Fluids in Rigid-Body Motion


Hyunse Yoon, Ph.D.

Associate Research Scientist
IIHR-Hydroscience & Engineering
Newton’s 2\textsuperscript{nd} Law of Motion

• In general, for a body of mass $m$,

\[ ma = \sum F \]

where, $a$ is the acceleration of the body and $\sum F$ is the vector sum of the external forces acting on the body.

• For a fluid element,

\[ ma = F_B + F_S \quad (1) \]

where,

- $F_B$ is the body force due to the gravity, i.e., the weight of the fluid element
- $F_S$ is the surface force due to the pressure and viscous friction on the surface of the fluid element

• In fluids, often times the motion equation is written for a unit volume by using the relationship $m = \rho \Psi$ and dividing Eq. (1) by the volume $\Psi$,

\[ \rho a = f_b + f_s \]

where, $f_b$ and $f_s$ are the body and surface forces per unit volume.
Newton’s 2nd Law of Motion – Contd.

- **Body force (Weight of the fluid)**
  \[
  F_B = -W\hat{k} = -\rho g V\hat{k}
  \]
  \[\therefore f_b = -\rho g \hat{k}\]

- **Surface force**
  \[f_s = f_p + f_v\]
  where,
  - \(f_p = -\nabla p\) due to the pressure
  - \(f_v = \nabla \cdot \tau\) due to the viscous shear stress

- **General motion equation for fluids**
  \[
  \rho a = -\rho g \hat{k} - \nabla p + \nabla \cdot \tau \quad (2)
  \]

Note: For one dimensional flow of Newtonian fluids, \(\tau = \mu \frac{du}{dy}\). This implies that the viscous shear stress (or the shear force) is caused by the relative motion between fluid particles.
Special Case: Fluids at Rest

- For fluids at rest, i.e., with no motion, Eq. (2) can be simplified as

$$\rho a = -\rho g \hat{k} - \nabla p + \nabla \cdot \tau = 0$$

or,

$$\nabla p = \rho g \quad (3)$$

where, $g = -g \hat{k}$.

- If rewrite Eq. (3) in components,

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g \quad (4)$$

Thus, $p$ is independent of $x$ and $y$ (i.e., the pressure remains constant in any horizontal direction) and varies only in the vertical direction $z$ as a result of gravity.

- If $\rho$ is constant, the solution of Eq. (4) becomes

$$p = -\gamma z$$

by taking $p = 0$ at $z = 0$. This is the hydrostatic pressure equation for incompressible fluids at rest.
Rigid Body Motion

• In rigid-body motion, all particles are in combined translation and rotation, and there is no relative motion between particles.

• With no relative motion, there are no strains or strain rates, so that the viscous term in Eq. (2) vanishes,

\[ \rho a = -\rho g \hat{k} - \nabla p + \nabla \cdot \tau = 0 \]

or,

\[ \nabla p = \rho (g - a) \quad (5) \]

where, \( g = -g \hat{k} \).

• Two simple rigid-motion cases of interest are
  a) Rigid body translation: Constant linear acceleration \( a = a_x \hat{i} + a_z \hat{k} \)
  b) Rigid body rotation: Constant rotation \( \Omega = \Omega \hat{k} \)
In case of uniform rigid-body acceleration, Eq. (5) applies, \( \mathbf{a} \) having the same magnitude and direction for all particles.

The vector sum of \( \mathbf{g} \) and \(-\mathbf{a}\) gives the direction of the pressure gradient or the greatest rate of increase of \( p \).

Then, the surfaces of constant pressure must be perpendicular to the direction of pressure gradient and are thus tilted at a downward angle \( \theta \).

\[
\nabla p = \rho (\mathbf{g} - \mathbf{a})
\]

where,

\[
\mathbf{g} = -g \mathbf{k}
\]

\[
\mathbf{a} = a_x \mathbf{i} + a_z \mathbf{k}
\]

Thus,

\[
\nabla p = \frac{\partial p}{\partial x} \mathbf{i} + \frac{\partial p}{\partial z} \mathbf{k} = -\rho a_x \mathbf{i} - (\rho g + \rho a_z) \mathbf{k}
\]

Equating like components,

\[
\frac{\partial p}{\partial x} = -\rho a_x, \quad \frac{\partial p}{\partial z} = -\rho (g + a_z)
\]

The angle of constant pressure lines,

\[
\theta = \tan^{-1} \frac{a_x}{g + a_z}
\]
Rigid Body Translation – Contd.

