Solution for review problems for Exam 2, 057:020-Fall 2007

Momentum Conservation:

Assumptions 1 The flow is steady, frictionless, one-dimensional, incompressible, and irrotational (so that the Bernoulli equation is applicable). 2 The weight of the elbow and the water in it is considered. 3 The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. 4 The momentum-flux correction factor for each inlet and outlet is given to be $\beta = 1.03$.

Properties We take the density of water to be 1000 kg/m^3.

Analysis The weight of the elbow and the water in it is

$$W = mg = (50\text{ kg})(9.81\text{ m/s}^2) = 490.5\text{ N} = 0.4905\text{ kN}$$

We take the elbow as the control volume, and designate the entrance by 1 and the outlet by 2. We also designate the horizontal coordinate by x (with the direction of flow as being the positive direction) and the vertical coordinate by z. The continuity equation for this one-inlet one-outlet steady flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m} = 30$ kg/s. Noting that $\dot{m} = \rho A V$, the inlet and outlet velocities of water are

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{30\text{ kg/s}}{(1000\text{ kg/m}^3)(0.0150\text{ m}^2)} = 2.0\text{ m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{30\text{ kg/s}}{(1000\text{ kg/m}^3)(0.0025\text{ m}^2)} = 12\text{ m/s}$$

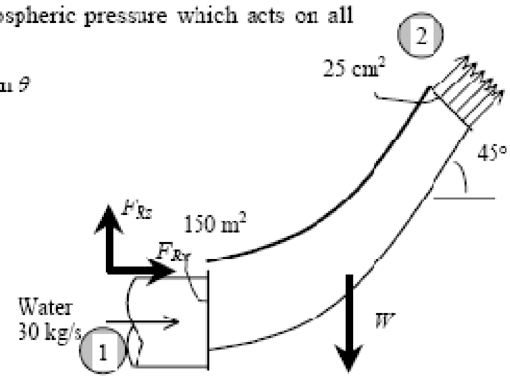
Taking the center of the inlet cross section as the reference level ($z_1 = 0$) and noting that $P_2 = P_{atm}$, the Bernoulli equation for a streamline going through the center of the reducing elbow is expressed as

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \;\rightarrow\; P_1 - P_2 = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2 - z_1\right) \;\rightarrow\; P_{1,\text{gage}} = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2\right)$$

Substituting,

$$P_{1,\text{gage}} = (1000\text{ kg/m}^3)(9.81\text{ m/s}^2)\left(\frac{(12\text{ m/s})^2 - (2\text{ m/s})^2}{2(9.81\text{ m/s}^2)} + 0.4\right)\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right) = 73.9\text{ kN/m}^2 = 73.9\text{ kPa}$$

The momentum equation for steady one-dimensional flow is $\sum \vec{F} = \sum_{out} \beta\dot{m}\vec{V} - \sum_{in} \beta\dot{m}\vec{V}$. We let the x- and z- components of the anchoring force of the elbow be F_{Rx} and F_{Rz}, and assume them to be in the positive directions. We also use gage pressures to avoid dealing with the atmospheric pressure which acts on all surfaces. Then the momentum equations along the x and z axes become



$$F_{Rx} + P_{1,\text{gage}}A_1 = \beta\dot{m}V_2\cos\theta - \beta\dot{m}V_1 \quad\text{and}\quad F_{Rz} - W = \beta\dot{m}V_2\sin\theta$$

