Review for Exam 2

Chapter 5: Mass, Momentum, and Energy equations

Flow rate and conservation of mass

general case

$$Q = \int_{CS} \underline{V} \cdot \underline{n} dA = \int_{CS} |\underline{V}| \cos\theta dA$$

$$\dot{m} = \int_{CS} \rho(\underline{V} \cdot \underline{n}) dA$$

average velocity: $\overline{V} = \dfrac{Q}{A}$

General Form of Continuity Equation

$$\frac{dM}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho d\forall + \int_{CS} \rho \underline{V} \cdot \underline{dA}$$

or

$$\underbrace{\int_{CS} \rho \underline{V} \cdot \underline{dA}}_{\text{net rate of outflow of mass across CS}} = \underbrace{-\frac{d}{dt}\int_{CV} \rho d\forall}_{\text{rate of decrease of mass within CV}}$$

net rate of outflow of mass across CS; rate of decrease of mass within CV

Simplification of Continuity Equation

1. Steady flow: $-\frac{d}{dt}\int_{CV}\rho d\forall = 0$

2. $\underline{V}$ = constant over discrete $\underline{dA}$ (flow sections):

$$\int_{CS}\rho\underline{V}\cdot\underline{dA} = \sum_{CS}\rho\underline{V}\cdot\underline{A}$$

3. Incompressible fluid (ρ = constant)

$$\int_{CS}\underline{V}\cdot\underline{dA} = -\frac{d}{dt}\int_{CV}d\forall$$ conservation of volume

4. Steady One-Dimensional Flow in a Conduit:

$$\sum_{CS}\rho\underline{V}\cdot\underline{A} = 0$$

$$-\rho_1V_1A_1 + \rho_2V_2A_2 = 0$$

for ρ = constant $\quad Q_1 = Q_2$

Momentum Equation

RTT with $B = M\underline{V}$ and $\beta = \underline{V}$

$$\sum\left[\underline{F}_S + \underline{F}_B\right] = \frac{d}{dt}\int_{CV}\rho\underline{V}d\forall + \int_{CS}\underline{V}\rho\underline{V}_R\cdot\underline{dA}$$

$\underline{V}$ = velocity referenced to an inertial frame (non-accelerating)
$\underline{V}_R$ = velocity referenced to control volume
$\underline{F}_S$ = surface forces + reaction forces (due to pressure and viscous normal and shear stresses)
$\underline{F}_B$ = body force (due to gravity)

Applications of the Momentum Equation

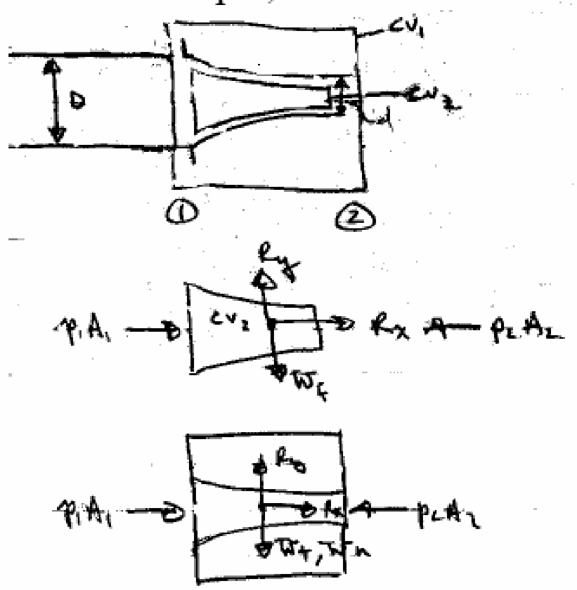
Initial Setup and Signs

1. Jet deflected by a plate or a vane
2. Flow through a nozzle
3. Forces on bends
4. Problems involving non-uniform velocity distribution
5. Motion of a rocket
6. Force on rectangular sluice gate
7. Water hammer

Important features for momentum equation:

1. Vector equations→ Need to derive component by component
2. Carefully define CV to include all external body and surface faces

   For example,



(R_x,R_y) = reaction force on fluid

(R_x,R_y) = reaction force on nozzle

3. Velocity must be referenced to a non-accelerating inertial frame.
4. Steady vs Unsteady flow
5. Uniform vs Nonuniform flow
6. Always use gage pressure
7. Pressure condition at a jet exit

