6.93 Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig. P6.93. The bottom plate is fixed and the upper plate moves with a constant velocity U. Determine the velocity at the interface. Express your answer in terms of U, μ_1, and μ_2. The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the x direction. The fluid velocity and shearing stress is continuous across the interface between the two fluids. Assume laminar flow.
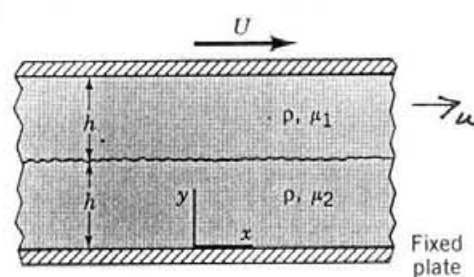

FIGURE P6.93

For the specified conditions, $v=0$, $w=0$, $\frac{\partial P}{\partial x}=0$, and $g_x=0$, so that the x-component of the Navier-Stokes equations (Eq. 6.127a) for either the upper or lower layer reduces to

$$\frac{d^2 u}{dy^2} = 0 \qquad (1)$$

Integration of Eq. (1) yields

$$u = A y + B$$

which gives the velocity distribution in either layer. In the upper layer at $y=2h$, $u=U$ so that

$$B_1 = U - A_1 (2h)$$

where the subscript 1 refers to the upper layer. For the lower layer at $y=0$, $u=0$ so that

$$B_2 = 0$$

where the subscript 2 refers to the lower layer. Thus,

$$u_1 = A_1 (y - 2h) + U$$

and

$$u_2 = A_2 y$$

At $y=h$, $u_1 = u_2$ so that

$$A_1 (h - 2h) + U = A_2 h$$

or

$$A_2 = -A_1 + \frac{U}{h} \qquad (2)$$

(con't)

Since the velocity distribution is linear in each layer the shearing stress

$$\tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{du}{dy}$$

is constant throughout each layer. For the upper layer

$$\tau_1 = \mu_1 A_1$$

and for the lower layer

$$\tau_2 = \mu_2 A_2$$

At the interface $\tau_1 = \tau_2$ so that

$$\mu_1 A_1 = \mu_2 A_2$$

or

$$\frac{A_1}{A_2} = \frac{\mu_2}{\mu_1} \qquad (3)$$

Substitution of Eq. (3) into Eq. (2) yields

$$A_2 = -\frac{\mu_2}{\mu_1} A_2 + \frac{U}{h}$$

or

$$A_2 = \frac{U/h}{1 + \mu_2/\mu_1}$$

Thus, velocity at the interface is

$$u_2 (y=h) = A_2 h = \frac{U}{1 + \frac{\mu_2}{\mu_1}}$$