Solving for F_{Rx} and F_{Rz}, and substituting the given values,

$$F_{Rx} = \beta\dot{m}(V_2\cos\theta - V_1) - P_{1,\text{gage}}A_1$$

$$= 1.03(30\text{ kg/s})[(12\cos 45° - 2)\text{ m/s}]\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right)$$

$$-(73.9\text{ kN/m}^2)(0.0150\text{ m}^2)$$

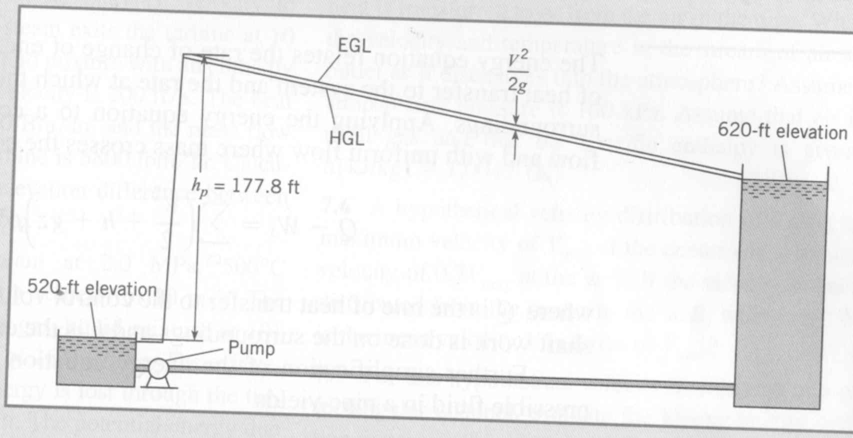
$$= -0.908\text{ kN}$$

$$F_{Rz} = \beta\dot{m}V_2\sin\theta + W = 1.03(30\text{ kg/s})(12\sin 45°\text{ m/s})\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right) + 0.4905\text{ kN} = 0.753\text{ kN}$$

$$F_R = \sqrt{F_{Rx}^2 + F_{Rz}^2} = \sqrt{(-0.908)^2 + (0.753)^2} = \mathbf{1.18\text{ kN}}, \quad \theta = \tan^{-1}\frac{F_{Rz}}{F_{Rx}} = \tan^{-1}\frac{0.753}{-0.908} = -\mathbf{39.7°}$$

Energy Equation and HGL/EGL:

A pump draws water from a reservoir, where the water-surface elevation is 520 ft, and forces the water through a pipe 5000 ft long and 1 ft in diameter. This pipe then discharges the water into a reservoir with water-surface elevation of 620 ft. The flow rate is 7.85 cfs, and the head loss in the pipe is given by $0.01(L/D)(V^2/2g)$. Determine the head supplied by the pump, h_p, and the power supplied to the flow, and draw the HGL and EGL for the system. Assume that the pipe is horizontal and is 510 ft in elevation.



Solution First solve for h_p by using the energy equation (one-dimensional flow assumed) written from the water surface in the lower reservoir to the water surface in the upper reservoir.

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$

In this example, p_1/γ, p_2/γ, V_1, and V_2 are all zero, but $z_1 = 520$ ft, and $z_2 = 620$ ft.

$$h_L = 0.01 \times \frac{5000}{1}\frac{V^2}{2g} \quad \text{and} \quad V = \frac{Q}{A_p} = 10 \text{ ft/s}$$

Then

$$h_p = 620 - 520 + 0.01 \times \frac{5000}{1}\frac{100}{64.4} \text{ ft-lbf/lbf} = 178 \text{ ft}$$

However, the product of the flow rate Q and specific weight will give us the weight rate of flow. Then the power supplied by the pump will be $h_p \times$ weight rate of flow, or

$$P = h_p Q \gamma \text{ ft-lbf/s} = \frac{h_p Q \gamma}{550} = 158 \text{ hp}$$

Differential Analysis of Fluid Flow:

Assumptions We number and list the assumptions for clarity:

1. The pipe is infinitely long in the x direction.
2. The flow is steady, i.e. any time derivative is zero.
3. This is a parallel flow (the r component of velocity, u_r, is zero).
4. The fluid is incompressible and Newtonian, and the flow is laminar.
5. The pressure is constant everywhere except for hydrostatic pressure.
6. The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and all derivatives with respect to θ are zero.

Analysis To obtain the velocity and pressure fields, we follow the step-by-step procedure outlined above.

Step 1 Lay out the problem and the geometry. See Fig. P9-100.

