Chapter 5 Mass, Momentum, and Energy Equations

1. Reynolds Transport Theorem (RTT)

$$\frac{dB_{sys}}{dt} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall}_{\substack{\text{time rate of change} \\ \text{of } B \text{ in } CV}} + \underbrace{\int_{CS} \beta\rho \underline{V}_R \cdot d\underline{A}}_{\substack{\text{net outflux of } B \\ \text{from } CV \text{ across } CS}}$$

where, $B = m\beta$, $\underline{V}_R = \underline{V} - \underline{V}_S$, $\underline{V}$ = fluid velocity, $\underline{V}_S = CS$ velocity, and

$d\underline{A} = \hat{\boldsymbol{n}}dA$ where $\hat{\boldsymbol{n}}$ is outward normal vector, $\underline{V} \cdot d\underline{A} = \underline{V} \cdot \hat{\boldsymbol{n}}dA$ (- inlet, + outlet)

For a fixed control volume, $\underline{V}_R = \underline{V}$ ($\underline{V}_S = 0$):

Parameter	B	β	RTT Equation
Mass	m	1	$\frac{dm}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \underline{V} \cdot d\underline{A}$
Momentum	$m\underline{V}$	$\underline{V}$	$\frac{d(m\underline{V})}{dt} = \frac{\partial}{\partial t}\int_{CV} \underline{V}\rho d\forall + \int_{CS} \underline{V}\rho \underline{V} \cdot d\underline{A}$
Energy	E	e	$\frac{dE}{dt} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho \underline{V} \cdot d\underline{A}$

2. Conservation of Mass – The Continuity Equation

$$\frac{dm}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \underline{V} \cdot d\underline{A} = 0$$

Special cases:

1) Steady flow: $\int_{CS} \rho \underline{V} \cdot d\underline{A} = 0$
2) Incompressible fluid (ρ =constant): $\int_{CS} \underline{V} \cdot d\underline{A} = -\frac{\partial}{\partial t}\int_{CV} d\forall$
3) $\underline{V}$ = constant over discrete $d\underline{A}$: $\int_{CS} \rho \underline{V} \cdot d\underline{A} = \sum_{CS} \rho \underline{V} \cdot \underline{A}$
4) Steady one-dimensional flow in a conduit: $\sum_{CS} \rho \underline{V} \cdot \underline{A} = 0 \Rightarrow$

$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \quad \Rightarrow$ if ρ = constant, $V_1 A_1 = V_2 A_2$ or $Q_1 = Q_2$

Some useful definitions:

- Mass flux (mass flow rate) $\quad \dot{m} = \int_A \rho \underline{V} \cdot d\underline{A} \quad$ (if ρ = constant , $\dot{m} = \rho Q$)
- Volume flux (flow rate) $\quad Q = \int_A \underline{V} \cdot d\underline{A} \quad$ (if $\underline{V}$ = constant, $Q = \underline{V} \cdot \underline{A}$)
- Average velocity $\quad \bar{V} = Q/A$

3. Newton's Second Law - Momentum Equation

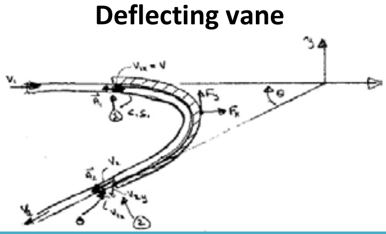
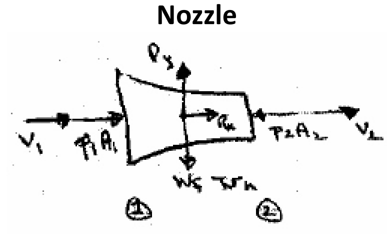
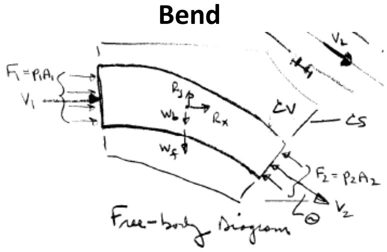
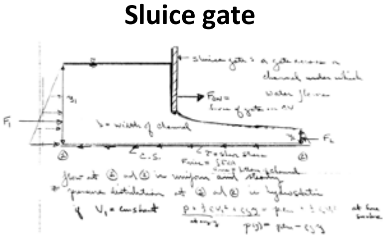
$$\underbrace{\frac{d(m\underline{V})}{dt} = \frac{\partial}{\partial t}\int_{CV} \underline{V}\rho d\forall + \int_{CS} \underline{V}\rho\underline{V}\cdot d\underline{A}}_{=m\underline{a}} = \sum \underline{F}$$

