Example problems on Momentum and Energy equations

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Momentum Eq.

- Newton's 2nd law*:

$$\frac{D}{Dt}\int_{sys} \underline{V}\,\underbrace{\rho d\forall}_{=dm} = \sum \underline{F}_{sys}$$

> * The rate of change of the momentum ($m\underline{V}$) of a particle is equal to the force ($\underline{F}$) acting on it.
>
> $$\frac{d(m\underline{V})}{dt} = \underline{F}$$

- RTT:

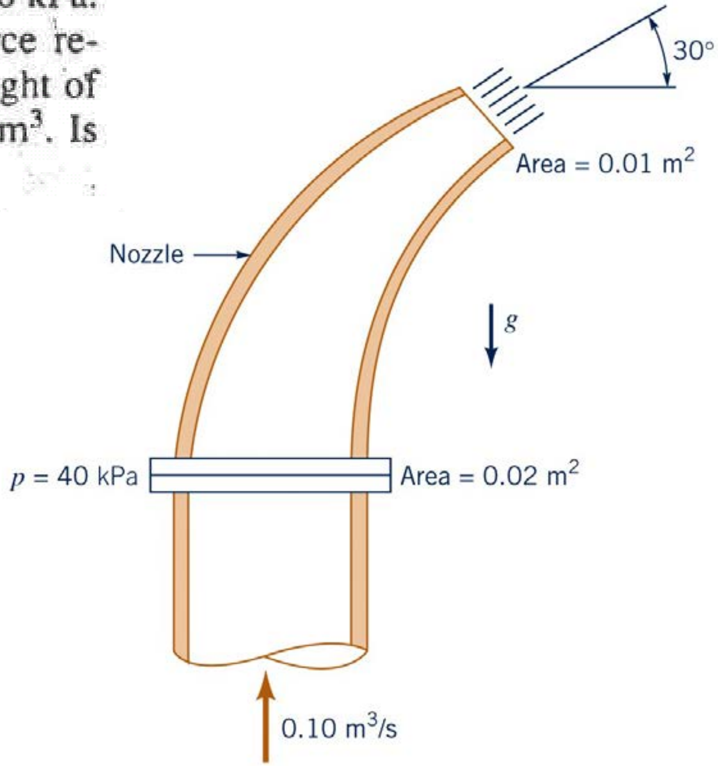
$$\underbrace{\frac{d}{dt}\int_{CV} \underline{V}\rho d\forall}_{\substack{\text{Rate of change}\\ \text{of momentum}\\ \text{in the CV}}} + \underbrace{\int_{CS} \underline{V}\,\underbrace{\rho \underline{V}_R \cdot d\underline{A}}_{=d\dot{m}}}_{\substack{\text{Net momentum}\\ \text{flux through the CS}}} = \underbrace{\sum \underline{F}_{CV}}_{\substack{\text{Net force}\\ \text{acting on the CV}}}$$

For steady flow with uniform flow across discrete CS's,

$$\sum \underline{F} = \sum\left(\dot{m}\underline{V}\right)_{\text{out}} - \sum\left(\dot{m}\underline{V}\right)_{\text{in}}$$

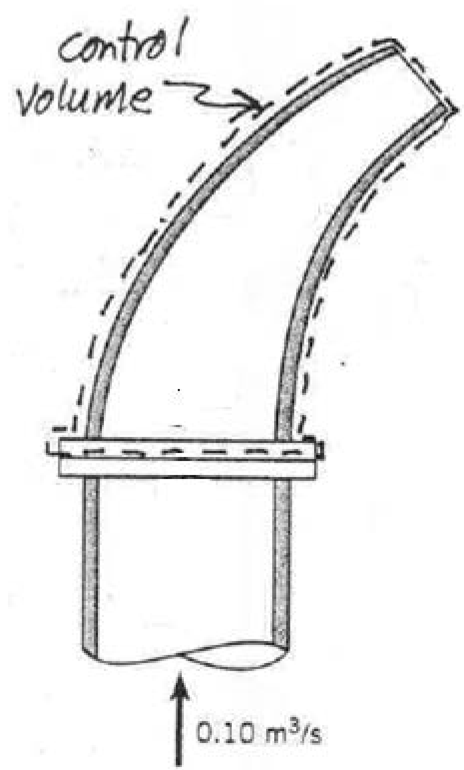
Momentum Eq. Example

5.50 A nozzle is attached to a vertical pipe and discharges water into the atmosphere as shown in Fig. P5.50. When the discharge is 0.1 m^3/s, the gage pressure at the flange is 40 kPa. Determine the vertical component of the anchoring force required to hold the nozzle in place. The nozzle has a weight of 200 N, and the volume of water in the nozzle is 0.012 m^3. Is the anchoring force directed upward or downward?



- Given:
  - Q = 0.1 m^3/s
  - p = 40 kPa at flange
  - W_n = 200 N
  - $V\!\!\!/_w$ = 0.012 m^3
- Find:
  - Vertical anchoring force, F_{Az}

Momentum Eq. Example – Contd.



