Chapter 10 Approximate Solutions of the NS Equations

10.1 The Creeping Flow Approximation

See Textbook P476

10.2 Approximation for Inviscid Regions of Flow

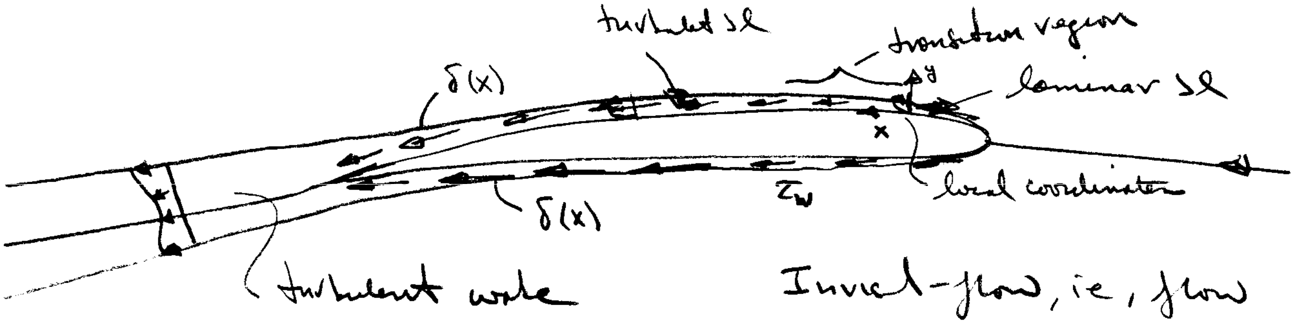
See Textbook P481

10.3 The Irrotational Flow Approximation

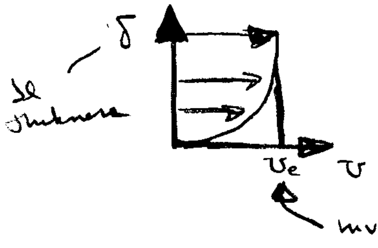
See Textbook P485

10.4 Qualitative Description of the Boundary Layer

Recall our previous description of the flow-field regions for high Re flow about slender bodies



Inviscid-flow, ie, flow external to the BL



viscous and inviscid flows merge smoothly

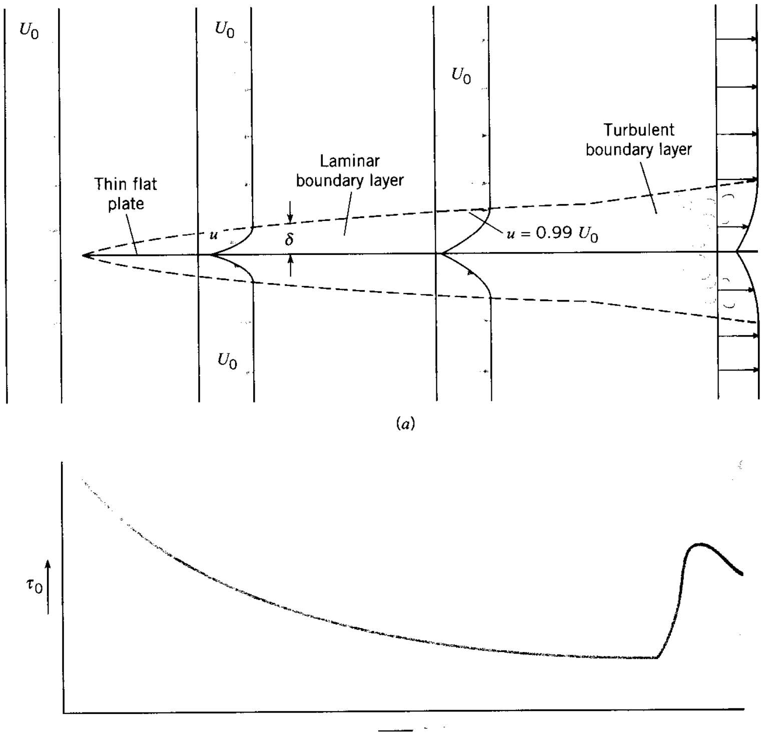


FIGURE 9.4
Development of boundary layer and distribution of shear stress along a thin, flat plate. (a) Flow pattern in boundary layers above and below the plate. (b) Shear-stress distribution on either side of the plate.

τ_w = shear stress

$\tau_w \propto$ rate of strain (velocity gradient)

$$=\mu \underbrace{\left.\frac{\partial u}{\partial y}\right|_{y=0}}_{\text{large near the surface where fluid undergoes large changes to satisfy the no-slip condition}}$$

Boundary layer theory is a simplified form of the complete NS equations and provides τ_w as well as a means of estimating C_{form}. Formally, boundary-layer theory represents the asymptotic form of the Navier-Stokes equations for high Re flow about slender bodies. As mentioned before, the NS equations are 2nd order nonlinear PDE and their solutions represent a formidable challenge. Thus, simplified forms have proven to be very useful.

Near the turn of the century (1904), Prandtl put forth boundary-layer theory, which resolved D'Alembert's paradox. As mentioned previously, boundary-layer theory represents the asymptotic form of the NS equations for high Re flow about slender bodies. The latter requirement is necessary since the theory is restricted to unseparated flow. In fact, the boundary-layer equations are singular at separation, and thus, provide no information at or beyond separation. However, the requirements of the theory are met in many practical situations and the theory has many times over proven to be invaluable to modern engineering.

The assumptions of the theory are as follows:

Variable	order of magnitude		
u	U	$O(1)$	
v	$\delta<<L$	$O(\varepsilon)$	$\varepsilon = \delta/L$
$\frac{\partial}{\partial x}$	L	$O(1)$	
$\frac{\partial}{\partial y}$	$1/\delta$	$O(\varepsilon^{-1})$	
ν	δ^2	ε^2	

The theory assumes that viscous effects are confined to a thin layer close to the surface within which there is a dominant flow direction (x) such that u ~ U and v << u. However, gradients across δ are very large in order to satisfy the no slip condition.