Energy Equation

The energy equation is derived from RTT with

B = E = total energy of the system

$\beta = e = E/M$ = energy per unit mass

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{CV} \rho\left(\frac{V^2}{2} + gz + u\right)d\forall + \int_{Cs} \rho\left(\frac{V^2}{2} + gz + u\right)\underline{V}\cdot\underline{dA}$$

- $\dot{Q}$: rate of heat transfer to sysem
- $\dot{W}$: rate of work done by system
- $\frac{d}{dt}\int_{CV}$ term: rate of change of energy in CV
- $\int_{Cs}$ term: flux of energy out of CV (ie, across CS)

Rate of Work Components: $\dot{W} = \dot{W}_s + \dot{W}_f$

For convenience of analysis, work is divided into shaft work W_s and flow work W_f

W_f = net work done on the surroundings as a result of normal and tangential stresses acting at the control surfaces

$= W_{f\,pressure} + W_{f\,shear}$

W_s = any other work transferred to the surroundings usually in the form of a shaft which either takes energy out of the system (turbine) or puts energy into the system (pump)

Simplified form of energy equation

Simplification:

- No acceleration normal to the stream lines, pressure is hydrostatically distributed.
- Internal energy u is considered as constant.
- Shaft work defined as $\dot{W}_S = \dot{W}_t - \dot{W}_p$
- Head loss defined as $h_L = \dfrac{u_2 - u_1}{g} - \dfrac{\dot{Q}}{g\dot{m}}$

The final form of simplified energy equation:

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

$$h_p = \dot{W}_p / \dot{m}g = \dot{W}_p / \gamma Q$$

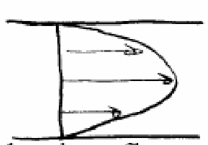
$$h_t = \dot{W}_t / \dot{m}g = \dot{W}_t / \gamma Q$$

$$\alpha = \frac{1}{A\bar{V}^2} \int_A V^3 dA = \text{kinetic energy correction factor}$$

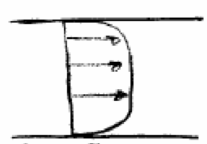
$$\bar{V} = \frac{1}{A} \int_A V dA = Q/A$$

V_1 & V_2 are average velocities

note that: $\alpha = 1$ if V is constant across the flow section
$\alpha > 1$ if V is nonuniform



laminar flow $\alpha = 2$



turbulent flow $\alpha = 1.05 \sim 1$ may be used

Application of the energy, momentum, and continuity equation in combination:

Energy:

$$\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2+h_t+h_L$$

Momentum:

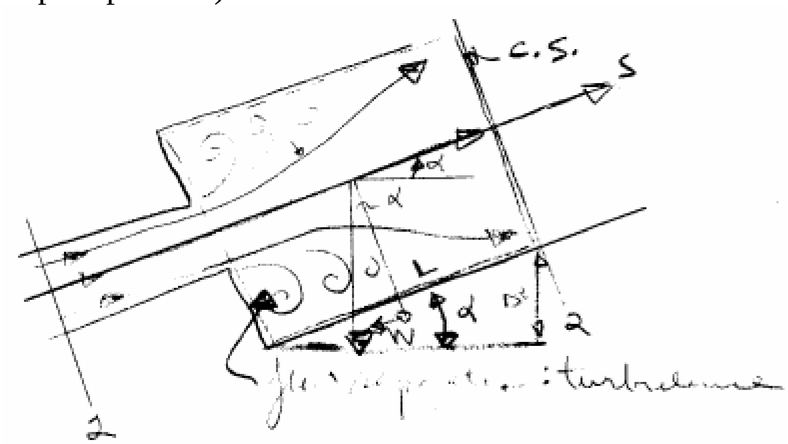
$$\sum F_s=\rho V_2^2 A_2-\rho V_1^2 A_1=\rho Q(V_2-V_1)$$

Continuity:

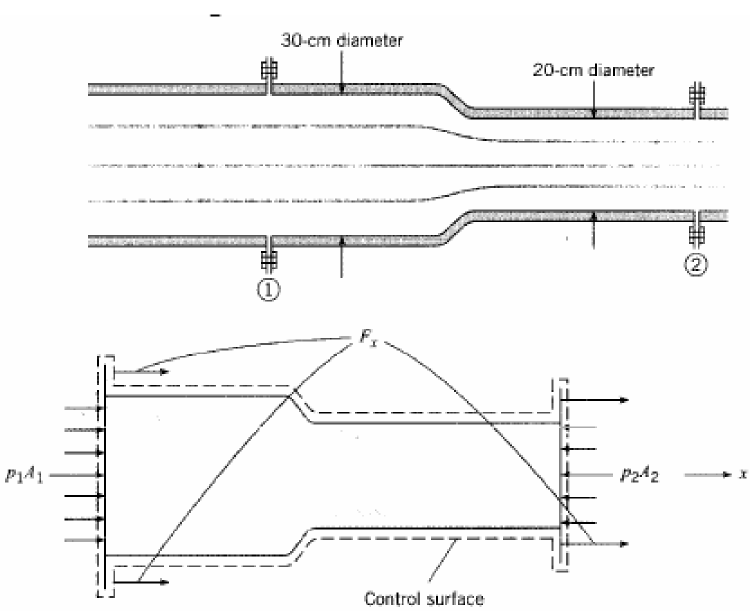
$$A_1V_1=A_2V_2=Q=\text{constant}$$

(Momentum and continuity: one inlet and one outlet, ρ = constant)

For instance, the equations above can be applied to the flow from a small pipe to a large pipe (abrupt expansion) or forces on transitions.



Example of abrupt expansion



$Q = .707\ m^3/s$

head loss $= .1\dfrac{V_2^2}{2g}$

(empirical equation)

Fluid = water

$p_1 = 250$ kPa

$D_1 = 30$ cm

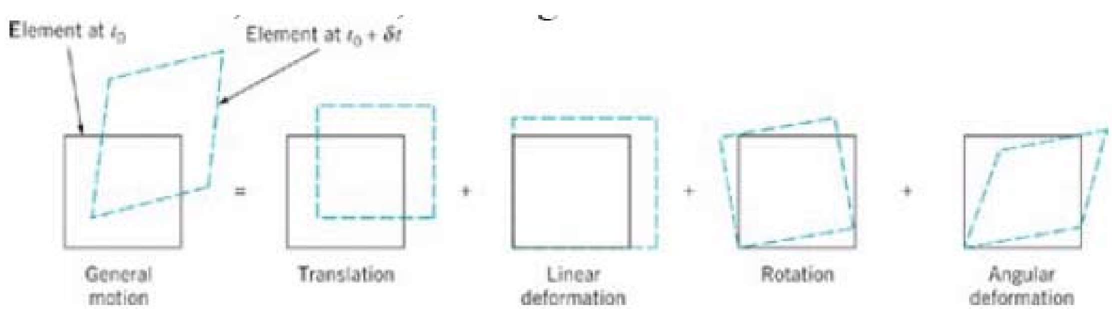
$D_2 = 20$ cm

$F_x = ?$

Example of forces on transitions

<u>Chapter 6: Differential Analysis of Fluid Flow</u>

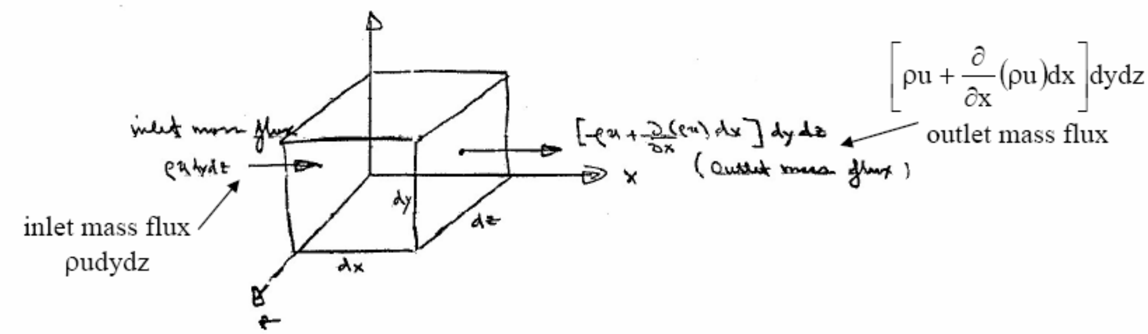
<u>Fluid element kinematics</u>



- Translation $\mathbf{V} = u\hat{i} + v\hat{j} + w\hat{k}$
- Rotation $\boldsymbol{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$
- Linear Strain $\varepsilon_{xx} = \frac{\partial u}{\partial x},\quad \varepsilon_{yy} = \frac{\partial v}{\partial y},\quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$
- Shear Strain $\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\quad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),\quad \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$
- Volumetric Strain Rate $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