- Control volume including:
  - Nozzle
  - Water in the nozzle

- Momentum eq.:

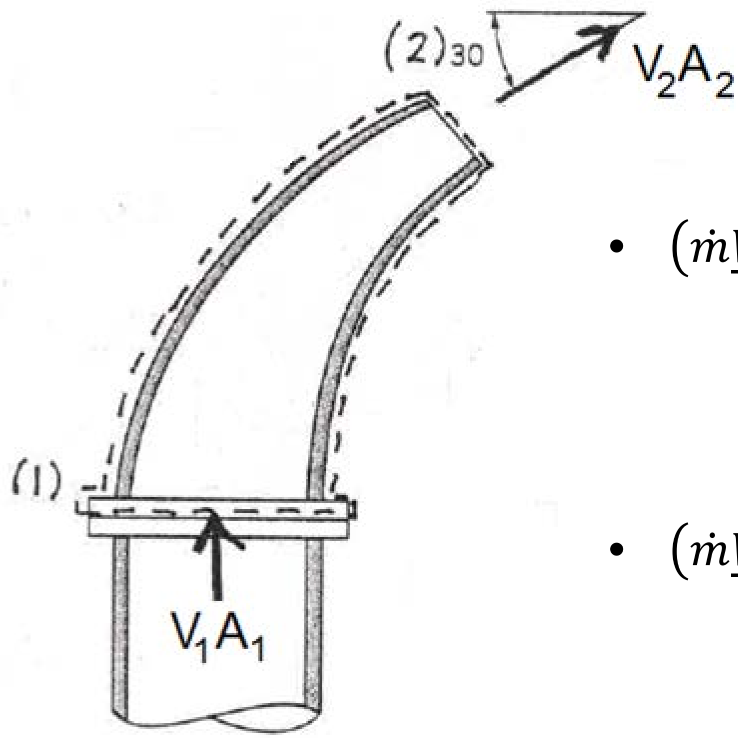
$$\sum \underline{F} = \left(\dot{m}\underline{V}\right)_{out} - \left(\dot{m}\underline{V}\right)_{in}$$

$$\dot{m} = \rho Q = \rho A V$$

$$\underline{V} = V_x \hat{\boldsymbol{\imath}} + V_z \widehat{\boldsymbol{k}}$$

Momentum Eq. Example – Contd.

- Momentum flux through inlet and outlet:



- $\left(\dot{m}\underline{V}\right)_{in}$:

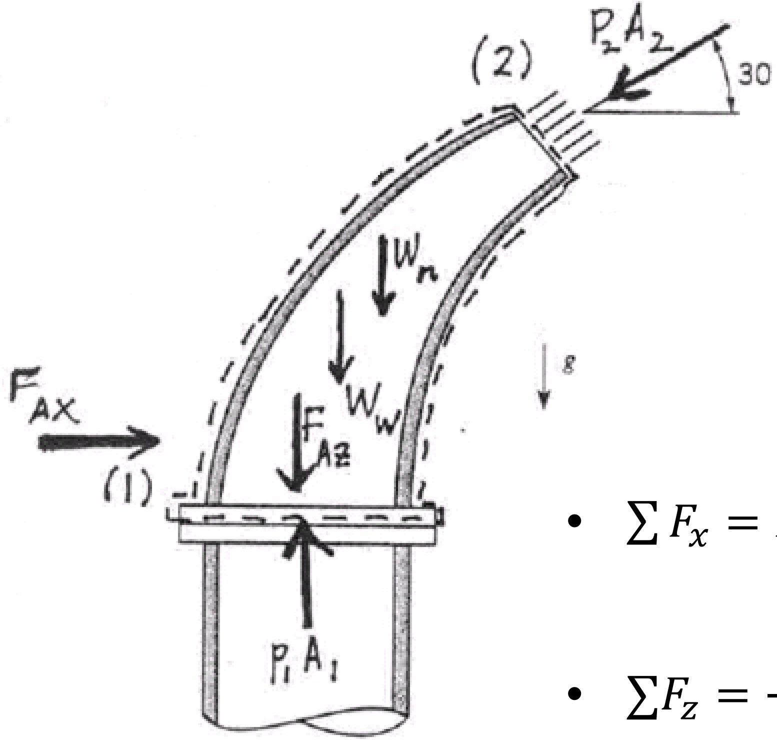
$$(\dot{m}V_x)_{in} = (\rho V_1 A_1)(0)$$
$$(\dot{m}V_z)_{in} = (\rho V_1 A_1)(V_1) \quad (1)$$

- $\left(\dot{m}\underline{V}\right)_{out}$:

$$(\dot{m}V_x)_{out} = (\rho V_2 A_2)(V_2 \cos 30°)$$
$$(\dot{m}V_z)_{out} = (\rho V_2 A_2)(V_2 \sin 30°\ ) \quad (2)$$

Momentum Eq. Example – Contd.

- Forces acting on the CV:



- $\sum F_x = F_{Ax} - p_2 A_2 \cos 30°$
- $\sum F_z = -F_{Az} + p_1 A_1 - p_2 A_2 \sin 30° \ - W_n - W_w \quad (3)$

Momentum Eq. Example – Contd.