where $\sum \underline{F} = \sum \underline{F_B} + \sum \underline{F_S}$ = vector sum of all external forces acting on CV including body forces $\sum \underline{F_B}$ (ex: gravity force) and surface forces $\sum \underline{F_S}$ (ex: pressure force, and shear forces, etc.)

Special cases:

1) Steady flow: $\frac{\partial}{\partial t}\int_{CV} \underline{V}\rho d\forall = 0$
2) Uniform flow across $\underline{A}$: $\int_{CS} \underline{V}\rho\underline{V}\cdot d\underline{A} = \sum \underline{V}\rho\underline{V}\cdot d\underline{A}$

Examples:

Flow type	$\sum \underline{F}$	$\sum \underline{V}\rho\underline{V}\cdot d\underline{A}$	Continuity Eq. or Bernoulli Eq.
Deflecting vane 	$\sum F_x = F_x$ $\sum F_y = F_y$	x-component: $\rho V_1(-V_1A_1)$ $+\rho(-V_2\cos\theta)(V_2A_2)$ y-component: $\rho(-V_2\sin\theta)(V_2A_2)$	$V_1A_1 = V_2A_2 = Q$
Nozzle 	$\sum F_x = R_x + p_1A_1$ $-p_2A_2$ $\sum F_y = R_y - W_{\text{Fluiud}}$ $-W_{\text{Nozzle}}$	x-component: $\rho V_1(-V_1A_1)$ $+\rho V_2(V_2A_2)$ y-component: 0	$A_1V_1 = A_2V_2 = Q$ $p_1 + \frac{\rho V_1^2}{2} = \frac{\rho V_2^2}{2}$ $(\because z_1 = z_1, p_2 = 0)$
Bend 	$\sum F_x = R_x + p_1A_1$ $-p_2A_2\cos\theta$ $\sum F_y$ $= R_y + p_2A_2\sin\theta$ $-W_{\text{Fluiud}} - W_{\text{Nozzle}}$	x-component: $\rho V_1(-V_1A_1)$ $+\rho(V_2\cos\theta)(V_2A_2)$ y-component: $\rho(-V_2\sin\theta)(V_2A_2)$	$A_1V_1 = A_2V_2 = Q$
Sluice gate 	$\sum F_x = F_{GW}$ $+\gamma\frac{y_1}{2}(y_1b)$ $-\gamma\frac{y_2}{2}(y_2b)$ $\sum F_y = 0$	x-component: $\rho V_1(-V_1A_1)$ $+\rho V_2(V_2A_2)$ y-component: 0	$V_1(y_1b) = V_2(y_2b)$ $= Q$ $\frac{V_1^2}{2g} + y_1$ $= \frac{V_2^2}{2g} + y_2 + h_L$ $(\because p_1 = p_2 = 0)$

4. First Law of Thermodynamics - Energy Equation

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho \underline{V}\cdot d\underline{A} = \dot{Q} - \dot{W}$$

where, $e = \breve{u} + e_k + e_p = \breve{u} + \frac{V^2}{2} + gz$ and $\dot{W} = \dot{W}_s + \left(\dot{W}_{\text{fp}} + \cancel{\dot{W}_{\text{fs}}}\right) = \dot{W}_s + \dot{W}_{\text{fp}} = \left(\dot{W}_{\text{t}} - \dot{W}_{\text{p}}\right) + \dot{W}_{\text{fp}}$

or

$$\dot{Q} - \dot{W}_s = \frac{\partial}{\partial t}\int_{CV} \rho\left(\frac{V^2}{2} + gz + \breve{u}\right) d\forall + \int_{CS} \rho\left(\frac{V^2}{2} + gz + \breve{u} + \frac{p}{\rho}\right)\underline{V}\cdot d\underline{A}$$