Next, we apply the above order of magnitude estimates to the NS equations.

$$\left.\begin{array}{l}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\
1\quad 1\qquad \varepsilon\quad \varepsilon^{-1}\qquad\qquad\quad \varepsilon^{2}\quad 1\qquad\quad \varepsilon^{-2}\\[2em]
u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}\right)\\
1\quad \varepsilon\qquad \varepsilon\quad 1\qquad\qquad\quad \varepsilon^{2}\quad 1\qquad\quad \varepsilon^{-1}\\[2em]
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\\
\quad 1\qquad 1
\end{array}\right\}\ \text{elliptic}$$

Retaining terms of O(1) only results in the celebrated boundary-layer equations

$$\left.\begin{array}{l}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}\\
\frac{\partial p}{\partial y}=0\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0
\end{array}\right\}\ \text{parabolic}$$

Some important aspects of the boundary-layer equations:

1) the y-momentum equation reduces to

$$\frac{\partial p}{\partial y} = 0$$

i.e., $p = p_e =$ constant across the boundary layer

edge value, i.e., inviscid flow value!

from the Bernoulli equation:

$$p_e + \frac{1}{2}\rho U_e^2 = \text{constant}$$

i.e., $$\frac{\partial p_e}{\partial x} = -\rho U_e \frac{\partial U_e}{\partial x}$$

Thus, the boundary-layer equations are solved subject to a specified inviscid pressure distribution

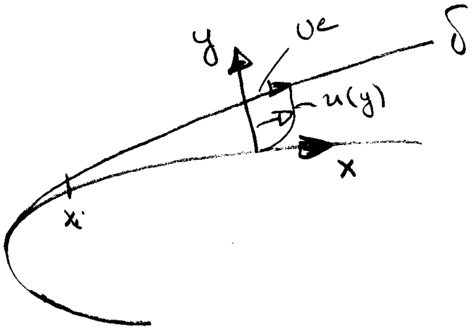
2) continuity equation is unaffected

3) Although NS equations are fully elliptic, the boundary-layer equations are parabolic and can be solved using marching techniques

4) Boundary conditions

$u = v = 0 \qquad y = 0$

$u = U_e \qquad y = \delta$



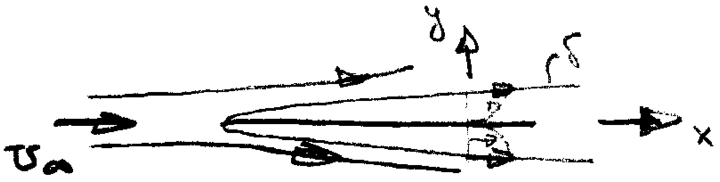
+ appropriate initial conditions @ x_i

There are quite a few analytic solutions to the boundary-layer equations. Also numerical techniques are available for arbitrary geometries, including both two- and three-dimensional flows. Here, as an example, we consider the simple, but extremely important case of the boundary layer development over a flat plate.

10.5 Quantitative Relations for the Laminar Boundary Layer

Laminar boundary-layer over a flat plate: Blasius solution (1908) student of Prandtl

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$



Note: $\frac{\partial p}{\partial x}=0$ for a flat plate

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$u = v = 0 \ @\ y = 0 \qquad u = U_\infty \quad @\ y = \delta$$

We now introduce a dimensionless transverse coordinate and a stream function, i.e.,

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \propto \frac{y}{\delta}$$

$$\psi = \sqrt{\nu x U_\infty} f(\eta)$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = U_\infty f'(\eta) \qquad f' = u / U_\infty$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left(\eta f' - f\right)$$

substitution into the boundary-layer equations yields

$$ff'' + 2f''' = 0 \qquad \text{Blasius Equation}$$

$$f = f' = 0 \quad @\ \eta = 0 \qquad f' = 1 \quad @\ \eta = 1$$

The Blasius equation is a 3rd order ODE which can be solved by standard methods (Runge-Kutta). Also, series solutions are possible. Interestingly, although simple in appearance no analytic solution has yet been found. Finally, it should be recognized that the Blasius solution is a similarity solution, i.e., the non-dimensional velocity profile f′ vs. η is independent of x. That is, by suitably scaling all the velocity profiles have neatly collapsed onto a single curve.

Now, lets consider the characteristics of the Blasius solution:

$$\frac{u}{U_\infty} \text{ vs. } y$$

FIGURE 9.5

Velocity distribution in laminar boundary layer. [After Blasius (3)].

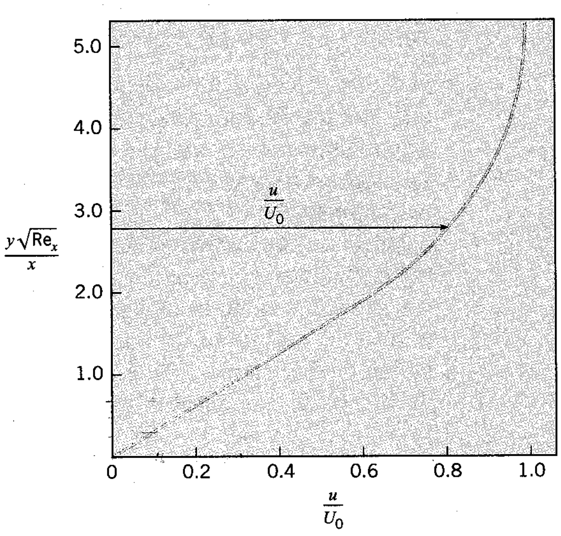


TABLE 9.1 RESULTS—δ AND τ_0 FOR DIFFERENT VALUES OF x

	$x = 0.1$ ft	$x = 1.0$ ft	$x = 2$ ft	$x = 4$ ft	$x = 6$ ft
$x^{1/2}$	0.316	1.00	1.414	2.00	2.45
τ_0, psf	0.552	0.174	0.123	0.087	0.071
δ, ft	0.005	0.016	0.022	0.031	0.039
δ, in.	0.060	0.189	0.270	0.380	0.466