- Vertical momentum eq. :

$$\sum F_z = (\dot{m}V_z)_{out} - (\dot{m}V_z)_{in}$$

Using (1), (2) and (3),

$$-F_{Az} + p_1A_1 - p_2A_2 \sin 30^\circ \ - W_n - W_w = (\rho V_2 A_2)(V_2 \sin 30^\circ\ ) - (\rho V_1 A_1)(V_1)$$

or

$$F_{Az} = p_1A_1 - W_n - W_w - \rho Q(V_2 \sin 30^\circ - V_1) \qquad (4)$$

Momentum Eq. Example – Contd.

With $V_1 = Q/A_1$, $V_2 = Q/A_2$, and $W_w = \gamma \forall_w$ in (4),

$$F_{Az} = p_1 A_1 - W_n - \gamma \forall_w - \rho Q \left(\frac{Q}{A_2} \sin 30° - \frac{Q}{A_1} \right)$$

$$= (40,000\ \text{Pa})(0.02\ \text{m}^2) - (200\ \text{N}) - \left(9790\ \frac{\text{N}}{\text{m}^3}\right)(0.012\ \text{m}^3)$$
$$- \left(999\ \frac{\text{kg}}{\text{m}^3}\right)\left(0.01\ \frac{\text{m}^3}{\text{s}}\right)\left(\frac{0.01\ \text{m}^3/\text{s}}{0.01\ \text{m}^2} \times \sin 30° - \frac{0.01\ \text{m}^3/\text{s}}{0.02\ \text{m}^2}\right)$$

$$\therefore F_{Az} = \mathbf{482.5\ N} \quad \text{(downward)}$$

Energy equation (Head form*)

*Energy per unit weight

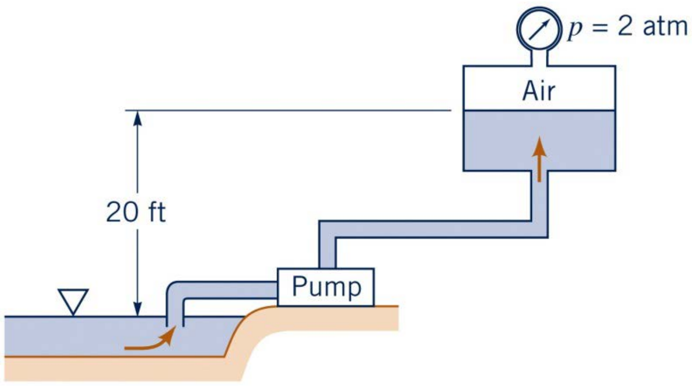
$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

- $h_p = \frac{\dot{W}_p}{\dot{m}g}$ Pump head $(\therefore \dot{W}_p = \dot{m}gh_p = \gamma Q h_p)$
- $h_t = \frac{\dot{W}_t}{\dot{m}g}$ Turbine head $(\therefore \dot{W}_t = \dot{m}gh_t = \gamma Q h_t)$
- h_L Head loss ($h_L > 0$)
- α Kinetic energy correction factor:

$$\alpha = \begin{cases} 2.0 & \text{Lamina flow} \\ 1.04 \sim 1.11 & \text{Turbulent flow} \\ \mathbf{1.0} & \textbf{Uniform flow} \end{cases}$$

Energy Eq. Example 1 (Pump)

5.116 A pump is to move water from a lake into a large, pressurized tank as shown in Fig. P5.116 at a rate of 1000 gal in 10 min or less. Will a pump that adds 3 hp to the water work for this purpose? Support your answer with appropriate calculations. Repeat the problem if the tank were pressurized to 3, rather than 2, atmospheres.



Given:

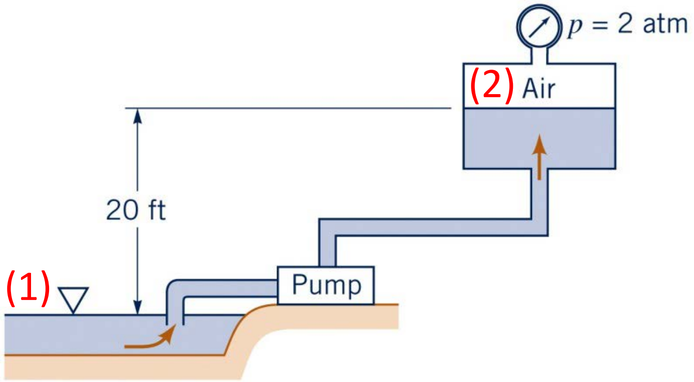
Q = 1000 gal per 10 min

p = 2 atm in the tank

$\dot{W}_p$ = 3 hp

Find: Will this pump work?

Energy Eq. Example 1 (Pump) – Contd.