Simplified Form of the Energy Equation (steady, one-dimensional pipe flow):

$$\frac{p_{out}}{\rho} + \frac{V_{out}^2}{2} + gz_{out} = \frac{p_{in}}{\rho} + \frac{V_{in}^2}{2} + gz_{in} + w_s - \text{loss}$$

where $w_s = \dot{W}_s/\dot{m}$, loss $= \breve{u}_{out} - \breve{u}_{in} - q$, and $q = Q/\dot{m}$.

For non-uniform flows,

$$\underbrace{\frac{p_{in}}{\gamma} + \alpha_{in}\frac{V_{in}^2}{2g} + z_{in} + h_p = \frac{p_{out}}{\gamma} + \alpha_{out}\frac{V_{out}^2}{2g} + z_{out} + h_t}_{\text{Mechanical energy}} + \underbrace{h_L}_{\substack{\text{Thermal}\\ \text{energy}}}$$

- pump head $h_p = \dot{W}_p/\dot{m}g = \dot{W}_p/\rho Q g = \dot{W}_p/\gamma Q$
- turbine head $h_t = \dot{W}_t/\dot{m}g$
- head loss $h_L = (\hat{u}_2 - \hat{u}_1)/g - \dot{Q}/\dot{m}g > 0$
- α : kinetic energy correction factor ($\alpha = 1$ for uniform flow across CS)
- V in energy equation refers to average velocity $\bar{V}$

Hydraulic and Energy Grade Lines

- Hydraulic Grade Line: $HGL = \frac{p}{\gamma} + z$
- Energy Grade Line: $EGL = \frac{p}{\gamma} + z + \alpha\frac{V^2}{2g}$

$$EGL_{in} + h_p = EGL_{out} + h_t + h_L$$

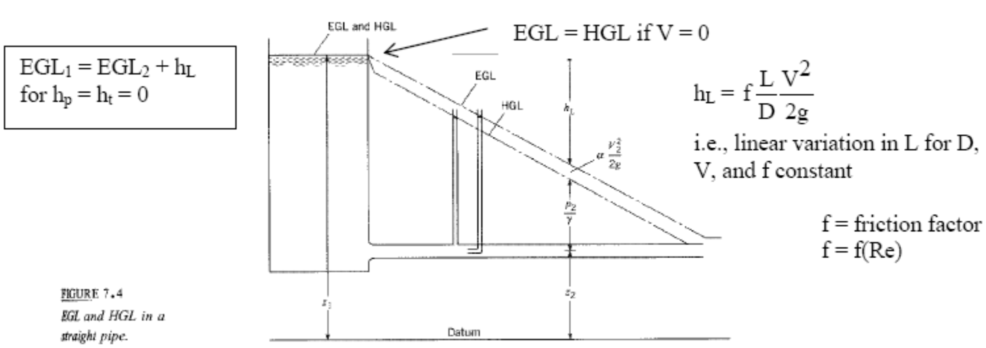
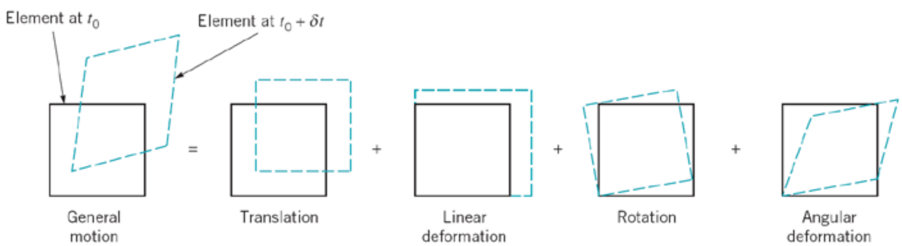


FIGURE 7.4 EGL and HGL in a straight pipe.

Chapter 6 Differential Analysis of Fluid Flow

1. Fluid Element Kinematics

Fluid element motion consists of translation, linear deformation, rotation, and angular deformation.



