Chapter 8 Viscous Flow in Pipes

1. Entrance Region and Fully Developed Flow

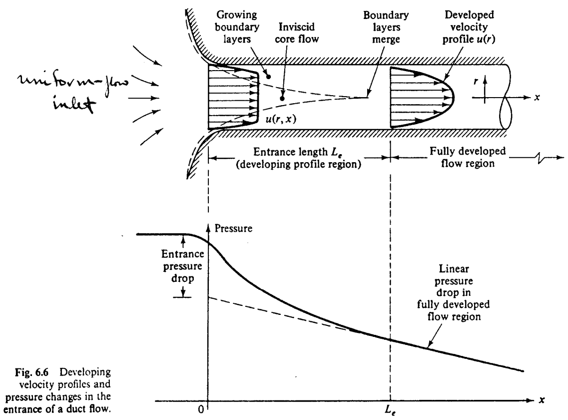


Fig. 6.6 Developing velocity profiles and pressure changes in the entrance of a duct flow.

Critical Reynolds number: $\mathrm{Re}_{crit} \sim 2000$

Entrance length Le:

$\frac{Le}{D} = 0.06\mathrm{Re}$ for laminar flow ($\mathrm{Re} < \mathrm{Re}_{crit}$)

$\frac{Le}{D} = 4.4(\mathrm{Re})^{1/6}$ for turbulent flow ($\mathrm{Re} > \mathrm{Re}_{crit}$)

2. Shear Stress and Head loss for a Pipe Flow

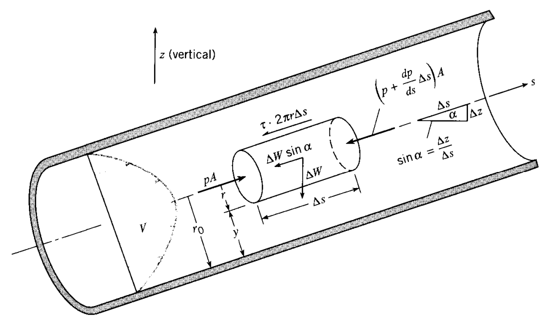


FIGURE 10.1
Variation of shear stress in a pipe.

Continuity: $V_1 = V_2$ where V = average velocity across pipe section

Momentum: $\tau = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right] \Rightarrow \tau_w = \frac{R}{2}\left[-\frac{d}{ds}(p + \gamma z)\right]$ where R = radius of pipe

Energy: $\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$

$\Rightarrow h_L = \frac{1}{\gamma}[(p_1 + \gamma z_1) - (p_2 + \gamma z_2)] = \frac{L}{\gamma}\left[-\frac{d}{ds}(p + \gamma z)\right]$ where $L = ds$ = length of pipe

Combining Energy with Momentum, $h_L = \frac{L}{\gamma}\left(\frac{2\tau_w}{R}\right)$

Friction factor (definition):

$$f = \frac{4\tau_w}{\frac{1}{2}\rho V^2}$$

Darcy-Weisbach equation:

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

where $h_L = h_f =$ loss due to pipe friction $\Rightarrow$ Need to know f

3. Laminar Flow

Exact solution to Navier-Stokes equations for a laminar flow in circular pipe:

$$u(r) = \frac{R^2-r^2}{4\mu}\left[-\frac{d}{ds}(p+\gamma z)\right] = V_c\left[1-\left(\frac{r}{R}\right)^2\right] \quad \text{where } V_c = \frac{R^2}{4\mu}\left[-\frac{d}{ds}(p+\gamma z)\right]$$

Flow rate: $Q = \int_0^R u(r)2\pi r dr = \frac{\pi R^4}{8\mu}\left[-\frac{d}{ds}(p+\gamma z)\right] = \frac{\pi R^2}{2}V_c$

Average velocity: $V = \frac{Q}{A} = \frac{R^2}{8\mu}\left[-\frac{d}{ds}(p+\gamma z)\right] = \frac{V_c}{2}$

Wall shear stress: $\tau_w = \mu\left.\frac{du}{dy}\right|_{y=0} = -\mu\left.\frac{du}{dr}\right|_{r=R} = \frac{R}{2}\left[-\frac{d}{ds}(p+\gamma z)\right] = \frac{R}{2}\left[\frac{8\mu V}{R^2}\right] = \frac{4\mu V}{R}$