$$\frac{V}{U_\infty}\sqrt{\frac{U_\infty}{V}} \text{ vs. } y$$

$$\delta = \frac{5x}{\sqrt{\mathrm{Re}}}$$ value of y where $u/U_\infty = .99$

$$\mathrm{Re}_x = \frac{U_\infty x}{\nu}$$

$$\tau_w = \frac{\mu U_\infty f''(0)}{\sqrt{2\nu x / U_\infty}}$$

i.e., $$c_f = \frac{2\tau_w}{\rho U_\infty^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} = \frac{\theta}{x}$$ ← see below

$$C_f = \frac{1}{L}\int_0^L c_f dx = 2c_f(L)$$

$$= \frac{1.328}{\sqrt{\mathrm{Re}_L}}$$

$$\frac{U_\infty L}{\nu}$$

Other:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy = 1.7208 \frac{x}{\sqrt{\mathrm{Re}_x}}$$ displacement thickness

measure of displacement of inviscid flow to due boundary layer

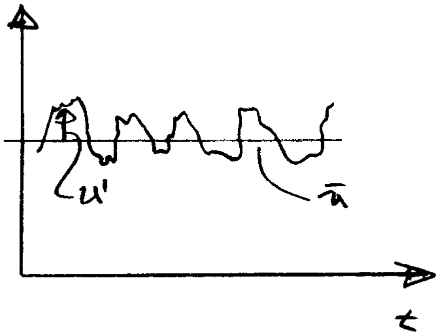
$$\theta = \int_0^\delta \left(1-\frac{u}{U_\infty}\right)\frac{u}{U_\infty}dy = 0.664\frac{x}{\sqrt{Re_x}} \qquad \text{momentum thickness}$$

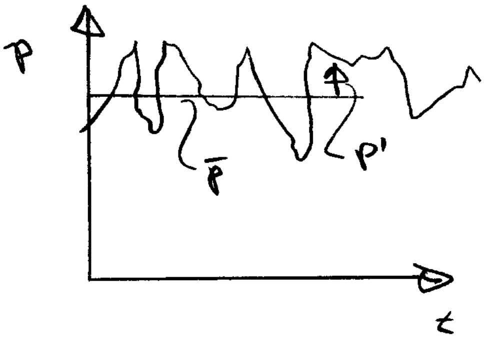
measure of loss of momentum due to boundary layer

$$H = \text{shape parameter} = \frac{\delta^*}{\theta} = 2.5916$$

10.6 Quantitative Relations for the Turbulent Boundary Layer

Description of Turbulent Flow





V and p are random functions of time in a turbulent flow The mathematical complexity of turbulence entirely precludes any exact analysis. A statistical theory is well developed; however, it is both beyond the scope of this course and not generally useful as a predictive tool. Since the time of Reynolds (1883) turbulent flows have been analyzed by considering the mean (time averaged) motion and the influence of turbulence on it; that is, we separate the velocity and pressure fields into mean and fluctuating components

$u = \bar{u} + u'$ $p = \bar{p} + p'$

$v = \bar{v} + v'$ and for compressible flow

$w = \bar{w} + w'$ $\rho = \bar{\rho} + \rho'$ and $T = \bar{T} + T'$

where (for example)

$$\bar{u} = \frac{1}{t_1}\int_{t_0}^{t_0+t_1} u\,dt$$

and t_1 sufficiently large that the average is independent of time

Thus by definition $\overline{u'} = 0$, etc. Also, note the following rules which apply to two dependent variables f and g

$$\bar{\bar{f}} = \bar{f} \qquad \overline{f+g} = \bar{f} + \bar{g}$$

$$\overline{\bar{f}\cdot g} = \bar{f}\cdot\bar{g}$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s} \qquad \overline{\int f ds} = \int \bar{f} ds$$

f = (u, v, w, p)
s = (x, y, z, t)

The most important influence of turbulence on the mean motion is an increase in the fluid stress due to what are called the apparent stresses. Also known as Reynolds stresses:

$$\tau'_{ij} = -\rho\overline{u'_i u'_j}$$

$$= \begin{bmatrix} -\rho\overline{u'^2} & -\rho\overline{u'v'} & -\rho\overline{u'w'} \\ -\rho\overline{u'v'} & -\rho\overline{v'^2} & -\rho\overline{v'w'} \\ -\rho\overline{u'w'} & -\rho\overline{v'w'} & -\rho\overline{w'^2} \end{bmatrix}$$

Symmetric
2nd order
tensor

The mean-flow equations for turbulent flow are derived by substituting $\underline{V} = \underline{\overline{V}} + \underline{V'}$ into the Navier-Stokes equations and averaging. The resulting equations, which are called the Reynolds-averaged Navier-Stokes (RANS) equations are:

Continuity $\nabla \cdot \underline{V} = 0$ i.e. $\nabla \cdot \underline{\overline{V}} = 0$ and $\nabla \cdot \underline{V'} = 0$

Momentum
$$\rho \frac{D\overline{V}}{Dt} + \rho \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = -\rho g\hat{k} - \nabla \overline{p} + \mu \nabla^2 \underline{\overline{V}}$$

or
$$\rho \frac{D\overline{V}}{Dt} = -\rho g\hat{k} - \nabla \overline{p} + \nabla \cdot \tau_{ij}$$

$$\tau_{ij} = \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \underbrace{- \rho\overline{u_i' u_j'}}_{\tau_{ij}'}$$

$u_1 = u$ $x_1 = x$
$u_2 = v$ $x_2 = y$
$u_3 = w$ $x_3 = z$

Comments:

1) equations are for the mean flow
2) differ from laminar equations by Reynolds stress terms = $\overline{u_i' u_j'}$
3) influence of turbulence is to transport momentum from one point to another in a similar manner as viscosity
4) since $\overline{u_i' u_j'}$ are unknown, the problem is indeterminate: the central problem of turbulent flow analysis is closure!

