Chapter 7 Dimensional Analysis and Modeling

The Need for Dimensional Analysis

Dimensional analysis is a process of formulating fluid mechanics problems in terms of nondimensional variables and parameters.

1. Reduction in Variables:

\[ F(A_1, A_2, \ldots, A_n) = 0, \]

If \( F(A_1, A_2, \ldots, A_n) = 0, \)

\[ A_i = \text{dimensional variables} \]

Then \( f(\Pi_1, \Pi_2, \ldots, \Pi_{r < n}) = 0 \)

\[ \Pi_j = \text{nondimensional parameters} \]

Thereby reduces number of experiments and/or simulations required to determine \( f \) vs. \( F \)

\[ = \Pi_j (A_i) \]

i.e., \( \Pi_j \) consists of nondimensional groupings of \( A_i \)’s

2. Helps in understanding physics

3. Useful in data analysis and modeling

4. Fundamental to concept of similarity and model testing

   Enables scaling for different physical dimensions and fluid properties
**Dimensions and Equations**

Basic dimensions: F, L, and t or M, L, and t

F and M related by $F = Ma = MLT^{-2}$

**Buckingham $\Pi$ Theorem**

In a physical problem including n dimensional variables in which there are m dimensions, the variables can be arranged into $r = n - \hat{m}$ independent nondimensional parameters $\Pi_r$ (where usually $\hat{m} = m$).

$$F(A_1, A_2, \ldots, A_n) = 0$$

$$f(\Pi_1, \Pi_2, \ldots \Pi_r) = 0$$

$A_i$’s = dimensional variables required to formulate problem

$(i = 1, n)$

$\Pi_j$’s = nondimensional parameters consisting of groupings of $A_i$’s $(j = 1, r)$

F, f represents functional relationships between $A_n$’s and $\Pi_r$’s, respectively

$\hat{m} = \text{rank of dimensional matrix}$

$= m$ (i.e., number of dimensions) usually
**Dimensional Analysis**

Methods for determining $\Pi_i$’s

1. Functional Relationship Method

Identify functional relationships $F(A_i)$ and $f(\Pi_j)$ by first determining $A_i$’s and then evaluating $\Pi_j$’s

   a. Inspection
   b. Step-by-step Method
   c. Exponent Method

2. Nondimensionalize governing differential equations and initial and boundary conditions

Select appropriate quantities for nondimensionalizing the GDE, IC, and BC e.g. for $M$, $L$, and $t$

Put GDE, IC, and BC in nondimensional form

Identify $\Pi_j$’s

Exponent Method for Determining $\Pi_j$’s

1) determine the n essential quantities
2) select $\hat{m}$ of the A quantities, with different dimensions, that contain among them the $\hat{m}$ dimensions, and use them as repeating variables together with one of the other A quantities to determine each $\Pi$. 
For example let \( A_1, A_2, \) and \( A_3 \) contain \( M, L, \) and \( t \) (not necessarily in each one, but collectively); then the \( \Pi_j \) parameters are formed as follows:

\[
\begin{align*}
\Pi_1 &= A_1^{x_1} A_2^{y_1} A_3^{z_1} A_4 \\
\Pi_2 &= A_1^{x_2} A_2^{y_2} A_3^{z_2} A_5 \\
\Pi_{n-m} &= A_1^{x_{n-m}} A_2^{y_{n-m}} A_3^{z_{n-m}} A_n
\end{align*}
\]

Determine exponents such that \( \Pi_i \)’s are dimensionless

3 equations and 3 unknowns for each \( \Pi_i \)

In these equations the exponents are determined so that each \( \Pi \) is dimensionless. This is accomplished by substituting the dimensions for each of the \( A_i \) in the equations and equating the sum of the exponents of \( M, L, \) and \( t \) each to zero. This produces three equations in three unknowns \( (x, y, t) \) for each \( \Pi \) parameter.