Friction factor: $f = \frac{8\tau_w}{\rho V^2} = \frac{8}{\rho V^2}\left(\frac{4\mu V}{R}\right) = \frac{64\mu}{\rho VD} = \frac{64}{Re_D} \Rightarrow f = \frac{64}{Re_D}$ exact solution

Since $h_L = \frac{L}{\gamma}\left[\frac{\Delta p-\gamma\Delta z}{L}\right] = f\frac{L}{D}\frac{V^2}{2g} = \left(\frac{64\mu}{\rho VD}\right)\frac{L}{D}\frac{V^2}{2g}$;

Horizontal pipe ($\Delta z = 0$)	Non-horizontal pipe ($\Delta z = L\sin\theta$)
$V = \frac{\Delta p D^2}{32\mu L}$	$V = \frac{(\Delta p - \gamma L\sin\theta)D^2}{32\mu L}$
$Q = \frac{\pi D^4\Delta p}{128\mu L}$	$Q = \frac{\pi(\Delta p - \gamma L\sin\theta)D^4}{128\mu L}$
$\tau_w = \frac{D}{4}\frac{\Delta p}{L}$	$\tau_w = \frac{D}{4}\frac{(\Delta p - \gamma L\sin\theta)}{L}$
$f = \frac{8\tau_w}{\rho V^2} = \frac{2}{\rho V^2}\frac{D}{L}\Delta p$	$f = \frac{8\tau_w}{\rho V^2} = \frac{2}{\rho V^2}\frac{D}{L}(\Delta p - \gamma L\sin\theta)$
$h_L = f\frac{L}{D}\frac{V^2}{2g} = \frac{4L\tau_w}{\gamma D}$	$h_L = f\frac{L}{D}\frac{V^2}{2g} = \frac{4L\tau_w}{\gamma D}$

4. Turbulent Flow

4.1 Description of Turbulent Flow

(1) Randomness and fluctuations: Turbulence is irregular, chaotic, and unpredictable. However, structurally stationary flow, such as steady flows, can be analyzed using Reynolds decomposition.

$$u = \bar{u} + u'$$

where, $\bar{u} = \frac{1}{T}\int_{t_0}^{t_0+T} u dT$ = $mean\ motion$

u' = superimposed random fluctuation, $\overline{u'} = 0$

$\overline{u'}^2 = \frac{1}{T}\int_{t_0}^{t_0+T} u'^2 dT$ = Reynolds stresses

(2) Nonlinearity: Reynolds stresses and 3D vortex stretching are direct results of nonlinear nature of turbulence.

(3) Diffusion: Large scale mixing of fluid particles greatly enhances diffusion of momentum (and heat).

(4) Vorticity/eddies/energy cascade: Turbulence is characterized by flow visualization as eddies, which vary in size from the largest (width of flow) to the smallest. Largest eddies contain most of energy, which break up into successively smaller eddies with energy transfer to yet smaller until Kolmogorov scale is reached and energy is dissipated by molecular viscosity (i.e. viscous diffusion).

(5) Dissipation: ε = rate of dissipation = energy/time. Dissipation occurs at smallest scales.

4.2 Reynolds-Averaged Navier-Stokes (RANS) equations:

Reynolds decomposition: $\underline{V} = \underline{\bar{V}} + \underline{V'}$, $p = \bar{p} + p'$, $\rho = \bar{\rho} + \rho'$

Continuity: $\nabla \cdot \underline{V} = 0$ i.e. $\nabla \cdot \underline{\bar{V}} = 0$ and $\nabla \cdot \underline{V'} = 0$

Momentum: $\rho \frac{D\underline{\bar{V}}}{Dt} + \rho \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = -\rho g \hat{k} - \nabla \bar{p} + \mu \nabla^2 \underline{\bar{V}}$

or $\rho \frac{D\underline{\bar{V}}}{Dt} = -\rho g \hat{k} - \nabla \bar{p} + \nabla \cdot \tau_{ij}$

where, $\tau_{ij} = \mu\left[\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}\right] - \underbrace{\rho \overline{u_i' u_j'}}_{\tau'_{ij}}$