4 equations and 4 + 6 = 10 unknowns

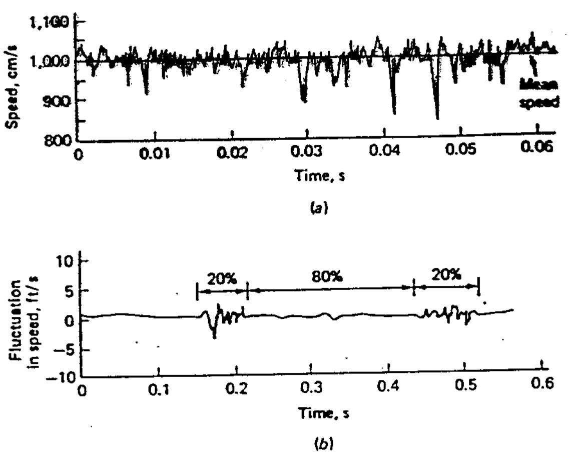


FIGURE 5-35
Hot-wire measurements showing turbulent velocity fluctuations: (*a*) typical trace of a single velocity component in a turbulent flow; (*b*) trace showing intermittent turbulence at the edge of a jet.

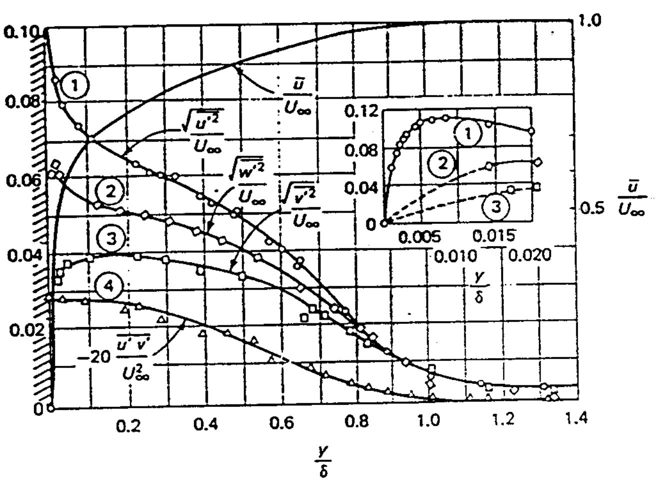


FIGURE 5-36
Flat-plate measurements of the fluctuating velocities u' (streamwise), (normal), and w' (lateral) and the turbulent shear $u'v'$. [*After Klebanoff (1955).*]

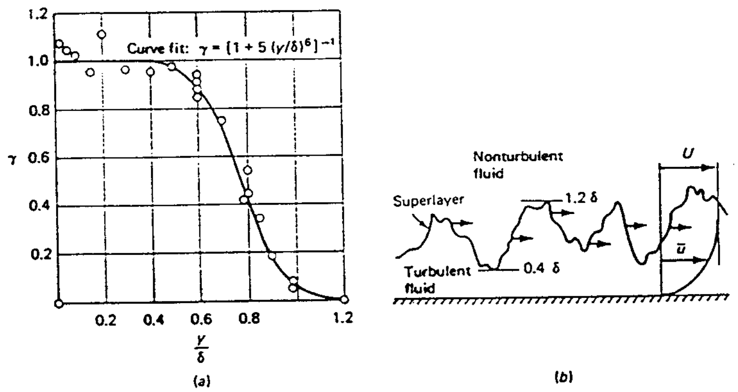


FIGURE 5-37
The phenomenon of intermittency in a turbulent boundary layer: (*a*) measured intermittency factors [*after Klebanoff (1955)*]; (*b*) the superlayer interface between turbulent and nonturbulent fluid.

$$\psi = \frac{\overline{u'\,v'}}{\sqrt{\overline{u'^2}}\cdot\sqrt{\overline{v'^2}}}$$

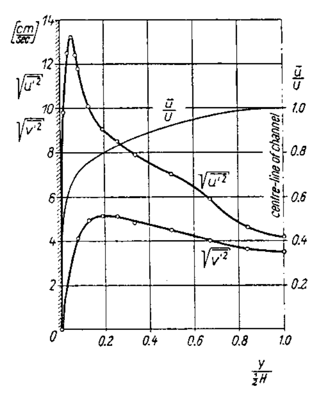


Fig. 18.3. Measurement of fluctuating turbulent components in a wind tunnel, at maximum velocity U = 100 cm/sec after Reichardt [41]

Root-mean-square of longitudinal fluctuation $\sqrt{\overline{u'^2}}$, transverse fluctuation $\sqrt{\overline{v'^2}}$, mean velocity $\bar{u}$

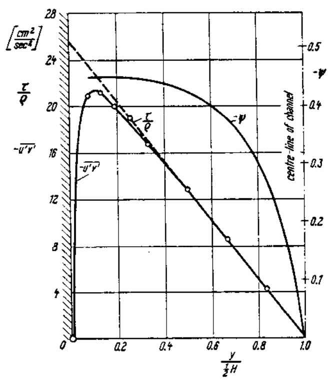


Fig. 18.4. Measurement of fluctuating components in a channel, after Reichardt [41]

The product $\overline{u'\,v'}$, the shearing stress τ/ϱ, and the correlation coefficient ψ

2-D Boundary-layer Form of RANS equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial}{\partial x}\left(\frac{p_e}{\rho}\right)+\nu\frac{\partial^2 u}{\partial y^2}\underbrace{-\frac{\partial}{\partial y}\left(\overline{u'v'}\right)}_{\text{requires modeling}}$$

Turbulence Modeling

Closure of the turbulent RANS equations require the determination of $-\rho\overline{u'v'}$, etc. Historically, two approaches were developed: (a) eddy viscosity theories in which the Reynolds stresses are modeled directly as a function of local geometry and flow conditions; and (b) mean-flow velocity profile correlations which model the mean-flow profile itself. The modern approaches, which are beyond the scope of this class, involve the solution for transport PDE's for the Reynolds stresses which are solved in conjunction with the momentum equations.

