assuming $\rho(p,T)$ given from ideal gas law, $T(z)$ known, and $g \neq g(z)$.

$p = \rho RT$ R = gas constant = 287 J/kg ·°K   dry air
p,T in absolute scale

$$\frac{dp}{dz} = -\frac{pg}{RT}$$

$$\frac{dp}{p} = \frac{-g}{R}\frac{dz}{T(z)}$$ which can be integrated for T(z) known



Pressure Variation in the Troposphere

$T = T_o - \alpha(z - z_o)$ linear decrease

$T_o = T(z_o)$ where $p = p_o(z_o)$ known

α = lapse rate = 6.5 °K/km

$$\frac{dp}{p} = -\frac{g}{R}\frac{dz}{[T_o - \alpha(z - z_o)]}$$

$$z' = T_o - \alpha(z - z_o)$$
$$dz' = \alpha dz$$

$$\ln p = \frac{g}{\alpha R}\ln[T_o - \alpha(z - z_o)] + \text{constant}$$

use reference condition

z_o = earth surface = 0

p_o = 101.3 kPa

T = 15°C

α = 6.5 °K/km

$$\ln p_o = \frac{g}{\alpha R}\ln T_o + \text{constant}$$

solve for constant

$$\ln\frac{p}{p_o} = \frac{g}{\alpha R}\ln\frac{T_o - \alpha(z - z_o)}{T_o}$$

$$\frac{p}{p_o} = \left[\frac{T_o - \alpha(z - z_o)}{T_o}\right]^{g/\alpha R}$$

i.e., p decreases for increasing z

Pressure Variation in the Stratosphere

$$T = T_s = -55°C$$

$$\frac{dp}{p} = -\frac{g}{R}\frac{dz}{T_s}$$

$$\ln p = -\frac{g}{RT_s}z + \text{constant}$$

use reference condition to find constant

$$\frac{p}{p_o} = e^{-(z-z_0)g/RT_s}$$

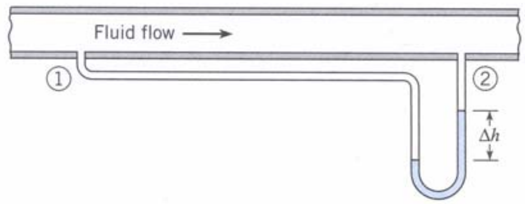
$$p = p_o \exp[-(z - z_o)g / RT_s]$$

i.e., p decreases exponentially for increasing z.

3.3 Pressure Measurements

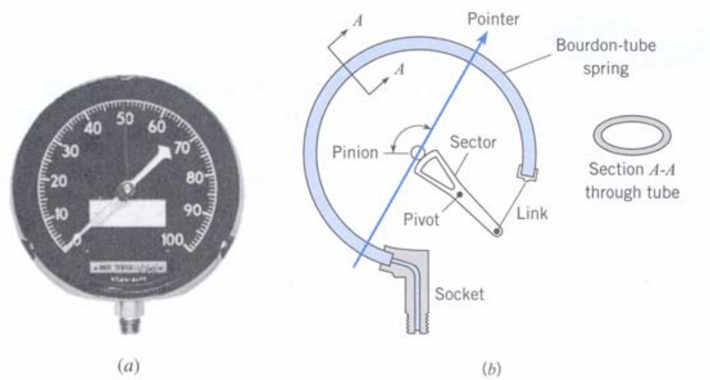
Pressure is an important variable in fluid mechanics and many instruments have been devised for its measurement. Many devices are based on hydrostatics such as barometers and manometers, i.e., determine pressure through measurement of a column (or columns) of a liquid using the pressure variation with elevation equation for an incompressible fluid.

Differential manometer



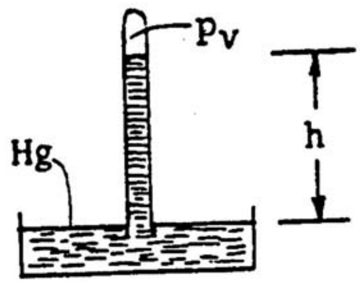
More modern devices include Bourdon-Tube Gage (mechanical device based on deflection of a spring) and pressure transducers (based on deflection of a flexible diaphragm/membrane). The deflection can be monitored by a strain gage such that voltage output is $\propto \Delta p$ across diaphragm, which enables electronic data acquisition with computers.

Bourdon-Tube Gage



In this course we will use both manometers and pressure transducers in EFD labs 2 and 3.

Manometry



1. Barometer

$p_v + \gamma_{Hg} h = p_{atm}$

$p_{atm} = \gamma_{Hg} h$ $p_v \sim 0$ i.e., vapor pressure Hg nearly zero at normal T

$h \sim 76$ cm

$\therefore$ $p_{atm} \sim 101$ kPa (or 14.6 psia)

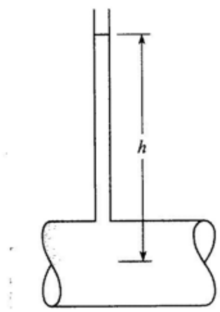
Note: p_{atm} is relative to absolute zero, i.e., absolute pressure. $p_{atm} = p_{atm}$(location, weather)

Consider why water barometer is impractical

$$\gamma_{Hg} h_{Hg} = \gamma_{H_2O} h_{H_2O}$$

$$h_{H_2O} = \frac{\gamma_{Hg}}{\gamma_{H_2O}} h_{Hg} = S_{Hg} h_{Hg} = 13.6 \times 76 = 1033.6\,\text{cm} = 34\,\text{ft.}$$