$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

with $p_1 = 0, V_1 \approx V_2 \approx 0, h_t = 0,$

$$0 + 0 + 0 + h_p = \frac{p_2}{\gamma} + 0 + (z_2 - z_1) + 0 + h_L$$

$$\therefore h_L = h_p - \frac{p_2}{\gamma} - (z_2 - z_1)$$

Energy Eq. Example 1 (Pump) – Contd.

$$h_L = h_p - \frac{p_2}{\gamma} - (z_2 - z_1)$$

1) $h_p = \frac{\dot{W}_p}{\gamma Q} = \frac{\left(3\ \text{hp}\times\frac{550\ \text{ft}\cdot\text{lb/s}}{1\ \text{hp}}\right)}{(62.4\ \text{lb/ft}^3)\left(\frac{1000\ \text{gal}}{10\ \text{min}}\times\frac{1\ \text{ft}^3}{7.48\ \text{gal}}\times\frac{1\ \text{min}}{10\ \text{s}}\right)} = 119\ \text{ft}$

2) $p_2 = \left(2\ \text{atm} \times \frac{14.7\ \text{lb/in}^2}{1\ \text{atm}} \times \frac{144\ \text{in}^2}{1\ \text{ft}^2}\right) = 4{,}230\ \ \text{lb/ft}^2$

3) $z_2 - z_1 = 20\ \text{ft}$

$$\therefore h_L = (119\ \text{ft})\ - \left(\frac{4{,}230\ \text{lb/ft}^2}{62.4\ \text{lb/ft}^3}\right) - (20\ \text{ft}) = 31.2\ \text{ft}\ \ > 0$$

The given pump WILL work.

Energy Eq. Example 1 (Pump) – Contd.

If $p_2 = 3\ \text{atm}$,

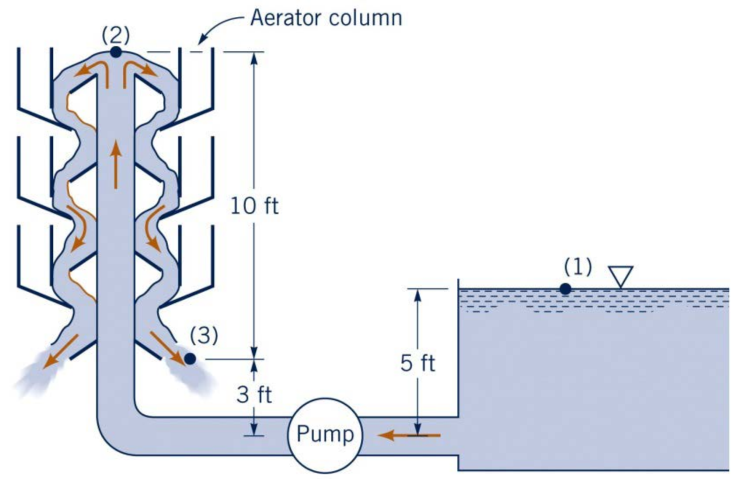
$$p_2 = \left(3\ \text{atm} \times \frac{14.7\ \text{lb/in}^2}{1\ \text{atm}} \times \frac{144\ \text{in}^2}{1\ \text{ft}^2}\right) = 6{,}350\ \ \text{lb/ft}^2$$

$$\therefore\ h_L = (119\ \text{ft})\ - \left(\frac{6{,}350\ \text{lb/ft}^2}{62.4\ \text{lb/ft}^3}\right) - (20\ \text{ft}) = -3\ \text{ft}\ \ {\color{red}< 0}$$

The given pump <u>WILL NOT</u> work.

Energy Eq. Example 2 (Pump)

5.122 Water is pumped from a tank, point (1), to the top of a water plant aerator, point (2), as shown in Video V5.16 and Fig. P5.122 at a rate of 3.0 ft^3/s. **(a)** Determine the power that the pump adds to the water if the head loss from (1) to (2) where $V_2 = 0$ is 4 ft. **(b)** Determine the head loss from (2) to the bottom of the aerator column, point (3), if the average velocity at (3) is $V_3 = 2$ ft/s.



Given:

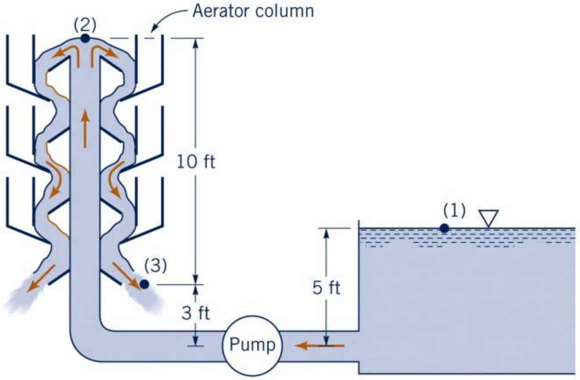
Q = 3.0 ft^3/s
h_L = 4 ft from (1) to (2)
V_2 = 0

Find:

(a) Pump power
(b) Head loss from (2) to (3)

Energy Eq. Example 2 (Pump) – Contd.