1) equations are for the mean flow
2) differ from laminar equations by Reynolds stress terms = $\overline{u_i' u_j'}$
3) influence of turbulence is to transport momentum from one position to another in a similar manner as viscosity
4) since $\overline{u_i' u_j'}$ are unknown, the problem is indeterminate: the central problem of turbulent flow analysis is closure. 4 equations and 4+6=10 unknowns

4.3 Turbulence Modeling

(a) eddy-viscosity theories:

In analogy with the laminar viscous stress, i.e., $\tau_t \propto$ mean-flow rate of strain

$$-\rho\overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$, where μ_t = eddy viscosity

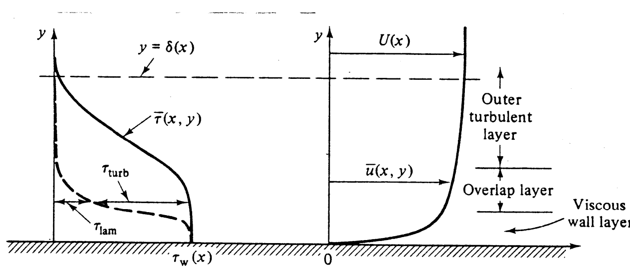
Mixing-length theory: based on kinetic theory of gasses

$$-\rho\overline{u'v'} = \rho \ell^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \frac{\partial \bar{u}}{\partial y}$$, where $\ell = \ell(y)$

One and two equation models:

$$\mu_t = \frac{C\rho k}{\varepsilon}$$, where C = constant, k = turbulent kinetic energy, ε = turbulence dissipation rate

(b) mean-flow velocity profile correlations



1. laminar sub-layer (viscous shear dominates)

Very near the wall: $0 < y^+ < 5$

$$u^+ = y^+$$ law-of-the-wall

where, $u^+ = u/u^*$, $y^+ = yu^*/\nu$, and u^* = friction velocity = $\sqrt{\tau_w/\rho}$

2. Outer layer (turbulent shear dominates)

$$\frac{U_e - u}{u^*} = f\left(\frac{y}{\delta}\right)$$ velocity defect law

3. overlap layer (viscous and turbulent shear important)

$20 < y^+ < 10^5$

$$u^+ = \frac{1}{\kappa}\ln y^+ + B$$ log-law

where, $\kappa = 0.41$ and $B = 5.5$

4.4 Smooth Pipes

Assuming log-law is valid across entire pipe, the velocity profile is

$$\frac{u(r)}{u^*} = \frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu} + B$$

Average velocity:

$$\frac{V}{u^*} = 2.44\ln\frac{Ru^*}{\nu} + 1.34$$

By using following relations,

$$\frac{V}{u^*} = \left(\frac{\rho V^2}{\tau_w}\right)^{1/2} = \left(\frac{8}{f}\right)^{1/2}$$

$$\frac{Ru^*}{\nu} = \frac{\frac{1}{2}VD}{\nu}\frac{u^*}{V} = \frac{1}{2}Re_D\left(\frac{f}{8}\right)^{1/2}$$

the relation between the friction factor and Reynolds number is (Prandtl, 1935),

$$\frac{1}{f^{1/2}} = 1.99\log\left(Re_D f^{1/2}\right) - 1.02$$

With slightly adjusted constants to fit friction data better:

$$\frac{1}{f^{1/2}} = 2.0\log\left(Re_D f^{1/2}\right) - 0.8$$

Alternative approximation: Power law (H. Blasius, 1911)

$$f = 0.316Re_D^{-1/4} \qquad 4000 < Re_D < 10^5$$

Then, the head loss is

$$h_f = f\frac{L}{D}\frac{V^2}{2g} \approx 0.316\left(\frac{\mu}{\rho VD}\right)^{1/4}\frac{L}{D}\frac{V^2}{2g}$$

$$\propto V^{1.75}$$

4.5 Rough Pipes

Non-dimensional roughness $\epsilon^+ = \frac{\epsilon u^*}{\nu}$

1) $\epsilon^+ < 5$ hydraulically smooth (no effect of roughness)
2) $5 < \epsilon^+ < 70$ transitional roughness (Re dependence)
3) $\epsilon^+ > 70$ fully rough (independent Re)