Step 2 List assumptions and boundary conditions. We have already listed six assumptions. The first boundary condition comes from imposing the no slip condition at the pipe wall: (1) at $r = R$, $\vec{V} = 0$. The second boundary condition comes from the fact that the centerline of the pipe is an axis of symmetry: (2) at $r = 0$, $du/dr = 0$.

Step 3 Write out and simplify the differential equations. We start with the continuity equation in cylindrical coordinates, a modified version of Eq. 9-62a,

Continuity:

$$\underbrace{\frac{1}{r}\frac{\partial (r u_r)}{\partial r}}_{\text{Assumption 3}} + \underbrace{\frac{1}{r}\frac{\partial (u_\theta)}{\partial \theta}}_{\text{Assumption 6}} + \frac{\partial u}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x} = 0 \qquad (1)$$

Equation 1 tells us that u is not a function of x. In other words, it doesn't matter where we place our origin – the flow is the same at any x location. This can also be inferred directly from Assumption 1, which tells us that there is nothing special about any x location since the pipe is infinite in length – the flow is fully developed. Furthermore, since u is not a function of time (Assumption 2) or θ (Assumption 6), we conclude that u is at most a function of r,

Result of continuity:

$$u = u(r) \text{ only} \qquad (2)$$

We now simplify the x momentum equation as far as possible:

x momentum:

$$\rho\left(\underbrace{\frac{\partial u}{\partial t}}_{\text{Assumption 2}} + \underbrace{u_r\frac{\partial u}{\partial r}}_{\text{Assumption 3}} + \underbrace{\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}}_{\text{Assumption 6}} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{Continuity}}\right)$$

$$= -\underbrace{\frac{\partial P}{\partial x}}_{\text{Assumption 5}} + \underbrace{\rho g_x}_{\rho g \sin\alpha} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}_{\text{Assumption 6}} + \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{Continuity}}\right)$$

or

Result of x momentum:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{-\rho g \sin\alpha}{\mu} \qquad (3)$$

As in previous examples the material acceleration (entire left hand side of the x momentum equation) is zero, implying that fluid particles are not accelerating at all in this flow field, and linearizing the Navier-Stokes equation. Also notice that we have replaced the partial derivative operators for the u derivatives with total derivative operators because of Eq. 2.

You can show in similar fashion that every term in the r momentum equation and in the θ momentum equation goes to zero.

Step 4 Solve the differential equations. Continuity, r momentum, and θ momentum have already been solved, and thus we are left with Eq. 3 (x momentum). After multiplying both sides by r, integrating, dividing by r, and integrating again,

Axial velocity component:

$$u = \frac{-\rho g \sin\alpha}{4\mu} r^2 + C_1 \ln r + C_2 \qquad (4)$$

where C_1 and C_2 are constants of integration.

Step 5 Apply boundary conditions from Step 2 above to obtain constants C_1 and C_2. We apply boundary condition (2) first:

Boundary condition (2):

$$\frac{du}{dr} = 0 + \frac{C_1}{0} = 0$$

Since $C_1/0$ is undefined (∞), the only way for du/dr to equal zero at $r = 0$ is for C_1 to equal 0. An alternative way to think of this boundary condition is to say that u must remain finite at the centerline of the pipe. Again this is possible only if constant C_1 is equal to 0.

$$C_1 = 0$$

Now we apply the first boundary condition,

Boundary condition (1):

$$u = \frac{-\rho g \sin\alpha}{4\mu} R^2 + 0 + C_2 = 0 \qquad \text{or} \qquad C_2 = \frac{\rho g \sin\alpha}{4\mu} R^2$$

Finally, Eq. 4 becomes

Final result for axial velocity:

$$u = \frac{\rho g \sin\alpha}{4\mu}\left(R^2 - r^2\right) \qquad (5)$$

The axial velocity profile is thus in the shape of a paraboloid, just as in Example 9-18.