The continuity equation in differential form

- In Cartesian coordinates:



$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

per unit ~~V~~
differential form of continuity equations

$$\frac{\partial \rho}{\partial t} + \underbrace{\nabla \cdot (\rho \underline{V})}_{\rho \nabla \cdot \underline{V} + \underline{V} \cdot \nabla \rho} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{V} = 0 \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{V} \cdot \nabla$$

Simplifications:

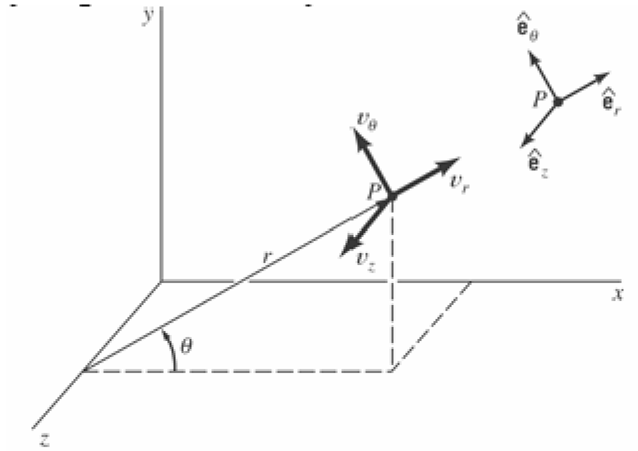
1. Steady flow: $\nabla \cdot (\rho \underline{V}) = 0$

2. ρ = constant: $\nabla \cdot \underline{V} = 0$

i.e., $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{3D}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{2D}$$

-In Cylindrical polar coordinates:



$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

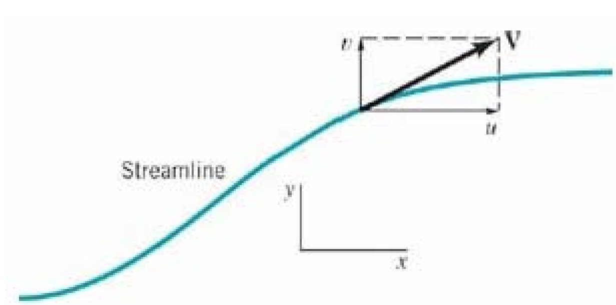
<u>The stream function ψ</u>

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

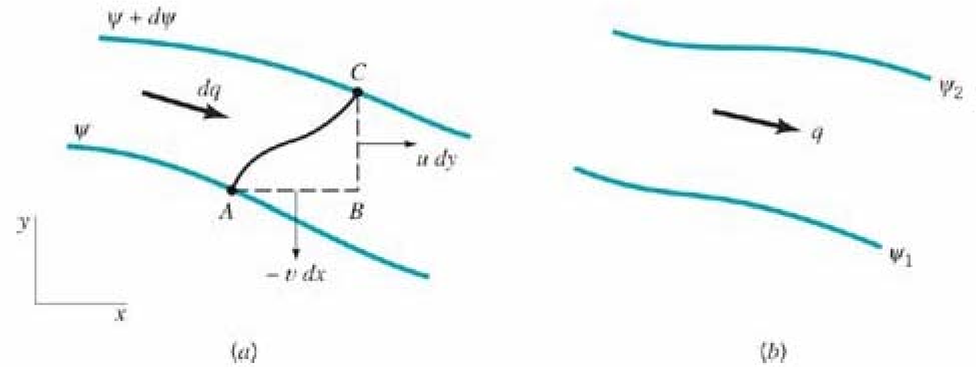
$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