For fully rough flow, $\epsilon^+ > 70$, the log law is modified as

$$u^+ = \frac{1}{\kappa}\ln\frac{y}{\epsilon} + 8.5$$

Then the average velocity becomes

$$\frac{V}{u^*} = 2.44\ln\frac{D}{\epsilon} + 3.2$$

or

$$\frac{1}{f^{1/2}} = -2\log\frac{\epsilon/D}{3.7}$$

Colebrook (1939) combined the smooth wall and fully rough relations,

$$\frac{1}{f^{1/2}} = -2.0\log\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{Re_D f^{1/2}}\right)$$

⇒ Moody Diagram

4.6 Three Canonical Types of Problems

1. Determine the Head Loss

$$h_f = f\frac{L}{D}\frac{V^2}{2g} = \Delta h = \left(\frac{p_1}{\gamma} + z_1\right) - \left(\frac{p_2}{\gamma} + z_2\right)$$

$$f = f\left(Re_D, \frac{\epsilon}{D}\right)$$

2. Determine the Flow Rate

$$V = \underbrace{\left[\frac{2g}{L/D}h_f\right]^{1/2}}_{\text{known from problem statement}} f^{-1/2}$$

Guess $f \to V \to Re \to f$, repeat until converged

3. Determine the Pipe Diameter

$$D = \underbrace{\left[\frac{8LQ^2}{\pi^2 g h_f}\right]^{1/5}}_{\text{known from problem statement}} f^{-1/5}$$

Guess $f \to D \to Re, \epsilon/D, \to f$, repeat until converged

4.7 Minor Losses

1. Entrance and exit effects
2. Expansions and contractions
3. Bends, elbows, tees, and other fittings
4. Valves (open or partially closed)

Energy equation:

$$\frac{p_1}{\gamma} + z_1 + \frac{1}{2g}\alpha_1 V_1^2 + h_p = \frac{p_2}{\gamma} + z_2 + \frac{1}{2g}\alpha_2 V_2^2 + h_t + h_f + \sum h_m$$

Head loss due to minor loss, h_m:

$$h_m = K_L \frac{V^2}{2g}$$

Loss coefficient, K_L:

$$K_L = \frac{h_m}{(V^2/2g)}$$

1) Flow in a bend: swirling flow and/or flow separation represent energy losses which must be added to head loss h_L
2) Valves: enormous losses
3) Entrances: depends on rounding of entrance
4) Exit (to a large reservoir): $K_L = 1$
5) Contraction and Expansions

   - Sudden

     Expansion: $h_L = \frac{(V_1 - V_2)^2}{2g} \Rightarrow K_{SE} = \left(1 - \frac{d^2}{D^2}\right)^2 = \frac{h_m}{V_1^2/2g}$

     Contraction: $K_{SC} = 0.42\left(1 - \frac{d^2}{D^2}\right)$ from experiment

   - Gradual

     Expansion (Diffuser): $K_L = C_{p_{ideal}} - C_p$

     where, $C_p = \frac{p_2 - p_1}{1/2\rho V_1^2}$ and $C_{p_{ideal}} = 1 - (A_1/A_2)^2$

Chapter 9 Flow over Immersed Bodies

1. Fluid flow categories:

1. Internal flow: bounded by walls or fluid interfaces
   Ex) duct/pipe, turbo machinery, open channel/river

2. External flow: unbounded or partially bounded. Viscous and inviscid flow regions
   Ex) Flow around vehicles and structures
   a. Boundary layer flow: high Reynolds number flow around streamlined bodies without flow separation
   b. Bluff body flow: flow around bluff bodies with flow separation

3. Free shear flow: absence of walls
   Ex) jets, wakes, mixing layers

2. Basic Considerations

Drag:

$$C_D = \frac{1}{1/2\,\rho V^2 A}\left\{\underbrace{\int_S (p - p_\infty)\underline{n}\cdot\hat{\imath}\,dA}_{C_{D_p}\text{:form drag}} + \underbrace{\int_S \tau_w \underline{t}\cdot\hat{\imath}\,dA}_{C_f\text{: skin−friction}}\right\}$$

Lift:

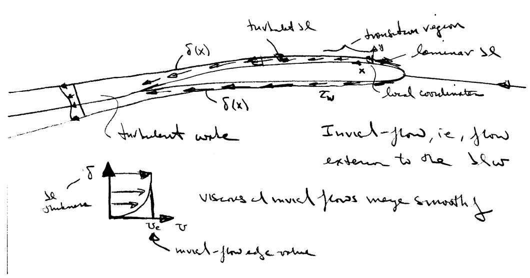
$$C_L = \frac{1}{1/2\,\rho V^2 A}\left\{\int_S (p - p_\infty)\underline{n}\cdot\hat{\jmath}\,dA\right\}$$

$t/c \ll 1$	$C_f \gg C_{D_p}$	streamlined body
$t/c \sim 1$	$C_{D_p} \gg C_f$	bluff body

Streamlining: One way to reduce the drag. Reducing vibration and noise.

3. Boundary Layer

Flow-field regions for high Re flow about slender bodies



- Boundary layer (BL) theory assumes that viscous effects are confined to a thin layer.
- There is a dominant flow direction (x) such that $u \sim U$ and $v \ll u$.
- Gradients across δ are very large in order to satisfy the no slip condition; thus, $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\tau = \begin{cases} \mu\frac{\partial u}{\partial y} & \text{laminar flow} \\ \mu\frac{\partial u}{\partial y} - \overline{\rho u'v'} & \text{turbulent flow} \end{cases}$$

1) $\frac{\partial p}{\partial y} = 0$, i.e. $p = p_e$ = constant across the boundary layer, where p_e = edge value (inviscid flow)
   Bernoulli equation: $p_e + \frac{1}{2}\rho U_e^2$ = constant, i.e., $\frac{\partial p_e}{\partial x} = -\rho U_e \frac{\partial U_e}{\partial x}$
2) Continuity equation is unaffected
3) Elliptic NS equations $\longrightarrow$ Parabolic BL equations which can be solved using marching techniques
4) Boundary conditions
   $u = v = 0$ at $y = 0$
   $u = U_e$ at $y = \delta$

3.1 Momentum Integral Method

$$\frac{\tau_w}{\rho U^2} = \frac{1}{2}c_f = \frac{d\theta}{dx} + \underbrace{(2+H)\frac{\theta}{U}\frac{dU}{dx}}_{\frac{dU}{dx}=0 \text{ for flat flates}}$$

- $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$ displacement thickness
- $\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$ momentum thickness
- $H = \frac{\delta^*}{\theta}$ shape parameter

For flat plates:

- $D = \int_0^x \tau_w dx = \rho U^2 \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \rho U^2 \theta$
- $C_f = \frac{D}{1/2\rho U^2 x} = \frac{2\theta}{x} = \frac{1}{x}\int_0^x c_f dx$
- $c_f = \frac{\tau_w}{1/2\rho U^2} \Rightarrow c_f = \frac{d}{dx}(xC_D) = 2\frac{d\theta}{dx}$
- $\tau_w = \rho U^2 \frac{d\theta}{dx}$

3.2 The Flat Plate Boundary Layer

3.2.1 Laminar Flow: Blasius solution (1908)

Introducing a dimensionless transverse coordinate and a stream function,

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\Psi = \sqrt{\nu x U_\infty} f(\eta)$$

Blasius equation:

$$ff'' + 2f''' = 0$$

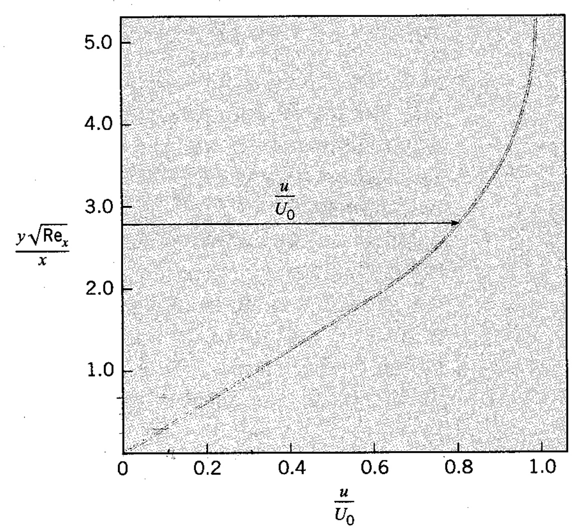
with B.C.'s, $f = f' = 0$ at $\eta = 0$ and $f' = 1$ at $\eta = 1$