2. Piezometer

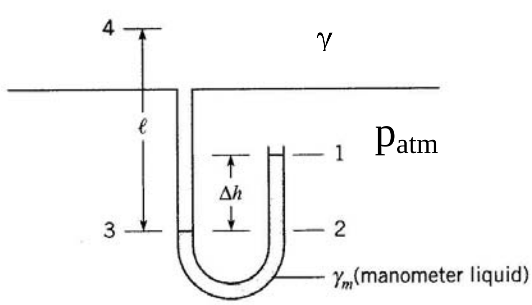


$p_{atm} + \gamma h = p_{pipe} = p$ absolute

$p = \gamma h$ gage

Simple but impractical for large p and vacuum pressures (i.e., $p_{abs} < p_{atm}$). Also for small p and small d, due to large surface tension effects, could be corrected using $\Delta h = 4\sigma/\gamma d$, but accuracy may be problem if $p/\gamma \sim \Delta h_\sigma$

3. U-tube or differential manometer



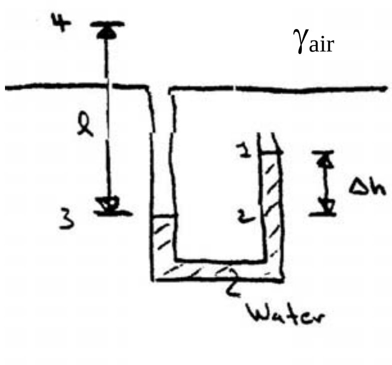
$p_1 + \gamma_m \Delta h - \gamma l = p_4$ $p_1 = p_{atm}$

$p_4 = \gamma_m \Delta h - \gamma l$ gage

$= \gamma_w [S_m \Delta h - S \ l]$

for gases S << S_m and can be neglected, i.e., can neglect Δp in gas compared to Δp in liquid in determining $p_4 = p_{pipe}$.
Example:

Air at 20 °C is in pipe with a water manometer. For given conditions compute gage pressure in pipe.



l = 140 cm

Δh = 70 cm

p_4 = ? gage (i.e., $p_1 = 0$)

Pressure same at 2&3 since same elevation & Pascal's law: in closed system pressure change produce at one part transmitted throughout entire system

$p_1 + \gamma\Delta h = p_3$ step-by-step method

$p_3 - \gamma_{air} l = p_4$

$\gamma\Delta h$

$p_1 + \gamma\Delta h - \gamma_{air} l = p_4$ complete circuit method

$\gamma\Delta h - \gamma_{air} l = p_4$ gage

$\gamma_{water}(20°C) = 9790\ N/m^3 \Rightarrow p_3 = \gamma\Delta h = 6853\ Pa\ [N/m^2]$

$\gamma_{air} = \rho g$

$$\rho = \frac{p}{RT} = \frac{(p_3 + p_{atm})}{R(°C + 273)} = \frac{6853 + 101300}{287(20 + 273)} = 1.286\ kg/m^3$$

($p_3 + p_{atm}$: p_{abs}; $°C + 273$: °K)

$\gamma_{air} = 1.286 \times 9.81 m/s^2 = 12.62\ N/m^3$

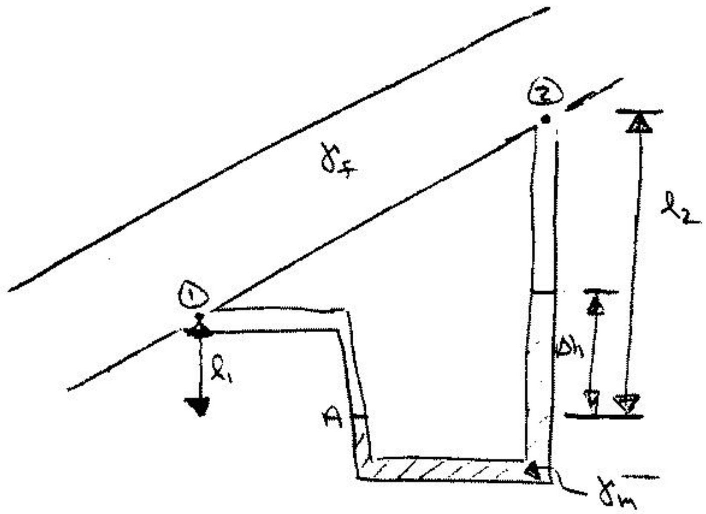
or could use Table A.3

note $\gamma_{air} << \gamma_{water}$

$$p_4 = p_3 - \gamma_{air} l = 6853 - \underbrace{12.62 \times 1.4}_{17.668} = 6835\ Pa$$

if neglect effect of air column $p_4 = 6853\ Pa$

A differential manometer determines the difference in pressures at two points ① and ② when the actual pressure at any point in the system cannot be determined.



$$p_1 + \gamma_f \ell_1 - \gamma_m \Delta h - \gamma_f (\ell_2 - \Delta h) = p_2$$
$$p_1 - p_2 = \gamma_f (\ell_2 - \ell_1) + (\gamma_m - \gamma_f) \Delta h$$

$$\underbrace{\left(\frac{p_1}{\gamma_f} + \ell_1\right) - \left(\frac{p_2}{\gamma_f} + \ell_2\right)} = \left(\frac{\gamma_m}{\gamma_f} - 1\right)\Delta h$$

difference in piezometric head

★if fluid is a gas $\gamma_f << \gamma_m$: $p_1 - p_2 = \gamma_m \Delta h$

★if fluid is liquid & pipe horizontal $\ell_1 = \ell_2$:

$$p_1 - p_2 = (\gamma_m - \gamma_f)\,\Delta h$$