In using the above method, the designation of \( \hat{m} = m \) as the number of basic dimensions needed to express the \( n \) variables dimensionally is not always correct. The correct value for \( \hat{m} \) is the rank of the dimensional matrix, i.e., the next smaller square subgroup with a nonzero determinant.
Dimensional matrix =

\[
\begin{bmatrix}
A_1 & \ldots & A_n \\
a_{11} & \ldots & a_{1n} \\
a_{31} & \ldots & a_{3n} \\
o & \ldots & o \\
\vdots & \ddots & \vdots \\
o & \ldots & o
\end{bmatrix}
\]

\(a_{ij} = \text{exponent of } M, L, \text{ or } t \text{ in } A_i\)

n x n matrix

Rank of dimensional matrix equals size of next smaller sub-group with nonzero determinant

Example: Hydraulic jump (see section 15.2)
Say we assume that

\[ V_1 = V_1(\rho, g, \mu, y_1, y_2) \]

or \[ V_2 = V_1 y_1 / y_2 \]

Dimensional analysis is a procedure whereby the functional relationship can be expressed in terms of \( r \) nondimensional parameters in which \( r < n = \) number of variables. Such a reduction is significant since in an experimental or numerical investigation a reduced number of experiments or calculations is extremely beneficial.
1) \( \rho, g \) fixed; vary \( \mu \) 
2) \( \rho, \mu \) fixed; vary \( g \) 
3) \( \mu, g \) fixed; vary \( \rho \) 

Represents many, many experiments

In general: \( F(A_1, A_2, \ldots, A_n) = 0 \) dimensional form

\( f(\Pi_1, \Pi_2, \ldots, \Pi_r) = 0 \) nondimensional form with reduced # of variables

It can be shown that

\[
F_r = \frac{V_1}{\sqrt{gy_1}} = F_r \left( \frac{y_2}{y_1} \right)
\]

neglect \( \mu \) (\( \rho \) drops out as will be shown)
thus only need one experiment to determine the functional relationship

\[
F_r^2 = \frac{1}{2}(x - x^2)
\]

\[
F_r = \left[ \frac{1}{2} x(1 + x) \right]^{1/2}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( F_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>0.61</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1.7</td>
</tr>
<tr>
<td>5</td>
<td>3.9</td>
</tr>
</tbody>
</table>

For this particular application we can determine the functional relationship through the use of a control volume analysis: (neglecting \( \mu \) and bottom friction)
x-momentum equation: \( \sum F_x = \sum V_x \rho V \cdot A \)
\[
\frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} = V_1 \rho (-V_1 y_1) + V_2 \rho (V_2 y_2)
\]
\[
\frac{\gamma}{2} (y_1^2 - y_2^2) = \frac{\gamma}{g} (V_2 y_2 - V_1 y_1)
\]

continuity equation: \( V_1 y_1 = V_2 y_2 \)

\[ V_2 = \frac{V_1 y_1}{y_2} \]

\[
\frac{\gamma y_1^2}{2} \left[ 1 - \left( \frac{y_2}{y_1} \right)^2 \right] = V_1^2 \frac{\gamma}{g} y_1 \left( \frac{y_1}{y_2} - 1 \right)
\]

pressure forces due to gravity = inertial forces

now divide equation by \( \frac{(1 - \frac{y_2}{y_1}) y_1^3}{g y_2} \)

\[
\frac{V_1^2}{g y_1} = \frac{1}{2} \frac{y_2}{y_1} \left( 1 + \frac{y_2}{y_1} \right)
\]

dimensionless equation

ratio of inertia forces/gravity forces = \((\text{Froude number})^2\)

note: \( F_r = F_r(y_2/y_1) \) do not need to know both \( y_2 \) and \( y_1 \), only ratio to get \( F_r \)

Also, shows in an experiment it is not necessary to vary \( \gamma, y_1, y_2, V_1, \) and \( V_2 \), but only \( F_r \) and \( y_2/y_1 \)
Next, can get an estimate of $h_L$ from the energy equation (along free surface from 1→2)

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + h_L$$

$$h_L = \frac{(y_2 - y_1)^3}{4y_1y_2}$$

$\neq f(\mu)$ due to assumptions made in deriving 1-D steady flow energy equations

Exponent method to determine $\Pi_j$’s for Hydraulic jump

use $V_1$, $y_1$, $\rho$ as repeating variables

$\Pi_1 = \frac{V_1^{x_1} y_1^{y_1} \rho^{z_1} \mu}{(L T^{-1})^{x_1} (L)^{y_1} (ML^{-3})^{z_1} ML^{-1} T^{-1}}$