<u>Important features of ψ</u>

- Curves of constant ψ are streamlines of the flow



- The difference in the value of ψ from one streamline to another is equal to the volume flow rate per unit width between the two streamlines



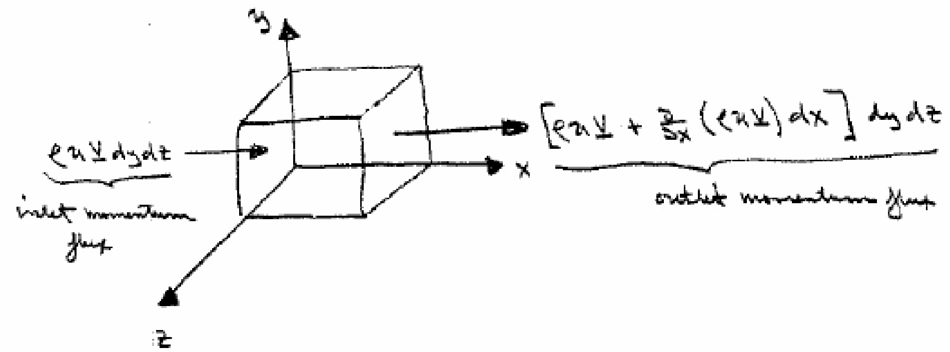
The flow between two streamlines

$$dq = udy - vdx = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = d\psi$$

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

<u>Navier-Stokes (NS) equations:</u>

NS equation is a differential form of the conservation of momentum!!



Start from 1-D flow approximation:

$$\sum \underline{F} = \underbrace{\frac{d}{dt}\int_{CV} \rho \underline{V} dV}_{\frown} + \underbrace{\int_{CS} \underline{V}\rho \underline{V}\cdot \underline{dA}}_{\frown} \qquad \text{1-D flow approximation}$$

$$= \sum (\dot{m}_i \underline{V}_i)_{out} - \sum (\dot{m}_i \underline{V}_i)_{in}$$

where $\underbrace{\dot{m} = \rho AV = \rho dydzu}_{\text{mass flux}}$ x-face

$$\sim \frac{d}{dt}(\rho \underline{V})dxdydz$$

$$= \left[\underbrace{\frac{\partial}{\partial x}(\rho u \underline{V})}_{\text{x-face}} + \underbrace{\frac{\partial}{\partial y}(\rho v \underline{V})}_{\text{y-face}} + \underbrace{\frac{\partial}{\partial z}(\rho w \underline{V})}_{\text{z-face}}\right] dxdydz$$

$$\sum \underline{F} = \rho \frac{D\underline{V}}{Dt} dxdydz$$

where $\sum \underline{F} = \sum \underline{F}_{body} + \sum \underline{F}_{surface}$

Notice that:

<u>Body force</u>→due to external fields such as gravity or magnetics

$$\sum \underline{F}_{body} = d\underline{F}_{grav} = \rho \underline{g} dxdydz$$

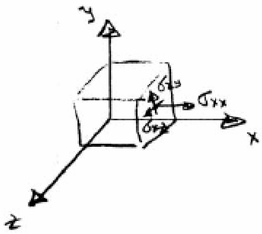
$$\text{and } \underline{g} = -g\hat{k} \quad \text{for } g\downarrow \; z\uparrow$$

$$\text{i.e., } \underline{f}_{body} = -\rho g \hat{k}$$

Surface force→due to the stresses acting on the sides of CS

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

normal pressure (→ $-p\delta_{ij}$) viscous stress (→ τ_{ij})



$$= \begin{bmatrix} -p+\tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p+\tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p+\tau_{zz} \end{bmatrix}$$

$$\underline{f}_{surf} = -\nabla p + \nabla \cdot \tau_{ij} = \nabla \cdot \sigma_{ij} \qquad \sigma_{ii} = -p\delta_{ii} + \tau_{ii}$$

$\delta_{ij} = 1$	$i = j$
$\delta_{ij} = 0$	$i \neq j$

Putting together the above results

$$\sum \underline{f} = \underline{f}_{body} + \underline{f}_{surf} = \rho \frac{D\underline{V}}{Dt}$$

$$\underline{f}_{body} = -\rho g \hat{k}$$

$$\underline{f}_{surface} = -\nabla p + \nabla \cdot \tau_{ij}$$

$$\underline{a} = \frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V}$$

The physical meanings of each term in NS equation are

$$\rho \underline{a} = -\rho g \bar{k} - \nabla p + \nabla \cdot \tau_{ij}$$

inertia force	body force due to gravity	surface force due to p	surface force due to viscous shear and normal stresses

