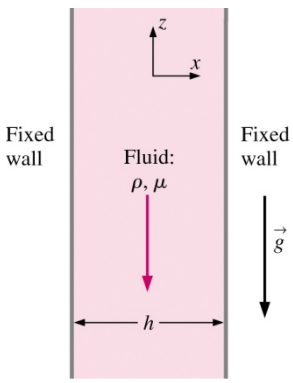
NAME

Fluids-ID

Quiz 7. Consider steady, incompressible, parallel, laminar flow of a viscous fluid falling between two infinite vertical walls. The distance between the walls is h, and gravity acts in the negative z-direction (downward in the figure). There is no applied (forced) pressure ($\nabla p = 0$) driving the flow – the fluid falls by gravity alone ($g_z = -g$). Calculate the centerline velocity (along x = 0 line) if the fluid is glycerin at 20°C and h = 2 mm. Assume the flow is purely two-dimensional (v = 0, $\partial/\partial y$ = 0) and parallel to the walls (u = 0).

(For glycerin @20°C, ρ = 1,260 kg/m^3, μ = 1.49 kg/m·s)



Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

z-momentum:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Solution

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{or} \quad \frac{\partial w}{\partial z} = 0 \qquad \because u = v = 0$$

Since $\frac{\partial w}{\partial t} = 0, \frac{\partial w}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \Rightarrow w = w(x)$ only.

z-momentum:

$$\rho\left(\underbrace{\frac{\partial w}{\partial t}}_{steady} + \underbrace{u\frac{\partial w}{\partial x}}_{u=0} + \underbrace{v\frac{\partial w}{\partial y}}_{\partial/\partial y=0} + \underbrace{w\frac{\partial w}{\partial z}}_{continuity}\right) = \underbrace{-\frac{\partial p}{\partial z}}_{\nabla p=0} + \underbrace{\rho g_z}_{-\rho g} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \underbrace{\frac{\partial^2 w}{\partial y^2}}_{\partial/\partial y=0} + \underbrace{\frac{\partial^2 w}{\partial z^2}}_{continuity}\right)$$

or

$$\frac{d^2 w}{dx^2} = \frac{\rho g}{\mu}$$

Integration of z-momentum equation gives,

$$w = \frac{\rho g}{2\mu} x^2 + C_1 x + C_2$$

B.C.

$w(x = h/2) = 0$:

$$0 = \frac{\rho g}{8\mu} h^2 + C_1 \frac{h}{2} + C_2$$

$w(x = -h/2) = 0$

$$0 = \frac{\rho g}{8\mu} h^2 - C_1 \frac{h}{2} + C_2$$

$\Rightarrow C_1 = 0, C_2 = -\frac{\rho g}{8\mu} h^2$

Thus,

$$w = \frac{\rho g}{2\mu} \left(x^2 - \left(\frac{h}{2}\right)^2 \right)$$

Centerline velocity:

$$w(x = 0) = \frac{\rho g}{2\mu} \left(-\left(\frac{h}{2}\right)^2 \right) = -\frac{\rho g h^2}{8\mu}$$

or

$$w = -\frac{1260\,\mathrm{kg/m^3} \times 9.81\,\mathrm{m/s^2} \times (0.002\,\mathrm{m})^2}{8 \times 1.49\,\mathrm{kg/m \cdot s}} = -4.15\,\mathrm{mm/s}$$