- Linear deformation(dilatation): $\nabla \cdot \underline{V} \Rightarrow$ if the fluid is incompressible, $\nabla \cdot \underline{V} = 0$
- Rotation(vorticity): $\underline{\xi} = 2\underline{\omega} = \nabla \times \underline{V} \Rightarrow$ if the fluid is irrotational, $\nabla \times \underline{V} = 0$
- Angular deformation is related to shearing stress: $\tau_{ij} = 2\mu\varepsilon_{ij}$

2. Mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

For a steady and incompressible flow: $\nabla \cdot \underline{V} = 0$

3. Momentum conservation

$$\underbrace{\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V}\right)}_{\underline{a}} = \underbrace{-\rho g \hat{k}}_{\text{body force due to gravity force}} \underbrace{\underbrace{-\nabla p}_{\text{pressure force}} + \underbrace{\nabla \cdot \tau_{ij}}_{\text{viscous shear force}}}_{\text{surface force}}$$

For Newtonian incompressible fluid the shear stress is propotional to the rate of strain, $\nabla \cdot \tau_{ij} = \mu \nabla^2 \underline{V}$.

4. Navier-Stokes Equations

1) Cartesian coordinates

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

2) Cylindrical coordinates:

Continuity:

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}=0$$

Momentum:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

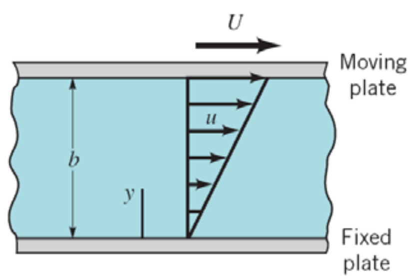
$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{\partial p}{\partial\theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

4. Exact solutions of NS Equations

Ex 1) Couette Flow (without pressure gradient)

Assumptions: laminar, steady, 2-D, incompressible, ignore gravity, no pressure gradient



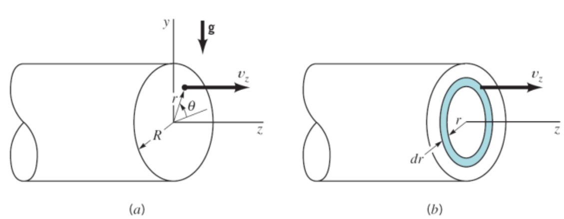
- Continuity: $\frac{\partial u}{\partial x}=0$
- Momentum: $0=\frac{\mu\,\partial^2 u}{\partial y^2}$
- B.C.: $u(h)=U, u(0)=0$

$\Rightarrow u(y)=\frac{U}{b}y$

Shear stress at the bottom wall: $\tau_w=\mu\left.\frac{du}{dy}\right|_{y=0}=\frac{\mu U}{b}$

Ex 2) Circular pipe (with constant pressure gradient)

Assumptions: laminar, steady, incompressible, fully-developed, constant pressure gradient



- Continuity: $\frac{1}{r}\frac{\partial(rv_r)}{\partial r}=0$
- z-Momentum: $0=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$
- B.C.: $v_r(r=0)=0, v_z(r=0)\neq\infty$, $v_z(r=R)=0$

$\Rightarrow v_z(r)=\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r^2-R^2)$

1) Flow rate: $Q=\int_0^R v_z dA=-\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right)=\frac{\pi R^4\Delta p}{8\mu\ell}$ $\quad\because\ -\frac{\partial p}{\partial z}=\frac{\Delta p}{\ell}$
2) Mean velocity: $\bar{V}=\frac{Q}{A}=\frac{R^2\Delta p}{8\mu\ell}$
3) Maximum velocity: $V_{max}=v_z(0)=\frac{R^2\Delta p}{4\mu\ell}=2\bar{V}$

Chapter 7 Dimensional Analysis and Modeling

1. Buckingham Pi Theorem

For any physically meaningful equation involving k variables, such as

$$u_1 = f(u_2, u_3, \cdots, u_k)$$

with minimum number of reference dimensions r, the equation can be rearranged into product of $k - r$ pi terms.