(a) Pump power



Energy equation **from (1) to (2)**

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

with $p_1 = p_2 = 0$, $V_1 = V_2 = 0$, and $h_t = 0$,

$$0 + 0 + z_1 + h_p = 0 + 0 + z_2 + 0 + h_L$$

or

$$h_p = (z_2 - z_1) + h_L$$

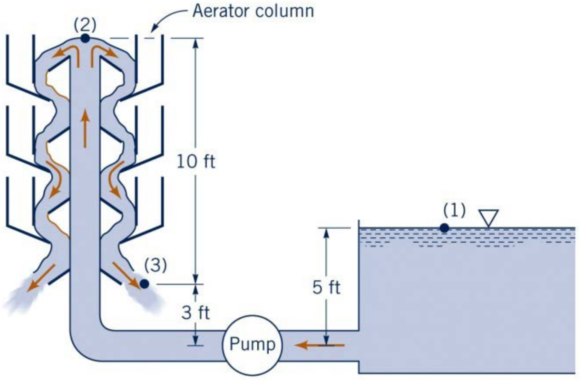
$$= (13\ \text{ft} - 5\ \text{ft}) + (4\ \text{ft}) = 12\ \text{ft}$$

$$\therefore \dot{W}_p = h_p \cdot \gamma Q = (12)(62.4)(3)\left(\frac{1}{550}\right) = \mathbf{4.1\ hp}$$

Energy Eq. Example 2 (Pump) – Contd.

(b) Head loss

Energy equation **from (2) to (3)**



$$\frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_p = \frac{p_3}{\gamma} + \frac{V_3^2}{2g} + z_3 + h_t + h_L$$

with $p_2 = p_3 = 0$, $V_2 = 0$, and $h_p = h_t = 0$,

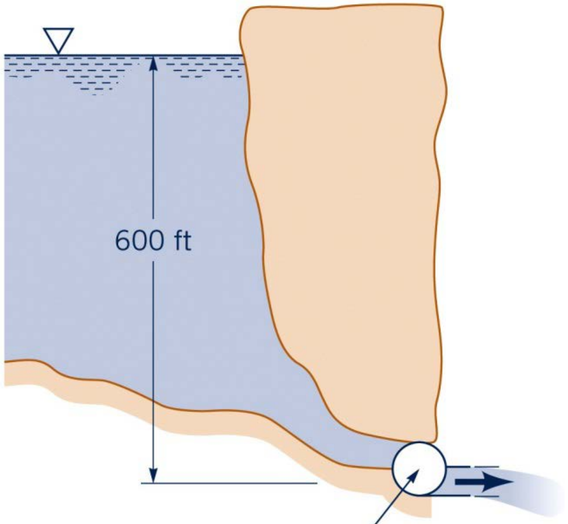
$$0 + 0 + z_2 + 0 = 0 + \frac{V_3^2}{2g} + z_3 + 0 + h_L$$

$$h_L = (z_2 - z_3) - \frac{V_3^2}{2g} = (13\text{ ft} - 3\text{ ft}) - \frac{(2)^2}{2(32.2)}$$

$$= 10\text{ ft} - 0.062\text{ ft} = \mathbf{9.94\ ft}$$

Example 3 (Turbine)

5.115 The hydroelectric turbine shown in Fig. P5.115 passes 8 million gal/min across a head of 600 ft. What is the maximum amount of power output possible? Why will the actual amount be less?



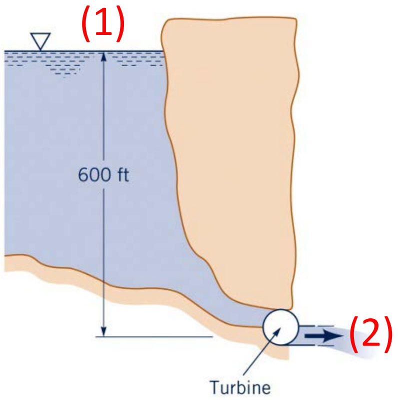
Given:

$$Q = \left(8 \times 10^6 \ \frac{\text{gal}}{\text{min}} \times \frac{1 \text{ ft}^3}{7.48 \text{ gal}} \times \frac{1 \text{ min}}{60 \text{ s}}\right) = 1.78 \times 10^4 \ \text{ft}^3/s$$

$$H = 600 \text{ ft}$$

Find: Maximum power output $\dot{W}_{\text{t}}$ possible

Example 3 (Turbine) – Contd.



$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

With $p_1 = p_2 = 0, V_1 \approx 0$, and $h_p = 0$,

$$0 + 0 + z_1 + 0 = 0 + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

or

$$h_t = (z_1 - z_2) - \frac{V_2^2}{2g} - h_L$$

Example 3 (Turbine) – Contd.

The maximum power would occur if there were no loss ($h_L = 0$) and negligible kinetic energy at the exit (i.e, $V_2 \approx 0$; large diameter outlet). Thus,

