

Chapter 5 Mass, Momentum, and Energy Equations

1. Reynolds Transport Theorem (RTT)

$$\frac{dB}{dt} = \underbrace{\frac{\partial}{\partial t} \int_{CV} \beta \rho dV}_{\text{time rate of change of } B_{\text{sys}} \text{ in } CV} + \underbrace{\int_{CS} \beta \rho \underline{V}_R \cdot d\underline{A}}_{\text{net outflux of } B \text{ from } CV \text{ across } CS}$$

where, $B = m\beta$, $\underline{V}_R = \underline{V} - \underline{V}_S$, \underline{V} = fluid velocity, \underline{V}_S = CS velocity, and

$d\underline{A} = \hat{n}dA$ where \hat{n} is outward normal vector, $\underline{V} \cdot d\underline{A} = \underline{V} \cdot \hat{n}dA$ (- inlet, + outlet)

For a fixed control volume, $\underline{V}_R = \underline{V}$ ($\underline{V}_S = \mathbf{0}$):

Parameter	B	β	RTT Equation
Mass	m	1	$\frac{dm}{dt} = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \underline{V} \cdot d\underline{A}$
Momentum	$m\underline{V}$	\underline{V}	$\frac{d(m\underline{V})}{dt} = \frac{\partial}{\partial t} \int_{CV} \underline{V} \rho dV + \int_{CS} \underline{V} \rho \underline{V} \cdot d\underline{A}$
Energy	E	e	$\frac{dE}{dt} = \frac{\partial}{\partial t} \int_{CV} e \rho dV + \int_{CS} e \rho \underline{V} \cdot d\underline{A}$

2. Conservation of Mass - The Continuity Equation

$$\frac{dm}{dt} = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \underline{V} \cdot d\underline{A} = 0$$

Special cases:

- 1) Steady flow: $\int_{CS} \rho \underline{V} \cdot d\underline{A} = 0$
- 2) Incompressible fluid ($\rho = \text{constant}$): $\int_{CS} \underline{V} \cdot d\underline{A} = -\frac{\partial}{\partial t} \int_{CV} dV$
- 3) $\underline{V} = \text{constant}$ over discrete $d\underline{A}$: $\int_{CS} \rho \underline{V} \cdot d\underline{A} = \sum_{CS} \rho \underline{V} \cdot \underline{A}$
- 4) Steady one-dimensional flow in a conduit: $\sum_{CS} \rho \underline{V} \cdot \underline{A} = 0 \Rightarrow$

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \quad \Rightarrow \text{if } \rho = \text{constant}, V_1 A_1 = V_2 A_2 \text{ or } Q_1 = Q_2$$

Some useful definitions:

- Mass flux (mass flow rate) $\dot{m} = \int_A \rho \underline{V} \cdot d\underline{A}$ (if $\rho = \text{constant}$, $\dot{m} = \rho Q$)
- Volume flux (flow rate) $Q = \int_A \underline{V} \cdot d\underline{A}$ (if $\underline{V} = \text{constant}$, $Q = \underline{V} \cdot \underline{A}$)
- Average velocity $\bar{V} = Q/A$

3. Newton's Second Law - Momentum Equation

$$\underbrace{\frac{d(m\underline{V})}{dt} = \frac{\partial}{\partial t} \int_{CV} \underline{V} \rho d\underline{V} + \int_{CS} \underline{V} \rho \underline{V} \cdot d\underline{A}}_{= m\underline{a}} = \underline{\Sigma F}$$

where $\underline{\Sigma F} = \underline{\Sigma F_B} + \underline{\Sigma F_S}$ = vector sum of all external forces acting on CV including body forces $\underline{\Sigma F_B}$ (ex: gravity force) and surface forces $\underline{\Sigma F_S}$ (ex: pressure force, and shear forces, etc.)

Special cases:

- 1) Steady flow: $\frac{\partial}{\partial t} \int_{CV} \underline{V} \rho d\underline{V} = 0$
- 2) Uniform flow across \underline{A} : $\int_{CS} \underline{V} \rho \underline{V} \cdot d\underline{A} = \underline{\Sigma V \rho V} \cdot d\underline{A}$

Examples:

Flow type	$\underline{\Sigma F}$	$\underline{\Sigma V \rho V} \cdot d\underline{A}$	Continuity Eq. or Bernoulli Eq.
Deflecting vane 	$\Sigma F_x = F_x$ $\Sigma F_y = F_y$	x-component: $\rho V_1(-V_1 A_1)$ $+ \rho(-V_2 \cos \theta)(V_2 A_2)$ y-component: $\rho(-V_2 \sin \theta)(V_2 A_2)$	$V_1 A_1 = V_2 A_2 = Q$
Nozzle 	$\Sigma F_x = R_x + p_1 A_1 - p_2 A_2$ $\Sigma F_y = R_y - W_{\text{fluid}} - W_{\text{nozzle}}$	x-component: $\rho V_1(-V_1 A_1)$ $+ \rho V_2(V_2 A_2)$ y-component: 0	$A_1 V_1 = A_2 V_2 = Q$ $p_1 + \frac{\rho V_1^2}{2} = \frac{\rho V_2^2}{2}$ ($\because z_1 = z_2, p_2 = 0$)
Bend 	$\Sigma F_x = R_x + p_1 A_1 - p_2 A_2 \cos \theta$ $\Sigma F_y = R_y + p_2 A_2 \sin \theta - W_{\text{fluid}} - W_{\text{bend}}$	x-component: $\rho V_1(-V_1 A_1)$ $+ \rho(V_2 \cos \theta)(V_2 A_2)$ y-component: $\rho(-V_2 \sin \theta)(V_2 A_2)$	$A_1 V_1 = A_2 V_2 = Q$
Sluice gate 	$\Sigma F_x = F_{GW} + \gamma \frac{y_1}{2} (y_1 b) - \gamma \frac{y_2}{2} (y_2 b)$ $\Sigma F_y = 0$	x-component: $\rho V_1(-V_1 A_1)$ $+ \rho V_2(V_2 A_2)$ y-component: 0	$V_1(y_1 b) = V_2(y_2 b) = Q$ $\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + h_L$ ($\because p_1 = p_2 = 0$)

4. First Law of Thermodynamics - Energy Equation

$$\frac{dE}{dt} = \frac{\partial}{\partial t} \int_{CV} e \rho dV + \int_{CS} e \rho \underline{V} \cdot d\underline{A} = \dot{Q} - \dot{W}$$

where, $e = \check{u} + e_k + e_p = \check{u} + \frac{V^2}{2} + gz$ and $\dot{W} = \dot{W}_{shaft} + \dot{W}_{flow} \approx \dot{W}_{shaft} = \dot{W}_{turbine} - \dot{W}_{pump}$

or

$$\dot{Q} - \dot{W}_s = \frac{\partial}{\partial t} \int_{CV} \rho \left(\frac{V^2}{2} + gz + \check{u} \right) dV + \int_{CS} \rho \left(\frac{V^2}{2} + gz + \check{u} \right) \underline{V} \cdot d\underline{A}$$

Simplified Form of the Energy Equation (steady, one-dimensional pipe flow):

$$\frac{p_{out}}{\rho} + \frac{V_{out}^2}{2} + gz_{out} = \frac{p_{in}}{\rho} + \frac{V_{in}^2}{2} + gz_{in} + w_s - \text{loss}$$

where $w_s = \dot{W}_s / \dot{m}$, $\text{loss} = \check{u}_{out} - \check{u}_{in} - q$, and $q = \dot{Q} / \dot{m}$.

For non-uniform flows,

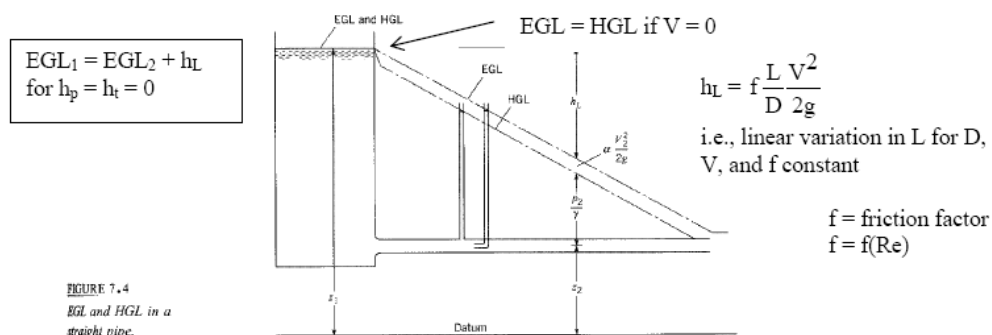
$$\underbrace{\frac{p_{in}}{\gamma} + \alpha_{in} \frac{V_{in}^2}{2g} + z_{in} + h_p}_{\text{Mechanical energy}} = \underbrace{\frac{p_{out}}{\gamma} + \alpha_{out} \frac{V_{out}^2}{2g} + z_{out} + h_t}_{\text{Mechanical energy}} + \underbrace{h_L}_{\text{Thermal energy}}$$