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \cdots, \Pi_{k-r})$$

Example – Exponent method:

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

where, $\Delta p_{ell} \doteq FL^{-3}$; $D \doteq L$; $\rho \doteq FL^{-4}T^2$; $\mu \doteq FL^{-2}T$; $V \doteq LT^{-1}$. Then, the number of pi terms = $k - r = 5 - 3 = 2$.

$$\Pi_1 = \Delta p_\ell D^a V^b \rho^c$$

It follows that

$$(FL^{-3})(L)^a(LT^{-1})^b(FL^{-4}T^2)^c = F^0L^0T^0$$

$$\begin{aligned} 1 + c &= 0 && \text{(for } F\text{)} \\ -3 + a + b - 4c &= 0 && \text{(for } L\text{)} \\ -b + 2c &= 0 && \text{(for } T\text{)} \end{aligned}$$

so that $a = 1$, $b = -2$, $c = -1$, and therefore

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2}$$

$$\Pi_2 = \mu D^a V^b \rho^c$$

It follows that

$$(FL^{-2}T)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c = F^0L^0T^0$$

Similarly for Π_1,

$$\Pi_2 = \frac{\mu}{DV\rho}$$

Then,

$$\frac{\Delta p_\ell D}{\rho V^2} = f\left(\frac{\mu}{DV\rho}\right)$$

2. Common Dimensionless Parameters for Fluid Flow Problems.

Variable	velocity	density	gravity	viscosity	Surface tension	compres sibility	Pressure change	Length
Symbol	V	ρ	g	μ	σ	K	Δp	L
Unit (SI)	$\mathrm{m/s}$	$\mathrm{kg/m^3}$	$\mathrm{m/s^2}$	$\mathrm{N\cdot s/m^2}$	$\mathrm{N/m}$	$\mathrm{N/m^2}$	$\mathrm{N/m^2}$	m
MLT	LT^{-1}	ML^{-3}	LT^{-2}	$ML^{-1}T^{-1}$	MT^{-2}	$ML^{-1}T^{-2}$	$ML^{-1}T^{-2}$	L
FLT	LT^{-1}	FT^2L^{-4}	LT^{-2}	FTL^{-2}	FL^{-1}	FL^{-2}	FL^{-2}	L

Dimensionless Groups	Symbol	Definition	Interpretation
Reynolds number	Re	$\dfrac{\rho VL}{\mu}$	$\dfrac{\text{inertia force}}{\text{viscous force}} = \dfrac{\rho V^2/L}{\mu V/L^2}$
Froude number	Fr	$\dfrac{V}{\sqrt{gL}}$	$\dfrac{\text{inertia force}}{\text{gravity force}} = \dfrac{\rho V^2/L}{\gamma}$
Weber number	We	$\dfrac{\rho V^2 L}{\sigma}$	$\dfrac{\text{inertia force}}{\text{surface tension force}} = \dfrac{\rho V^2/L}{\sigma/L^2}$
Mach number	Ma	$\dfrac{V}{\sqrt{K/\rho}} = \dfrac{V}{a}$	$\sqrt{\dfrac{\text{indertia force}}{\text{compressibility force}}}$
Euler number	$\mathrm{C_p}$	$\dfrac{\Delta p}{\rho V^2}$	$\dfrac{\text{pressure force}}{\text{inertia force}} = \dfrac{\Delta p/L}{\rho V^2/L}$

3. Similarity and Model Testing

If all relevant dimensionless parameters have the same corresponding values for model and prototype, flow conditions for a model test are completely similar to those for prototype.

$$\Pi_{\text{model}} = \Pi_{\text{prototype}}$$

Model Testing

1) Fr similarity $\mathrm{Fr}_m = \mathrm{Fr}_p$

$$\frac{V_m}{\sqrt{gL_m}} = \frac{V_p}{\sqrt{gL_p}} \Rightarrow V_m = \sqrt{\alpha}V_p \text{ Froude scaling, where } \alpha = L_m/L_p$$

2) Re similarity $\mathrm{Re}_m = \mathrm{Re}_p$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \Rightarrow \frac{\nu_m}{\nu_p} = \frac{V_m L_m}{V_p L_p} = \alpha^{\frac{3}{2}}$$