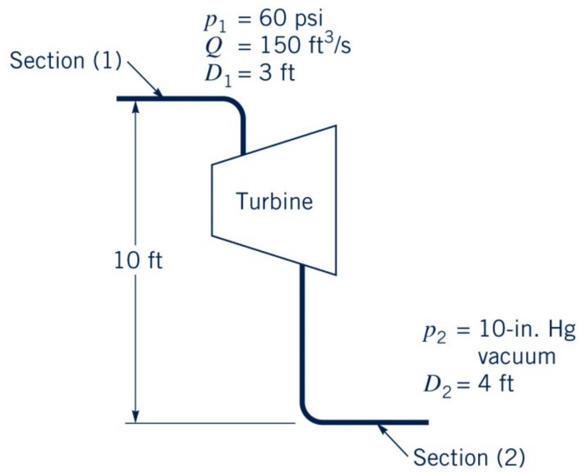
$$h_t = (z_1 - z_2) - \frac{V_2^2}{2g} - h_L = 600 \text{ ft}$$

$$\therefore \dot{W}_{t,\max} = h_t \cdot \gamma Q = (600 \text{ ft}) \left(62.4 \ \frac{\text{lb}}{\text{ft}^3}\right) \left(1.78 \times 10^4 \ \frac{\text{ft}^3}{\text{s}}\right) \left(\frac{1 \text{ ft}\cdot\text{lb/s}}{550 \text{ hp}}\right)$$

$$= \mathbf{1.21 \times 10^6 \ hp}$$

Example 4 (Turbine)

5.117 Water is supplied at 150 ft^3/s and 60 psi to a hydraulic turbine through a 3-ft inside diameter inlet pipe as indicated in Fig. P5.117. The turbine discharge pipe has a 4-ft inside diameter. The static pressure at section (2), 10 ft below the turbine inlet, is 10-in. Hg vacuum. If the turbine develops 2500 hp, determine the power lost between sections (1) and (2).

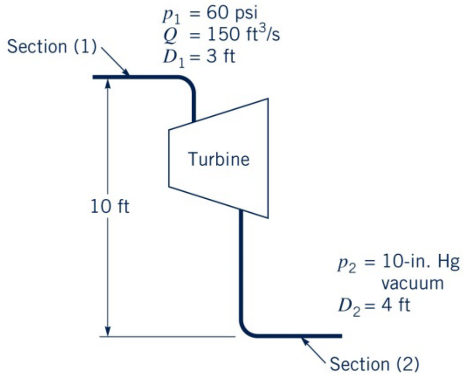


Given:

$$\dot{W}_t = 2500\ hp$$

Find: Power loss $\dot{W}_{\text{loss}}$ between sections (1) and (2)

Example 4 (Turbine) – Contd.



$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2+h_t+h_L$$

$$p_1=(60\,\mathrm{lb/in^2})(144\,\mathrm{in^2/ft^2})=8{,}640\,\mathrm{lb/ft^2}$$

$$p_2=-\frac{(10\text{ in.Hg})(13.6)(1.94\text{ slugs/ft}^3)\left(32\text{ft/s}^2\right)\left(1\text{ lb/slugs}\cdot\frac{\text{ft}}{\text{s}}\right)}{(12\text{ in./ft})}=-704\ \mathrm{lb/ft^2}$$

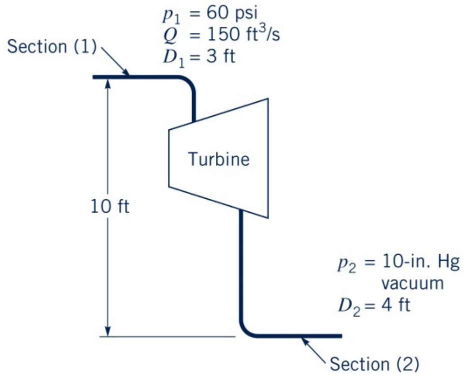
$$V_1=\frac{Q}{A_1}=\frac{150\mathrm{ft^3\ /s}}{\pi(3\text{ ft})^2/4}=21.22\,\mathrm{ft/s}$$

$$V_2=\frac{Q}{A_2}=\frac{150\mathrm{ft^3\ /s}}{\pi(4\text{ ft})^2/4}=11.94\,\mathrm{ft/s}$$

$$h_t=\frac{\dot{W}_t}{\gamma Q}=\frac{(2500\text{ hp})\left(550\,\frac{\text{ft}\cdot\text{lb/s}}{\text{hp}}\right)}{(62.4\,\mathrm{lb/ft^3})(150\mathrm{ft^3\ /s})}=146.9\text{ ft}$$

$$h_p=0$$

Example 4 (Turbine) – Contd.



Solve Energy eq. for h_L with h_p = 0,

$$h_L = \frac{(p_1 - p_2)}{\gamma} + \frac{(V_1^2 - V_2^2)}{2g} + (z_1 - z_2) - h_t$$

or

$$h_L = \frac{\left(8640 - (-703)\right)}{62.4} + \frac{(21.22^2 - 11.94^2)}{2 \times 32.2} + (10) - (146.9)$$

$$= 149.7 + 4.8 + 10 - 146.9 = 17.6 \text{ ft}$$

$$\therefore \text{Power loss} = h_L \cdot \gamma Q = (17.6)(62.4)(150)\left(\frac{1}{550}\right) = \mathbf{300\ hp}$$