Step 6 Verify the results. You can plug in the velocity field to verify that all the differential equations and boundary conditions are satisfied.

The volume flow rate through the pipe is found by integrating Eq. 5 through the whole cross-sectional area of the pipe,

Volume flow rate:

$$\dot{V} = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} u\,dr = \frac{2\pi\rho g \sin\alpha}{4\mu}\int_{r=0}^{R}\left(R^2 - r^2\right) r\,dr = \frac{\pi R^4}{8\mu}\rho g \sin\alpha \qquad (6)$$

Since volume flow rate is also equal to the average axial velocity times cross-sectional area, we can easily determine the average axial velocity, V:

Average axial velocity:

$$V = \frac{\dot{V}}{A} = \frac{\dfrac{\pi R^4}{8\mu}\rho g \sin\alpha}{\pi R^2} = \frac{R^2}{8\mu}\rho g \sin\alpha \qquad (7)$$

Dimensional Analysis 1:

EXAMPLE 5.4

At low velocities (laminar flow), the volume flow Q through a small-bore tube is a function only of the tube radius R, the fluid viscosity μ, and the pressure drop per unit tube length dp/dx. Using the pi theorem, find an appropriate dimensionless relationship.

Solution

Write the given relation and count variables:

$$Q = f\left(R, \mu, \frac{dp}{dx}\right) \quad \text{four variables } (n = 4)$$

Make a list of the dimensions of these variables from Table 5.1 using the $\{MLT\}$ system:

Q	R	μ	dp/dx
$\{L^3T^{-1}\}$	$\{L\}$	$\{ML^{-1}T^{-1}\}$	$\{ML^{-2}T^{-2}\}$

There are three primary dimensions (M, L, T), hence $j \leq 3$. By trial and error we determine that R, μ, and dp/dx cannot be combined into a pi group. Then $j = 3$, and $n - j = 4 - 3 = 1$. There is only *one* pi group, which we find by combining Q in a power product with the other three:

$$\Pi_1 = R^a\mu^b\left(\frac{dp}{dx}\right)^c Q^1 = (L)^a(ML^{-1}T^{-1})^b(ML^{-2}T^{-2})^c(L^3T^{-1})$$
$$= M^0L^0T^0$$

Equate exponents:

Mass: $\quad b + c = 0$

Length: $\quad a - b - 2c + 3 = 0$

Time: $\quad -b - 2c - 1 = 0$

Solving simultaneously, we obtain $a = -4$, $b = 1$, and $c = -1$. Then

$$\Pi_1 = R^{-4}\mu^1\left(\frac{dp}{dx}\right)^{-1} Q$$

or

$$\Pi_1 = \frac{Q\mu}{R^4(dp/dx)} = \text{const} \qquad \textit{Ans.}$$

Since there is only one pi group, it must equal a dimensionless constant. This is as far as dimensional analysis can take us. The laminar flow theory of Sec. 4.11 shows that the value of the constant is $-\frac{\pi}{8}$.

Dimensional Analysis 2:

7-98

Solution We are to determine the relationship between four established nondimensional parameters, and then try to form the Stanton number by some combination of only *two* other established dimensionless parameters.

Analysis We manipulate Re, Nu, and Pr, guided by the known result. After some trial and error,

Stanton number:

$$\mathrm{St} = \frac{\mathrm{Nu}}{\mathrm{Re} \times \mathrm{Pr}} = \frac{\frac{Lh}{k}}{\frac{\rho VL}{\mu} \times \frac{\mu c_P}{k}} = \frac{h}{\rho c_p V} \qquad (1)$$

We recognize from Table 7-5 (or from Problem 7-97) that *Peclet number* is equal to the product of Reynolds number and Prandtl number. Thus,

Stanton number:

$$\mathrm{St} = \frac{\mathrm{Nu}}{\mathrm{Pe}} = \frac{\frac{Lh}{k}}{\frac{\rho LVc_P}{k}} = \frac{h}{\rho c_p V} \qquad (2)$$