- pump head $h_p = \dot{W}_p / \dot{m}g = \dot{W}_p / \rho Qg = \dot{W}_p / \gamma Q$
- turbine head $h_t = \dot{W}_t / \dot{m}g$
- head loss $h_L = (\hat{u}_2 - \hat{u}_1) / g - \dot{Q} / \dot{m}g > 0$
- α : kinetic energy correction factor ($\alpha = 1$ for uniform flow across CS)
- V in energy equation refers to average velocity \bar{V}

Hydraulic and Energy Grade Lines

- Hydraulic Grade Line: $HGL = \frac{p}{\gamma} + z$
- Energy Grade Line: $EGL = \frac{p}{\gamma} + z + \alpha \frac{V^2}{2g}$

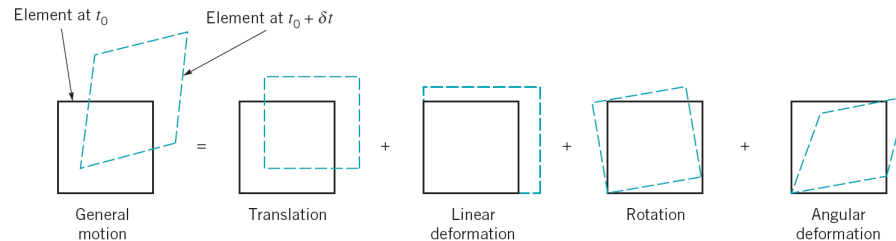
$$EGL_{in} + h_p = EGL_{out} + h_t + h_L$$



Chapter 6 Differential Analysis of Fluid Flow

1. Fluid Element Kinematics

Fluid element motion consists of translation, linear deformation, rotation, and angular deformation.



- Linear deformation(dilatation): $\nabla \cdot \underline{V} \Rightarrow$ if the fluid is incompressible, $\nabla \cdot \underline{V} = 0$
- Rotation(vorticity): $\underline{\xi} = 2\underline{\omega} = \nabla \times \underline{V} \Rightarrow$ if the fluid is irrotational, $\nabla \times \underline{V} = 0$
- Angular deformation is related to shearing stress: $\tau_{ij} = 2\mu\varepsilon_{ij}$

2. Mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

For a steady and incompressible flow: $\nabla \cdot \underline{V} = 0$

3. Momentum conservation

$$\rho \underbrace{\left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right)}_{\underline{a}} = \underbrace{-\rho g \hat{k}}_{\text{body force due to gravity force}} + \underbrace{\left(-\nabla p + \nabla \cdot \underline{\tau}_{ij} \right)}_{\text{surface force}}$$

pressure force
viscous shear force

For Newtonian incompressible fluid, $\nabla \cdot \underline{\tau}_{ij} = \mu \nabla^2 \underline{V} \Rightarrow$ Navier-Stokes eq.

4. Navier-Stokes Equations

1) Cartesian coordinates

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum:

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

$$\rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right]$$

$$\rho \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = -\frac{\partial p}{\partial z} - \rho g + \mu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right]$$

2) Cylindrical coordinates:

Continuity:

$$\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Momentum:

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \rho g_r + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right]$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right]$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right]$$

4. Exact solutions of NS Equations

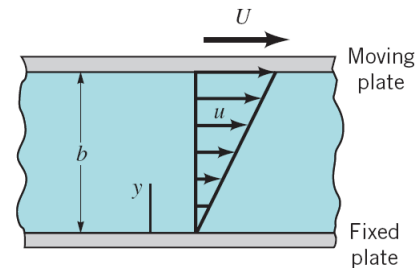
Ex 1) Couette Flow (without pressure gradient)

Assumptions: laminar, steady, 2-D, incompressible, ignore gravity, no pressure gradient

- Continuity: $\frac{\partial u}{\partial x} = 0$
- Momentum: $0 = \mu \frac{\partial^2 u}{\partial y^2}$
- B.C.: $u(h) = U, u(0) = 0$

$$\Rightarrow u(y) = \frac{U}{b} y$$

$$\text{Shear stress at the bottom wall: } \tau_w = \mu \left. \frac{du}{dy} \right|_{y=0} = \frac{\mu U}{b}$$

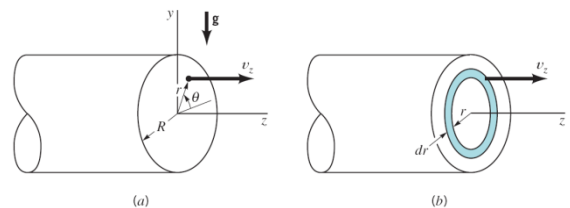


Ex 2) Circular pipe (with constant pressure gradient)

Assumptions: laminar, steady, incompressible, fully-developed, constant pressure gradient