$$\delta = \frac{5x}{\sqrt{Re_x}}$$

$$\delta^* = 1.7208\frac{x}{\sqrt{Re_x}}$$

$$\theta = 0.664\frac{x}{\sqrt{Re_x}}$$

$$H = \frac{\delta^*}{\theta} = 2.5916$$



$$\tau_w = 0.332U_\infty^{3/2}\sqrt{\frac{\rho\mu}{x}} = 0.332\frac{\mu U_\infty}{x}\sqrt{Re_x}$$

$$c_f = \frac{\tau_w}{1/2\,\rho U_\infty^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}$$

$$C_f = \frac{1}{L}\int_0^L c_f dx = 2c_f(L) = \frac{1.328}{\sqrt{Re_L}}$$

where $Re_x = \frac{U_\infty x}{\nu}$

3.2.2 Transition to Turbulent Flow

- $Re_{x,tr} = 5 \times 10^5$

3.2.3 Turbulent Flow

From the momentum integral theorem for flat plates,

$$c_f = 2\frac{d\theta}{dx}$$

By using a power-low approximation (Prandtl),

$$c_f \approx 0.02Re_\delta^{-1/6}$$

$$\frac{u}{U_\infty} \approx \left(\frac{y}{\delta}\right)^{1/7}$$

$$\Rightarrow \ \theta = \int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left[1-\left(\frac{y}{\delta}\right)^{1/7}\right]dy = \frac{7}{72}\delta$$

Then, the momentum equation becomes,

$$0.02Re_\delta^{-1/6} = 2\frac{d}{dx}\left(\frac{7}{72}\delta\right)$$

or

$$Re_\delta^{-1/6} = 9.72\frac{d\delta}{dx}$$

Separate the variables and integrate, assuming $\delta = 0$ at $x = 0$:

$$\delta = \frac{0.16x}{Re_x^{1/7}}$$

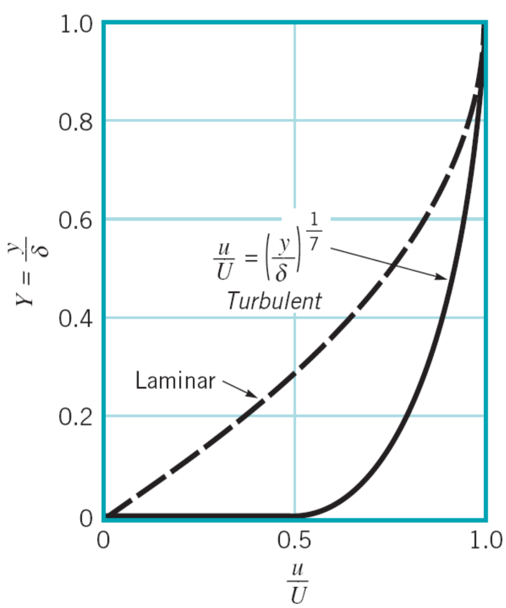
$$\delta^* = \frac{1}{8}\delta$$

$$\theta = \frac{7}{72}\delta$$

$$H = \frac{\delta^*}{\theta} = 1.3$$

$$c_f = \frac{0.027}{Re_x^{1/7}}$$

$$C_f = \frac{0.031}{Re_L^{1/7}} = \frac{7}{6}c_f(L)$$



4. Drag

4.1 Flat plates

1) Parallel to the flow

$$C_{D_p} = \frac{1}{1/2\,\rho V^2 A}\int_S (p - p_\infty)\underline{n}\cdot \hat{\imath} dA = 0$$

$$C_f = \frac{1}{1/2\,\rho V^2 A}\int_S \tau_w \underline{t}\cdot \hat{\imath} dA = \begin{cases} \dfrac{1.33}{Re_L^{1/2}} & \text{laminar flow} \\ \dfrac{0.074}{Re_L^{1/5}} & \text{turbulent flow} \end{cases}$$