- One of the tilted lines (the surfaces of constant pressure) is the free surface, which is found by the requirement that the fluid retain its volume unless it spills.
- The rate of increase of pressure in the direction $\mathbf{g} - \mathbf{a}$ is greater than in the ordinary hydrostatics and is given by

$$\frac{dp}{ds} = \rho G \quad \text{where} \quad G = \sqrt{a_x^2 + (g + a_z)^2}$$

Tilting of constant-pressure surfaces in a tank of liquid in rigid-body acceleration.

\[
\nabla p = \rho (\mathbf{g} - \mathbf{a}) \quad (5)
\]

where,

\[
\mathbf{g} = -g\mathbf{k} \\
\mathbf{a} = a_x\mathbf{i} + a_z\mathbf{k}
\]

Thus,

\[
\nabla p = \frac{\partial p}{\partial x} \mathbf{i} + \frac{\partial p}{\partial z} \mathbf{k} = -\rho a_x\mathbf{i} - \rho (g + a_z)\mathbf{k}
\]

Equating like components,

\[
\frac{\partial p}{\partial x} = -\rho a_x \quad \frac{\partial p}{\partial z} = -\rho (g + a_z)
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The angle of constant pressure lines,

\[
\theta = \tan^{-1}\frac{a_x}{g + a_z}
\]
Rigid Body Translation – Example

**Example 2.13**

A drag racer rests her coffee mug on a horizontal tray while she accelerates at 7 m/s². The mug is 10 cm deep and 6 cm in diameter and contains coffee 7 cm deep at rest. (a) Assuming rigid-body acceleration of the coffee, determine whether it will spill out of the mug. (b) Calculate the gage pressure in the corner at point A if the density of coffee is 1010 kg/m³.

\[ \theta = \tan^{-1} \frac{a_x}{g} = \tan^{-1} \frac{7}{9.81} = 35.5^\circ \]

\[ \Delta z = (3)(\tan 35.5^\circ) = 2.14 \text{ cm} < 3 \text{ cm} \text{ (no spilling)} \]

\[ p_A = \rho g \Delta s = (1010)\sqrt{(7)^2 + (9.81)^2} [(0.07 + 0.0214) \cos 35.5^\circ] = 906 \text{ Pa} \]

(Note: When at rest, \( p_A = \rho g h_{\text{rest}} = (1010)(9.81)(0.07) = 694 \text{ Pa} \))

Alternatively, since \( a_z = 0 \) thus \( \frac{\partial p}{\partial z} = -\rho g, \)

\[ p_A = \rho g \Delta z = (1010)(9.81)(0.07 + 0.0214) = 906 \text{ Pa} \]

The coffee tilted during the acceleration.
Rigid Body Rotation

- For a fluid rotating about the z axis at a constant rate $\Omega$ without any translation, the fluid acceleration will be a centripetal term,
  $$\mathbf{a} = -r\Omega^2 \mathbf{\hat{i}}_r$$

- From Equation (5) written in a cylindrical coordinate system,
  $$\nabla p = \frac{\partial p}{\partial r} \mathbf{\hat{i}}_r + \frac{\partial p}{\partial z} \mathbf{\hat{k}} = \rho (\mathbf{g} - \mathbf{a}) = \rho (r\Omega^2 \mathbf{\hat{i}}_r - g\mathbf{\hat{k}})$$

- Equating like components,
  $$\frac{\partial p}{\partial r} = \rho r\Omega^2$$
  $$\frac{\partial p}{\partial z} = -\rho g \quad (6)$$

- By solving the two 1st-order PDE’s in Eq. (6),
  $$p = p_0 - \rho gz + \frac{1}{2} \rho r^2 \Omega^2 \quad (7)$$

  where, $p_0$ is the pressure at $(r, z) = (0, 0)$.

- The pressure is linear in $z$ and quadratic (parabolic) in $r$.

Development of paraboloid constant-pressure surfaces in a fluid in rigid-body rotation. The dashed line along the direction of maximum pressure increase is an exponential curve.
Rigid Body Rotation – Contd.

• If we wish to plot a constant-pressure surface, say \( p = p_1 \), Equation (7) becomes

\[
z = \frac{p_0 - p_1}{\rho g} + \frac{r^2 \Omega^2}{2g} = a + br^2
\]

• Thus, the surfaces are paraboloids of revolution, concave upward, with their minimum points on the axis of rotation.

 Similarly as in rigid body translation case, the position of the free surface is found by conserving the volume of fluid.

• Since the volume of a paraboloid is one-half of the base area times its height, the still-water level is exactly halfway between the high and low points of the free surface.

• The center of the fluid drops an amount

\[
\frac{h}{2} = \frac{\Omega^2 R^2}{4g}
\]

and the edges rise an equal amount.

Determining the free surface position for rotation of a cylinder of fluid about its central axis.
Rigid Body Rotation – Example

The coffee cup in Example 2.13 is removed from the drag racer, placed on a turntable, and rotated about its central axis until a rigid-body mode occurs. Find (a) the angular velocity that will cause the coffee to just reach the lip of the cup and (b) the gage pressure at point A for this condition.

\[
\frac{h}{2} = \frac{\Omega^2 R^2}{4g} = \frac{\Omega^2 (0.03)^2}{4(9.81)} = 0.03
\]

∴ \(\Omega = 36.2 \text{ rad/s} = 345 \text{ rpm}\)

Since point A is at \((r, z) = (3 \text{ cm}, -4 \text{ cm})\) and by putting the origin of coordinates \(r\) and \(z\) at the bottom of the free-surface depression, thus \(p_0 = 0\) (i.e., gage pressure),

\[
p_A = p_0 - \rho g z + \frac{1}{2} \rho r^2 \Omega^2
\]

\[
= 0 - (1010)(9.81)(-0.04) + \frac{1}{2} (1010)(0.03)^2 (36.2)^2 = 990 \text{ Pa}
\]

(Note: This is about 43% greater than the still-water pressure \(p_A = 694 \text{ Pa}\))