Write viscous shear and normal stresses in the form as

$$\tau_{ij} = \mu \varepsilon_{ij} \qquad \mu = \text{coefficient of viscosity}$$

$$\varepsilon_{ij} = \text{rate of strain tensor}$$

$$= \begin{bmatrix} \dfrac{\partial u}{\partial x} & \left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & \left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) \\ \left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) \\ \left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) & \left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

After some mathematical manipulation, NS and continuity equations are obtained as:

$$\text{x:} \quad \rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

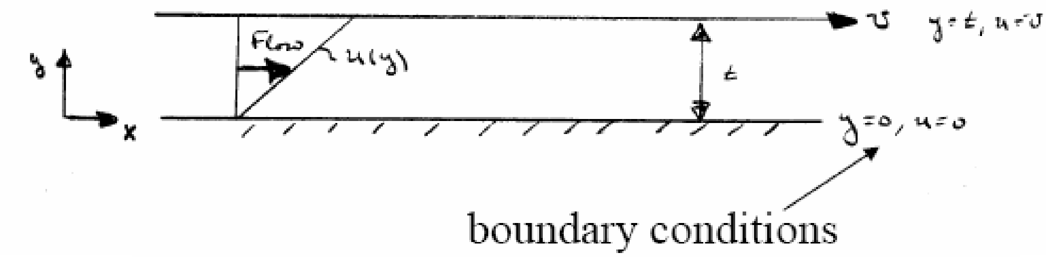
$$\text{y:} \quad \rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]$$

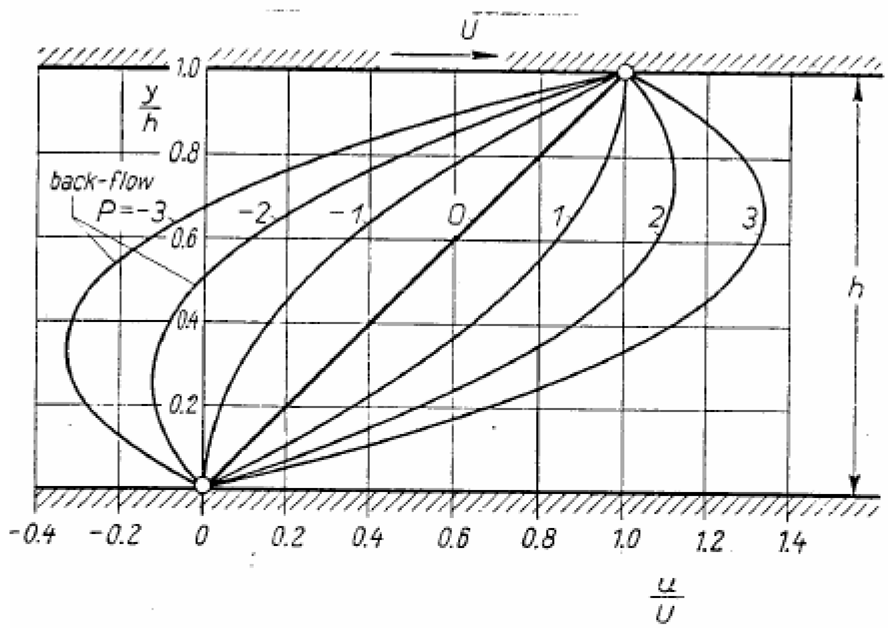
$$\text{z:} \quad \rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