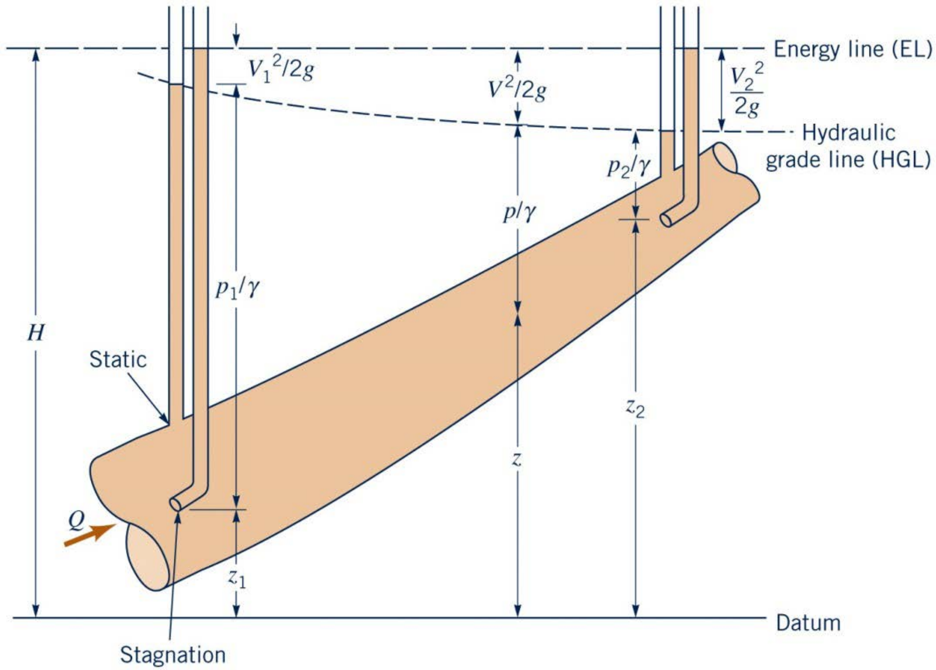
Hydraulic and Energy Grad Lines

- A way of graphical representation of the level of mechanical energy in a flow:

$$\mathrm{HGL} \equiv \frac{p}{\gamma} + z$$

$$\mathrm{EGL} \equiv \frac{p}{\gamma} + z + \frac{V^2}{2g} \quad \left(= \mathrm{HGL} + \frac{V^2}{2g}\right)$$

Hydraulic and Energy Grad Lines – Contd.



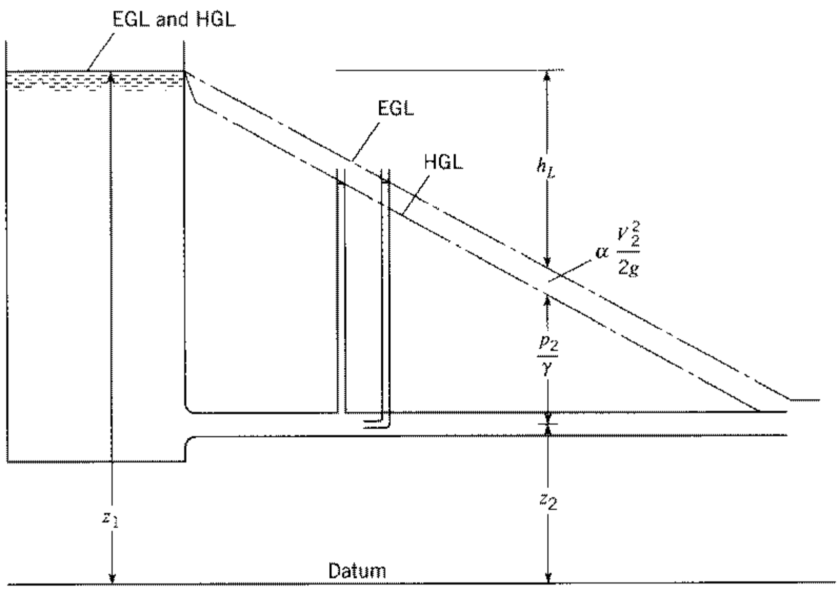
(An example of ideal frictionless flow)

- HGL: If a piezometer is tapped into a pipe (i.e., a pressure tap), the liquid would rise to a height of p/γ above the pipe center, i.e. $\mathrm{HGL} = z + p/\gamma$.

- EGL: If a stagnation tube is inserted into a pipe, the liquid would rise to a height of $p/\gamma + V^2/2g$ above the pipe center, i.e., $\mathrm{EGL} = z + p/\gamma + V^2/2g$.

- For ideal frictionless (Bernoulli-type) flows, EGL is horizontal and its height remains constant.

Hydraulic and Energy Grad Lines – Contd.



EGL and HGL in a straight pipe (friction flow)

- The EGL and HGL coincide with the free surface for stationary bodies since the velocity is zero and the static pressure (gage) is zero.
- The EGL is always a distance $V^2/2g$ above the HGL.
- At a pipe exit, the pressure head is zero (atmospheric pressure) and the HGL coincides with the pipe outlet.
- The head loss causes the EGL and HGL to slope down in the direction of flow.

Hydraulic and Energy Grad Lines – Contd.