- Continuity: $\frac{1}{r} \frac{\partial(rv_r)}{\partial r} = 0$
- z-Momentum: $0 = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right]$
- B.C.: $v_r(r=0) = 0, v_z(r=0) \neq \infty, v_z(r=R) = 0$

$$\Rightarrow v_z(r) = \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) (r^2 - R^2)$$



$$1) \text{ Flow rate: } Q = \int_0^R v_z dA = -\frac{\pi R^4}{8\mu} \left(\frac{\partial p}{\partial z} \right) = \frac{\pi R^4 \Delta p}{8\mu \ell} \quad \therefore -\frac{\partial p}{\partial z} = \frac{\Delta p}{\ell}$$

$$2) \text{ Mean velocity: } \bar{V} = \frac{Q}{A} = \frac{R^2 \Delta p}{8\mu \ell}$$

$$3) \text{ Maximum velocity: } V_{max} = v_z(0) = \frac{R^2 \Delta p}{4\mu \ell} = 2\bar{V}$$

Chapter 7 Dimensional Analysis and Modeling

1. Buckingham Pi Theorem

For any physically meaningful equation involving k variables, such as

$$u_1 = f(u_2, u_3, \dots, u_k)$$

with minimum number of reference dimensions r , the equation can be rearranged into product of $k - r$ pi terms.

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \dots, \Pi_{k-r})$$

Methods for determining Π 's

- Inspection
- Step-by-step method
- Exponent Method

Example - Exponent method:

$$F_D = f(\mu, V, L)$$

$$F_D \doteq MLT^{-2}$$

$$\mu \doteq ML^{-1}T^{-1}$$

$$V \doteq LT^{-1}$$

$$L \doteq L$$

Number of pi terms = $k - r = 4 - 3 = 1$

$$\Pi = F_D \mu^a V^b L^c$$

It follows that

$$(MLT^{-2})(ML^{-1}T^{-1})^a(LT^{-1})^b(L)^c = M^0L^0T^0$$

$$1 + c = 0 \quad (\text{for } M)$$

$$-2 - a - c = 0 \quad (\text{for } T)$$

$$1 + a + b - c = 0 \quad (\text{for } L)$$

so that $a = -1$, $b = -1$, $c = -1$, and therefore

$$\Pi = \frac{F_D}{\mu V L}$$

Thus, $\frac{F_D}{\mu V L} = \text{constant}$

2. Common Dimensionless Parameters for Fluid Flow Problems.

Variable	velocity	density	gravity	viscosity	Surface tension	compressibility	Pressure change	Length
Symbol	V	ρ	g	μ	σ	K	Δp	L
Unit (SI)	m/s	kg/m ³	m/s ²	N · s/m ²	N/m	N/m ²	N/m ²	m
MLT	LT^{-1}	ML^{-3}	LT^{-2}	$ML^{-1}T^{-1}$	MT^{-2}	$ML^{-1}T^{-2}$	$ML^{-1}T^{-2}$	L
FLT	LT^{-1}	FT^2L^{-4}	LT^{-2}	FTL^{-2}	FL^{-1}	FL^{-2}	FL^{-2}	L

Dimensionless Groups	Symbol	Definition	Interpretation
Reynolds number	Re	$\frac{\rho VL}{\mu}$	$\frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho V^2/L}{\mu V/L^2}$
Froude number	Fr	$\frac{V}{\sqrt{gL}}$	$\frac{\text{inertia force}}{\text{gravity force}} = \frac{\rho V^2/L}{\gamma}$
Weber number	We	$\frac{\rho V^2 L}{\sigma}$	$\frac{\text{inertia force}}{\text{surface tension force}} = \frac{\rho V^2/L}{\sigma/L^2}$
Mach number	Ma	$\frac{V}{\sqrt{K/\rho}} = \frac{V}{a}$	$\sqrt{\frac{\text{inertia force}}{\text{compressibility force}}}$
Euler number	C_p	$\frac{\Delta p}{\rho V^2}$	$\frac{\text{pressure force}}{\text{inertia force}} = \frac{\Delta p/L}{\rho V^2/L}$

3. Similarity and Model Testing

If all relevant dimensionless parameters have the same corresponding values for model and prototype, flow conditions for a model test are completely similar to those for prototype.

$$\Pi_{\text{model}} = \Pi_{\text{prototype}}$$

Model Testing

1) Fr similarity $Fr_m = Fr_p$

$$\frac{V_m}{\sqrt{gL_m}} = \frac{V_p}{\sqrt{gL_p}} \Rightarrow V_m = \sqrt{\alpha} V_p \text{ Froude scaling, where } \alpha = L_m/L_p$$

2) Re similarity $Re_m = Re_p$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \Rightarrow \frac{\nu_m}{\nu_p} = \frac{V_m L_m}{V_p L_p} = \alpha^{\frac{3}{2}}$$