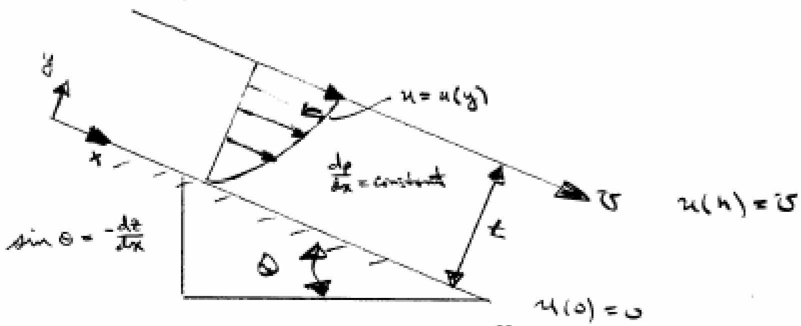
Differential analysis of fluid flow

Couette Flow



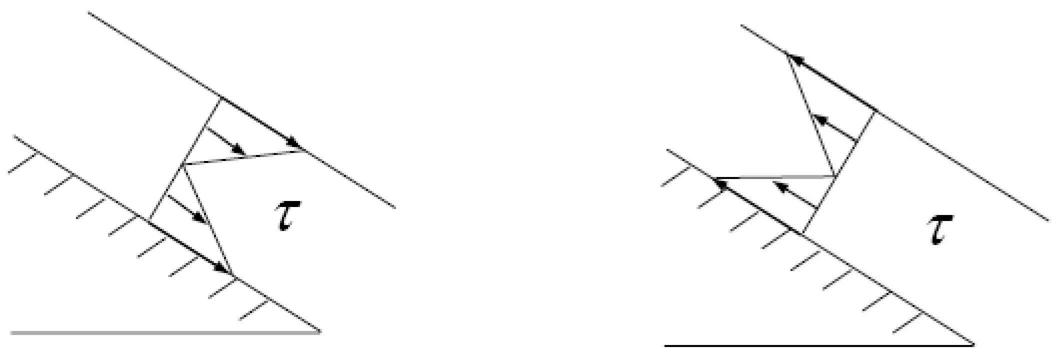


Generalization for inclined flow with a constant pressure gradient



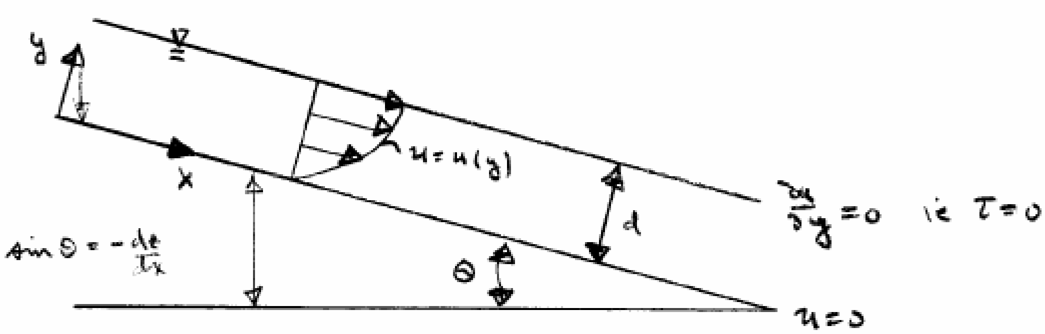
For favorable pressure gradient, $\frac{dh}{dx} < 0$, $\tau > 0$

For adverse pressure gradient, $\frac{dh}{dx} > 0$, $\tau < 0$



$\frac{dh}{dx} < 0$ $\frac{dh}{dx} > 0$

Flow down an inclined plane



Chapter 7: Dimensional Analysis and Modeling

- Buckingham Π theorem

$$F\left(A_1, \cdots A_n\right) = 0$$

$$\Rightarrow f\left(\pi_1, \cdots \pi_{\underbrace{n-\hat{m}}_{r}}\right) = 0 \quad \hat{m} = m \text{ usually}$$

significant reduction in number of variables which reduces number of experiments or calculations required