(a) eddy-viscosity: theories
(mainly used with differential methods)

$$-\rho\overline{u'v'}=\mu_t\frac{\partial\overline{u}}{\partial y}$$

In analogy with the laminar viscous stress, i.e., $\tau_t \propto$ mean-flow rate of strain

The problem is reduced to modeling μ_t, i.e.,

$\mu_t = \mu_t(\underline{x}$, flow at hand)

Various levels of sophistication presently exist in modeling μ_t

$$\mu_t = \rho V_t L_t$$

turbulent velocity scale

turbulent length scale

where V_t and L_t are based an large scale turbulent motion

The total stress is

$$\tau_{total} = (\mu + \mu_t)\frac{\partial \bar{u}}{\partial y}$$

molecular viscosity

eddy viscosity (for high Re flow $\mu_t >> \mu$)

<u>Mixing-length theory</u> (Prandtl, 1920)

$$-\rho\overline{u'v'} = c\rho\sqrt{\overline{u'}^2}\sqrt{\overline{v'}^2}$$

based on kinetic theory of gases

$$\sqrt{\overline{u'}^2} = \ell_1 \frac{\partial \bar{u}}{\partial y}$$

$$\sqrt{\overline{v'}^2} = \ell_2 \frac{\partial \bar{u}}{\partial y}$$

ℓ_1 and ℓ_2 are mixing lengths which are analogous to molecular mean free path, but much larger

$$\Rightarrow \quad -\rho\overline{u'v'} = \rho\ell^2\left|\frac{\partial \bar{u}}{\partial y}\right|\frac{\partial \bar{u}}{\partial y}$$

distance across shear layer

$$\ell = \ell(y)$$

$$= \text{f(boundary layer, jet, wake, etc.)}$$

Known as a zero equation model since no additional PDE's are solved, only an algebraic relation

Although mixing-length theory has provided a very useful tool for engineering analysis, it lacks generality. Therefore, more general methods have been developed.

One and two equation models

$$\mu_t = \frac{C\rho k^2}{\varepsilon}$$

C = constant

k^2 = turbulent kinetic energy

$= \overline{u'^2} = \overline{u'^2 + v'^2 + w'^2}$

ε = turbulent dissipation rate

Governing PDE's are derived for k and ε which contain terms that require additional modeling. Although more general then the zero-equation models, the k-ε model also has definite limitation; therefore, recent work involves the solution of PDE's for the Reynolds stresses themselves. Difficulty is that these contain triple correlations that are very difficult to model.

(b) mean-flow velocity profile correlations
(mainly used with integral methods)

As an alternative to modeling the Reynolds stresses one can model mean flow profile directly. For simple 2-D flows this approach is quite food and will be used in this course. For complex and 3-D flows generally not successful. Consider the shape of turbulent velocity profiles.

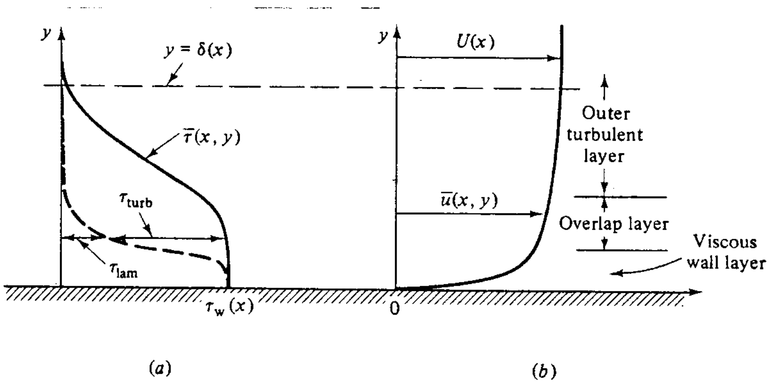


Fig. 6.8 Typical velocity and shear distributions in turbulent flow near a wall: (*a*) shear; (*b*) velocity. (measurements)

Note that very near the wall τ_{laminar} must dominate since $-\rho\overline{u_i u_j} = 0$ at the wall (y = 0) and in the outer part turbulent stress will dominate. This leads to the three layer concept:

Inner layer: viscous stress dominates

Outer layer: turbulent stress dominates

Overlap layer: both types of stress important

1) Inner layer (Prandtl, 1930)

$u = f(\mu, \tau_w, \rho, y)$ note: not $f(\delta)$

From dimensional analysis $u^+ = f(y^+)$ law-of-the-wall

$u^+ = y^+$

where: $u^+ = \dfrac{u}{u^*}$

$u^* =$ friction velocity $= \sqrt{\tau_w / \rho}$

$y^+ = \dfrac{yu^*}{\nu}$

very near the wall:

$\tau \sim \tau_w \sim \text{constant} = \mu \dfrac{du}{dy} \quad \Rightarrow \quad u = cy \quad$ or $\quad u^+ = y^+$

2) <u>Outer layer (Karmen, 1933)</u>

$$(U_e - u)_{outer} = g(\delta, \tau_w, \rho, y)$$

note: independent of μ and actually also depends on $\dfrac{dp}{dx}$

From dimensional analysis $\dfrac{U_e - u}{u^*} = f\left(\dfrac{y}{\delta}\right)$ velocity defect law

3) <u>Overlap layer (Milliken, 1937)</u>

In order for the inner and outer layers to merge smooth

$$\frac{u}{u^*}=\frac{1}{\underset{\substack{\uparrow\\ .41}}{\kappa}}\ln y^+ + \underset{\substack{\uparrow\\ 5}}{B} \qquad \text{log-law}$$

κ and B from experiments and independent of dp/dx

FIGURE 10.5
Velocity distribution for smooth pipes. [After Schlichting (36)]