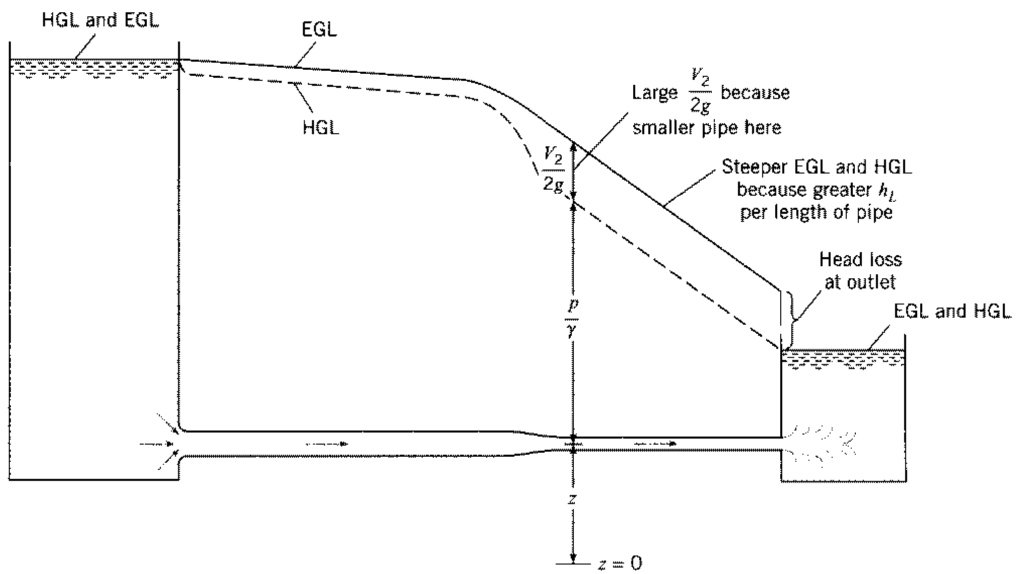


FIGURE 7.8

Change in EGL and HGL due to change in diameter of pipe.

- The EGL and HGL approach each other as the velocity decreases, and they diverge as the velocity increases.
- The height of HGL decreases as the velocity increases, and vice versa.

Hydraulic and Energy Grad Lines – Contd.

FIGURE 7.5

Rise in EGL and HGL due to pump.

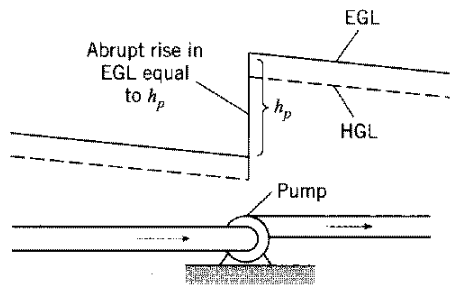
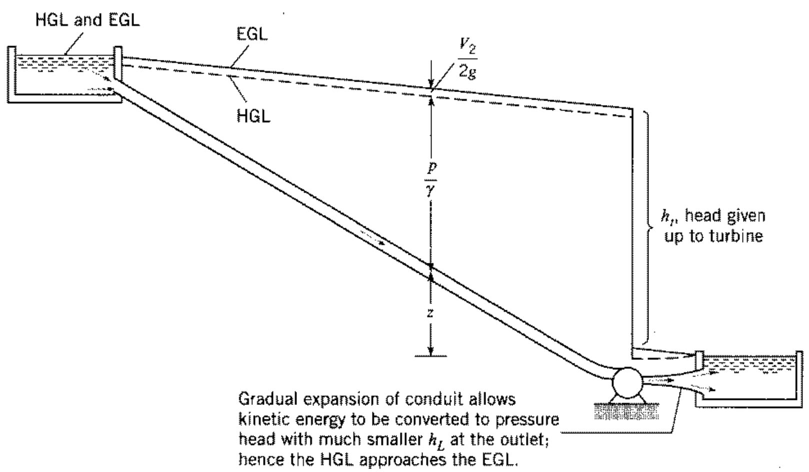


FIGURE 7.6

Drop in EGL and HGL due to turbine.



- A steep jump occurs in EGL and HGL whenever mechanical energy is added to the fluid (e.g., by a pump).
- A steep drop occurs in EGL and HGL whenever mechanical energy is removed from the fluid (e.g., by a turbine).

Hydraulic and Energy Grad Lines – Contd.

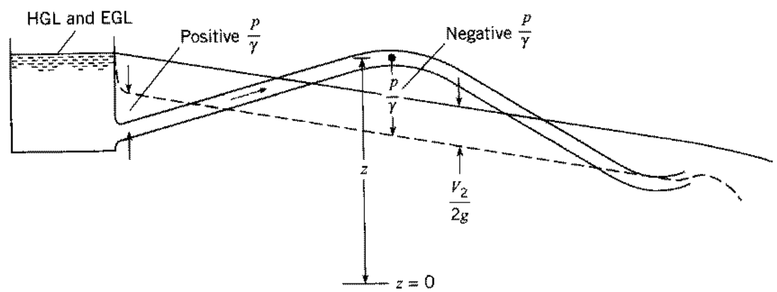


FIGURE 7.9
Subatmospheric pressure when pipe is above HGL.

- The pressure (gage) of a fluid is zero at locations where the HGL intersects the fluid.
- The pressure in a flow section that lies above the HGL is negative (vacuum), and the pressure in a section that lies below the HGL is positive.
- Situations where the pressure drops below the vapor pressure of the liquid cause cavitation.