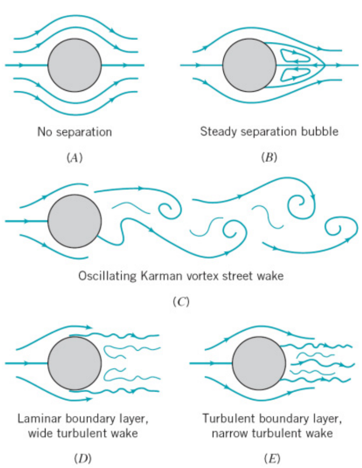
2) Normal to the flow

$$C_{D_p} = \frac{1}{1/2\,\rho V^2 A}\int_S (p - p_\infty)\underline{n}\cdot \hat{\imath} dA = \frac{1}{A}\int_S C_p dA = 2$$

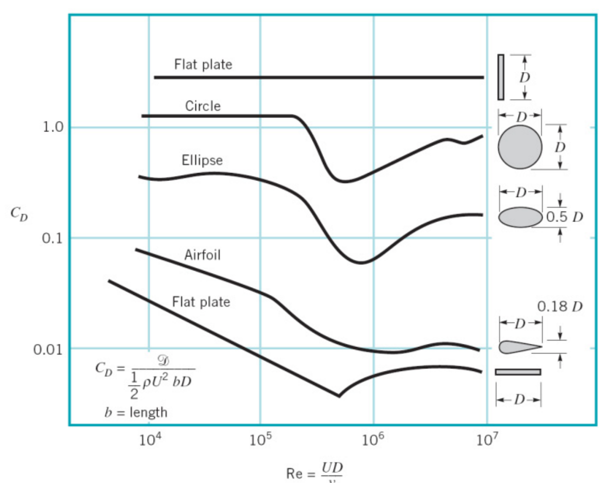
$$C_f = 0$$

4.2 Bluff Body

$$C_D = \frac{\text{Drag}}{1/2\,\rho V^2 A}$$



(b)



<u>Flow separation</u>: The fluid stream detaches itself from the surface of the body at sufficiently high velocities. Only appeared in viscous flow.

- Inside the separation region:
  - low pressure, existence of recirculating/backflows
  - viscous and rotational effects are the most significant
- Important physics related to flow separation:
  - 'Stall' for airplane, vortex shedding

Terminal velocity is the maximum velocity attained by a falling body when the drag reaches a magnitude such that the sum of all external forces on the body is zero.

$$\sum \underline{F} = F_d + F_b - F_g = m\underline{a} = 0$$

or

$$\frac{1}{2}\rho V_0^2 C_D A_p = (\gamma_{sphere} - \gamma_{fluid})\forall_{sphere}$$

For a sphere, $A_p = \frac{\pi}{4}d^2$ and $\forall = \frac{\pi}{6}d^3$

The terminal velocity is:

$$V_0 = \left[\frac{(\gamma_{sphere} - \gamma_{fluid})(4/3)d}{C_D \rho_{fluid}}\right]^{1/2}$$

Effect of Compressibility on Drag:

- $Ma \sim 1$: C_D increases due to shock waves and wave drag
- $Ma \geq 1$: upstream flow is not warned of approaching disturbance which results in the formation of shock waves across which flow properties and streamlines change discountinously
- $Ma_{critical}$(sphere) ~ 0.6, $Ma_{critical}$(slender bodies) ~ 1

5. Lift

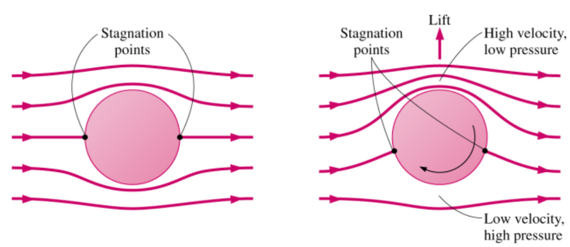
$$C_L = \frac{\text{Lift}}{1/2\,\rho V^2 A}$$

Lift force: the component of the net force (viscous+pressure) that is perpendicular to the flow direction

The minimum flight velocity: Total weight W of the aircraft be equal to the lift

$$W = F_L = \frac{1}{2}C_{L,max}\rho V_{min}^2 A \;\rightarrow\; V_{min} = \sqrt{\frac{2W}{\rho C_{L,max} A}}$$

Magnus effect: Lift generation by spinning. Breaking the symmetry causes the lift.



(*a*) Potential flow over a stationary cylinder (*b*) Potential flow over a rotating cylinder