- Methods for determining Π_i's

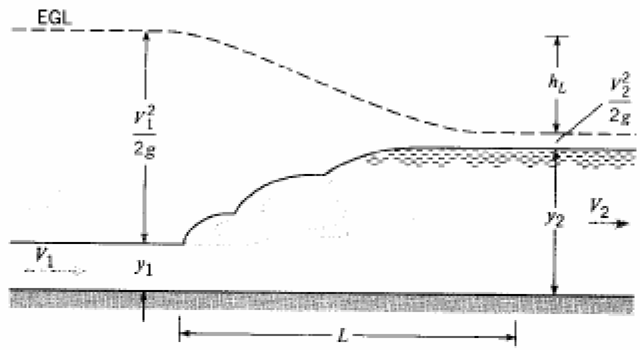
1. Functional Relationship Method

Identify functional relationships $F(A_i)$ and $f(\Pi_j)$by first determining A_i's and then evaluating Π_j's

a. Inspection	intuition
b. Step-by-step Method	text
c. Exponent Method	class

2. Nondimensionalize governing differential equations and initial and boundary conditions

Example: Hydraulic jump



we assume that

$V_1 = V_1(\rho, g, \mu, y_1, y_2)$
or $V_2 = V_1 y_1 / y_2$

Exponent method to determine Π_j's for Hydraulic jump

use V, y_1, ρ as repeating variables

$F(g, V_1, y_1, y_2, \rho, \mu) = 0$

$\frac{L}{T^2}\ \frac{L}{T}\ L\ L\ \frac{M}{L^3}\ \frac{M}{LT}$

$m = 3 \Rightarrow r = n - m = 3$

$\Pi_1 = V^{x1}\, y_1^{y1}\, \rho^{z1}\, \mu$
$= (LT^{-1})^{x1}\,(L)^{y1}\,(ML^{-3})^{z1}\, ML^{-1}T^{-1}$

L	$x_1 + y_1 - 3z_1 - 1 = 0$	$y_1 = 3z_1 + 1 - x_1 = -1$
T	$-x_1 \quad - 1 = 0$	$x_1 = -1$
M	$z_1 \quad + 1 = 0$	$z_1 = -1$

$$\Pi_1 = \frac{\mu}{\rho y_1 V} \quad \text{or} \quad \Pi_1^{-1} = \frac{\rho y_1 V}{\mu} = \text{Reynolds number} = \text{Re}$$

$\Pi_2 = V^{x2}\, {y_1}^{y2}\, \rho^{z2}\, g$

$\quad = (LT^{-1})^{x2}\, (L)^{y2}\, (ML^{-3})^{z2}\, LT^{-2}$

L $\quad x_2 + y_2 - 3z_2 + 1 = 0 \qquad y_2 = -1 - x_2 = 1$

T $\quad -x_2 \qquad\qquad -2 = 0 \qquad x_2 = -2$

M $\quad z_2 = 0$

$$\Pi_2 = V^{-2} y_1 g = \frac{g y_1}{V^2} \qquad \Pi_2^{-1/2} = \frac{V}{\sqrt{g y_1}} = \text{Froude number} = Fr$$

$\Pi_3 = (LT^{-1})^{x3}\, (L)^{y3}\, (ML^{-3})^{z3}\, y_2$

L $\quad x_3 + y_3 + 3z_3 + 1 = 0 \qquad y_3 = -1$

T $\quad -x_3 = 0$

M $\quad -3z_3 = 0$

$$\Pi_3 = \frac{y_2}{y_1} \qquad\qquad \Pi_3^{-1} = \frac{y_1}{y_2} = \text{depth ratio}$$

$f(\Pi_1, \Pi_2, \Pi_3) = 0$

or, $\quad \Pi_2 = \Pi_2(\Pi_1, \Pi_3)$

i.e., $\quad F_r = F_r(Re, y_2/y_1)$

Common Dimensionless Parameters for Fluid Flow Problems

Re, Fr, We, Ma, C_p, etc

Nondimensionalization of the Basic Equation

$$\nabla \cdot \mathbf{V} = 0$$

$$\frac{D\mathbf{V}}{Dt} = -\nabla p + Re^{-1} \nabla^2 \mathbf{V}$$

$$\mathbf{V} = \mathbf{V}/U_0$$

$$\mathbf{X} = \mathbf{X}/L$$

$$t = tU_0/L$$

$$p = (p + \rho g z)/\rho U_0^2$$

Similarity and Model testing

$$\Pi_{i\ model} = \Pi_{i\ prototype} \qquad i = 1, r = n - \hat{m}\ (m)$$

Similarity is classified as:

1. Geometric Similarity (similar length scale) : $\alpha = L_m / L_p$
2. Kinematic Similarity (similar length and time scale)
3. Dynamic Similarity (similar length, time, and force scales) (Re& Fr scaling etc)