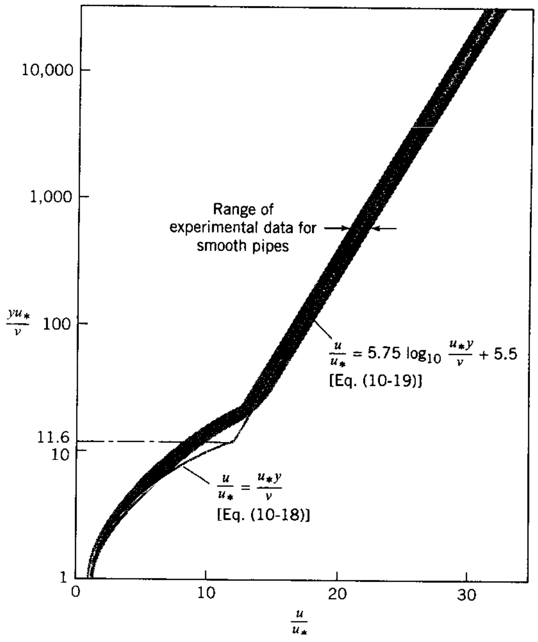
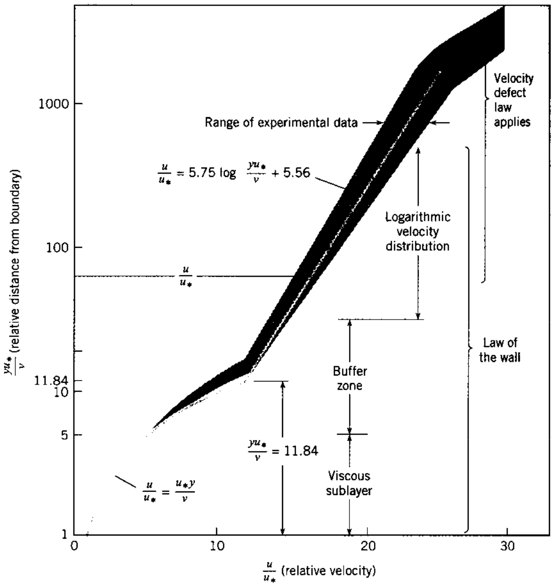


FIGURE 9.9
Velocity distribution in a turbulent boundary layer.



Note that the y^+ scale is logarithmic and thus the inner law only extends over a very small portion of δ

Inner law region $< .2\delta$

And the log law encompasses most of the boundary-layer. Thus as an approximation one can simply assume

$$\frac{u}{u^*} = \frac{1}{\kappa}\ln y + B \qquad u^+ = \sqrt{\tau_w/\rho} \qquad y^+ = \frac{yu^*}{\nu}$$

is valid all across the shear layer. This is the approach used in this course for turbulent flow analysis. The approach is a good approximation for simple and 2-D flows (pipe and flat plate), but does not work for complex and 3-D flows.

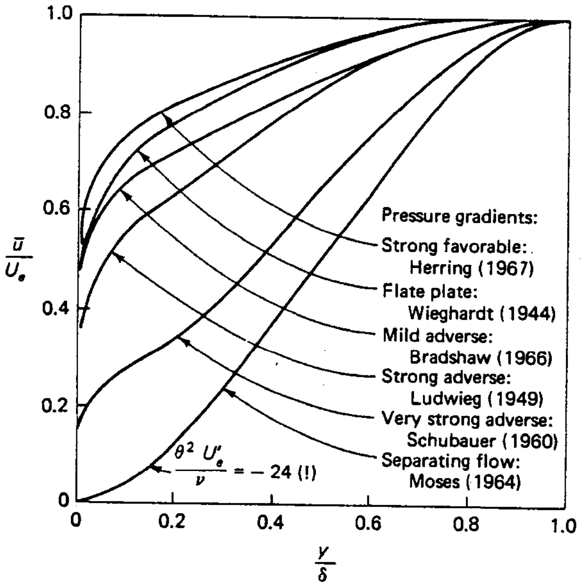


FIGURE 6-4
Experimental turbulent-boundary-layer velocity profiles for various pressure gradients.

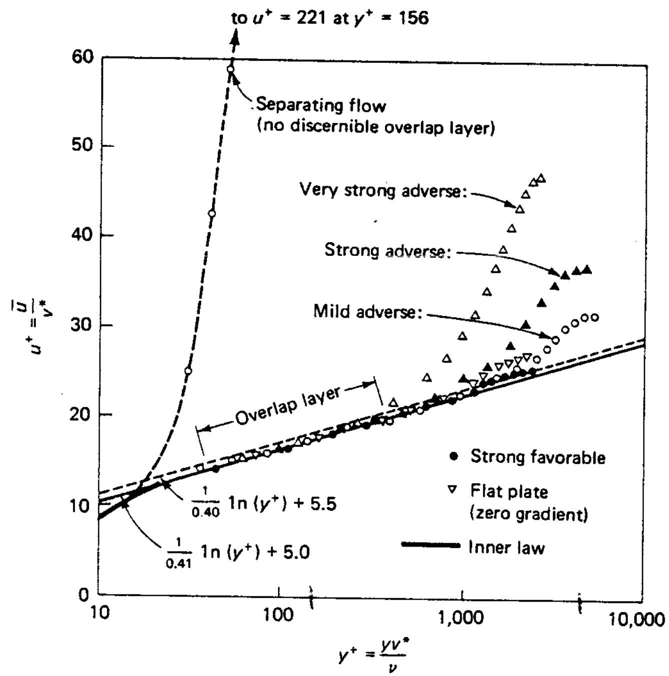


FIGURE 6-5
Replot of the velocity profiles of Fig. 6-4 using inner-law variables y^+ and u^+.

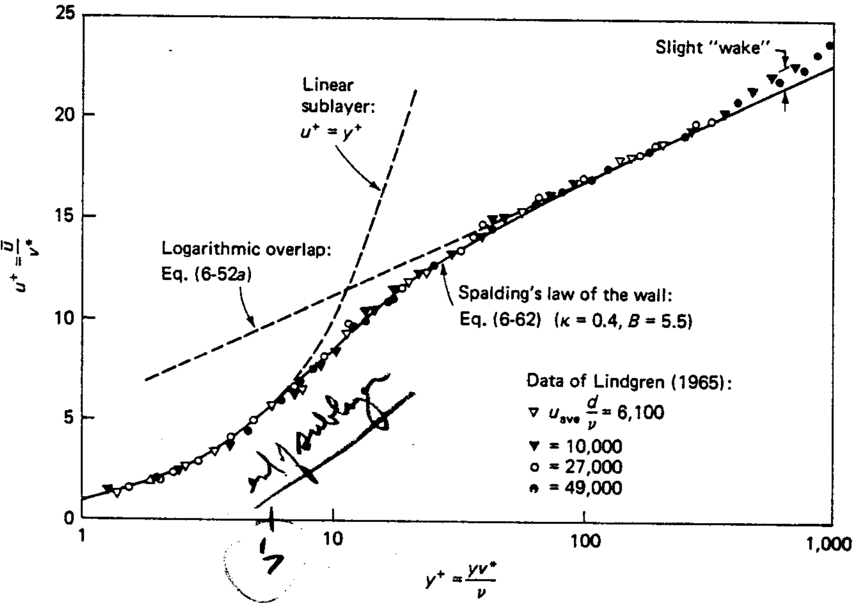


FIGURE 6-6
Comparison of Spalding's inner-law expression with the pipe-flow data of Lindgren (1965).

Momentum Integral Analysis

Background: History and Modern Approach: FD

To obtain general momentum integral relation which is valid for both laminar and turbulent flow

$$\int_{y=0}^{\overset{\text{For flat plate or } \delta \text{ for general case}}{\infty}} \left(\text{momentum equation} + (u - v)\,\text{continuity}\right) dy$$

$$\underbrace{\frac{\tau_w}{\rho U^2} = \frac{1}{2} c_f = \frac{d\theta}{dx}}_{\text{flat plate equation } \frac{dU}{dx} = 0} + (2 + H)\frac{\theta}{U}\frac{dU}{dx} \qquad -\frac{dp}{dx} = \rho U \frac{dU}{dx}$$

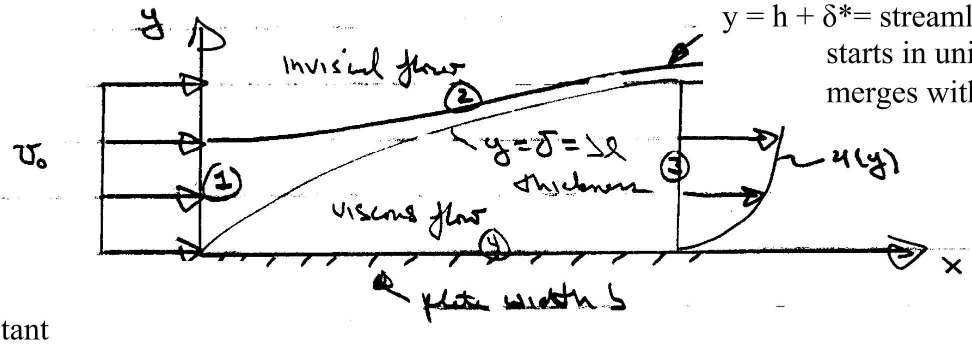
$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$ momentum thickness

$$H = \frac{\delta^*}{\theta}$$ shape parameter

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$ displacement thickness

Can also be derived by CV analysis as shown next for flat plate boundary layer.

Momentum Equation Applied to the Boundary Layer



y = h + δ*= streamline
starts in uniform flow
merges with δ at 3

Steady
ρ = constant
neglect g
v << u = u$_o$ ⇒ p = constant
i.e., -∇p = 0

CV = 1, 2, 3, 4

$$-\underset{\substack{\uparrow\\ \text{force on CV}}}{D} = \text{drag} = b\int_0^x \underset{\substack{\nwarrow\\ \text{wall shear stress}}}{\tau_w} dx$$

pressure force = 0 for v << U$_o$

u ~ U$_o$

$$\sum F_x = -D = \rho\int_1 u(\underline{V}\cdot\underline{dA}) + \rho\int_3 u(\underline{V}\cdot\underline{dA})$$

$$= \rho(-U_o^2 bh) + \rho b\int_3 u^2 dy$$

$$D(x) = \rho U_o^2 bh - \rho b\int_0^\delta u^2 dy$$

next eliminate h using continuity

$$0 = \rho \int_1 \underline{V} \cdot \underline{dA} + \rho \int_3 \underline{V} \cdot \underline{dA}$$

$$\rho U_o bh = \rho b \int_0^\delta u dy$$

depends on u(y)

$$U_o h = \int_0^\delta u dy$$

$$D(x) = \rho b U_o \int_0^\delta u dy - \rho b \int_0^\delta u^2 dy$$

$$= \rho b \int_0^\delta u(U_o - u) dy$$

$$C_D = \frac{D}{\frac{1}{2}\rho U_o^2 bL} = \frac{2}{L} \underbrace{\int_0^\delta \frac{u}{U_o}\left(1 - \frac{u}{U_o}\right) dy}_{\theta = \text{momentum thickness}}$$

$$C_D = \frac{2\theta}{L}$$

$$C_D = \frac{D}{\frac{1}{2}\rho U_o^2 A} = \frac{b\int_0^x \tau_w dx}{\frac{1}{2}\rho U_o^2 bL} = \frac{2\theta}{L}$$

$$\int_0^x \frac{\tau_w}{\frac{1}{2}\rho U_o^2}(x)dx = 2\theta(x)$$

$$\frac{1}{2}\left(\frac{\tau_w}{\frac{1}{2}\rho U_o^2}\right) = \frac{d\theta}{dx}$$

$$\frac{c_f}{2} = \underset{\nwarrow}{\frac{d\theta}{dx}}$$

c_f = local skin friction coefficient

momentum integral relation for flat plate boundary layer

$$\theta = \int_0^\delta \frac{u}{u_o}\left(1 - \frac{u}{u_o}\right)dy$$

Approximate solution for a laminar boundary-layer

Assume cubic polynomial for u(y)

$$\frac{u}{U_\infty} = A + By + Cy^2 + Dy^3$$

$$\left.\begin{array}{ll} u = \dfrac{\partial^2 u}{\partial y^2} = 0 & y = 0 \\ u = U_\infty ; \dfrac{\partial u}{\partial y} = 0 & y = \delta \end{array}\right\} \qquad \begin{array}{ll} A = 0 & B = \dfrac{3}{2}\delta \\ C = 0 & D = -\dfrac{1}{2}\delta^3 \end{array}$$

i.e., $\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ $\qquad u_y = U\left(\frac{3}{2\delta} + \frac{3}{2}\frac{y^2}{\delta}\right)\Bigg|_{y=0} = \frac{U3}{2\delta}$

$\boxed{\frac{\tau_w}{\rho U^2} = \frac{1}{2}c_f = \frac{d\theta}{dx}}$ momentum integral equation for $\frac{dp}{dx} = 0$

$$\frac{1}{\rho U^2}\underbrace{\left[\mu U \frac{3}{2\delta}\right]}_{\tau_w = \mu \frac{du}{dy}} = .139\frac{d\delta}{dx} \qquad \theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

Compare with

Exact Blassius

i.e., $\delta = \frac{4.65x}{\sqrt{Re_x}}$ $\frac{5x}{\sqrt{Re_x}}$ 7% ↓

$\tau_w = \frac{.323\rho V^2}{\sqrt{Re_x}}$ $\frac{.332\rho U^2}{\sqrt{Re_x}}$ 3%↓

$c_f = \frac{.646}{\sqrt{Re_x}}$ $\frac{.664}{\sqrt{Re_x}}$

$C_f = \frac{1.29}{\sqrt{Re_L}}$ $\frac{1.33}{\sqrt{Re_L}}$

$$\underbrace{C_f = \frac{1}{\frac{1}{2}\rho U^2 \underset{\nwarrow \text{span length}}{b} L}\int_0^L \tau_w(x)dx}_{\text{total skin-friction drag coefficient}}$$

Approximate solution Turbulent Boundary-Layer

$\mathrm{Re}_t \sim 3 \text{ X } 10^6$ for a flat plate boundary layer

$\mathrm{Re}_{crit} \sim 500{,}000$

$$\frac{c_f}{2} = \frac{d\theta}{dx}$$

as was done for the approximate laminar flat plate boundary-layer analysis, solve by expressing $c_f = c_f(\delta)$ and $\theta = \theta(\delta)$ and integrate, i.e.

assume log-law valid across entire turbulent boundary-layer

$$\frac{u}{u^*} = \frac{1}{\kappa}\ln\frac{yu^*}{\nu} + B$$

neglect laminar sub layer and velocity defect region

at $y = \delta$, $u = U$

$$\frac{U}{u^*} = \frac{1}{\kappa}\ln\underbrace{\frac{\delta u^*}{\nu}}_{\mathrm{Re}_\delta\left(\frac{c_f}{2}\right)^{1/2}} + B$$

$$\left.\begin{array}{l} \text{or} \quad \left(\dfrac{2}{c_f}\right)^{1/2} = 2.44\ln\left[\mathrm{Re}_\delta\left(\dfrac{c_f}{2}\right)^{1/2}\right] + 5 \\[2em] \qquad c_f \cong .02\,\mathrm{Re}_\delta^{-1/6} \text{ power-law fit} \end{array}\right\} \quad c_f(\delta)$$

Next, evaluate

$$\frac{d\theta}{dx} = \frac{d}{dx}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

can use log-law or more simply a power law fit

$$\left.\begin{aligned} \frac{u}{U} &= \left(\frac{y}{\delta}\right)^{1/7} \\ \theta &= \frac{7}{72}\delta = \theta(\delta) \end{aligned}\right\}$$

Note: can not be used to obtain $c_f(\delta)$ since $\tau_w \to \infty$

$$\Rightarrow \qquad \tau_w = c_f \frac{1}{2}\rho U^2 = \rho U^2 \frac{d\theta}{dx} = \frac{7}{72}\rho U^2 \frac{d\delta}{dx}$$

$$\mathrm{Re}_\delta^{-1/6} = 9.72\frac{d\delta}{dx}$$

or $\dfrac{\delta}{x} = .16\,\mathrm{Re}_x^{-1/7}$

$\delta \propto x^{6/7}$ almost linear

i.e., much faster growth rate than laminar boundary layer

$$c_f = \frac{.027}{\mathrm{Re}_x^{1/7}}$$

$$C_f = \frac{.031}{\mathrm{Re}_L^{1/7}} = \frac{7}{6}C_f(L)$$

Alternate forms given in text depending on experimental information and power-law fit used, etc. (i.e., dependent on Re range.)

Some additional relations given in texts for larger Re are as follows:

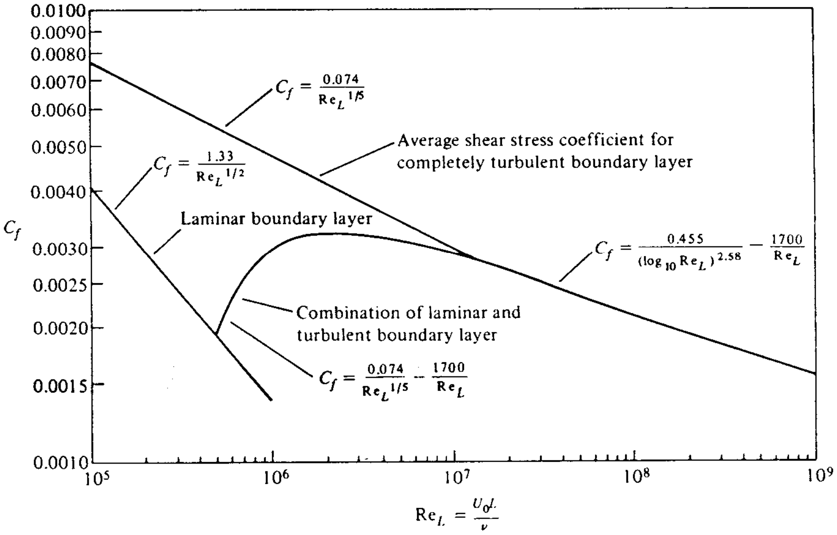
Total
shear-stress
coefficient

$$C_f = \frac{.455}{(\log_{10} Re_L)^{2.58}} \frac{-1700}{Re_L} \qquad Re > 10^7$$

$$\frac{\delta}{L} = c_f(.98 \log Re_L - .732)$$

Local
shear-stress
coefficient

$$c_f = (2 \log Re_x - .65)^{-2.3}$$



Finally, a composite formula that takes into account both the initial laminar boundary-layer (with translation at $Re_{CR} = 500{,}000$) and subsequent turbulent boundary layer is $C_f = \frac{.074}{Re_L^{1/5}} - \frac{1700}{Re_L}$ $\qquad 10^5 \le Re \le 10^7$