$L x_1 + y_1 - 3z_1 - 1 = 0 \quad y_1 = 3z_1 + 1 - x_1 = -1$

$T -x_1 - 1 = 0 \quad x_1 = -1$

$M z_1 + 1 = 0 \quad z_1 = -1$

$\Pi_1 = \frac{\mu}{\rho y_1 V_1}$ or $\Pi_1^{-1} = \frac{\rho y_1 V_1}{\mu} = \text{Reynolds number} = \text{Re}$

Assume $m = n$ to avoid evaluating rank of 6 x 6 dimensional matrix
\[ \Pi_2 = V_1^{x_2} y_1^{y_2} \rho^{z_2} g \]
\[ = (LT^{-1})^{x_2} (L)^{y_2} (ML^{-3})^{z_2} LT^{-2} \]

\[ L \quad x_2 + y_2 - 3z_2 + 1 = 0 \quad y_2 = -1 - x_2 = 1 \]
\[ T \quad -x_2 - 2 = 0 \quad x_2 = -2 \]
\[ M \quad z_2 = 0 \]
\[ \Pi_2 = V_1^{-2} y_1^g \pi_1 = \frac{g y_1}{V_1^2} \quad \Pi_2^{-1/2} = \frac{V_1}{\sqrt{g y_1}} = \text{Froude number} = Fr \]

\[ \Pi_3 = V_1^{x_3} y_1^{y_3} \rho^{z_3} y_2 \]
\[ = (LT^{-1})^{x_3} (L)^{y_3} (ML^{-3})^{z_3} L \]

\[ L \quad x_3 + y_3 - 3z_3 + 1 = 0 \quad y_3 = -1 \]
\[ T \quad -x_3 = 0 \]
\[ M \quad -3z_3 = 0 \]
\[ \Pi_3 = \frac{y_2}{y_1} \quad \Pi_3^{-1} = \frac{y_1}{y_2} = \text{depth ratio} \]
\[ f(\Pi_1, \Pi_2, \Pi_3) = 0 \]

or, \[ \Pi_2 = \Pi_2(\Pi_1, \Pi_3) \]

i.e., \[ Fr = Fr(Re, y_2/y_1) \]

if we neglect \( \mu \) then Re drops out
\[ Fr = \frac{V_1}{\sqrt{g y_1}} = f \left( \frac{y_2}{y_1} \right) \]

Note that dimensional analysis does not provide the actual functional relationship. Recall that previously we used control volume analysis to derive
\[ \frac{V_1^2}{gy_1} = \frac{1}{2} \frac{y_2}{y_1} \left( 1 + \frac{y_2}{y_1} \right) \]

the actual relationship between \( F \) vs. \( y_2/y_1 \)

\[ F = F(\text{Re}, \text{Fr}, y_1/y_2) \]

or

\[ \text{Fr} = \text{Fr}(\text{Re}, y_1/y_2) \]

dimensional matrix:

\[
\begin{bmatrix}
g & V_1 & y_1 & y_2 & \rho & \mu \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 3 & -1 \\
-2 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Size of next smaller subgroup with nonzero determinant = 3 = rank of matrix
Common Dimensionless Parameters for Fluid Flow Problems

Most common physical quantities of importance in fluid flow problems are: (without heat transfer)

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>V</td>
<td>ρ</td>
<td>g</td>
<td>μ</td>
<td>σ</td>
<td>K</td>
<td>Δp</td>
<td>L</td>
</tr>
</tbody>
</table>

velocity density gravity viscosity surface compressibility pressure length
tension change

\[ n = 8 \quad m = 3 \quad \Rightarrow \quad 5 \text{ dimensionless parameters} \]

1) **Reynolds number**
\[ \text{Re} = \frac{\rho V L}{\mu} \]
\[ R_{\text{crit}} \] distinguishes among flow regions: laminar or turbulent
value varies depending upon flow situation

2) **Froude number**
\[ \text{Fr} = \frac{V}{\sqrt{g L}} \]
important parameter in free-surface flows

3) **Weber number**
\[ \text{We} = \frac{\rho V^2 L}{\sigma} \]
important parameter at gas-liquid or liquid-liquid interfaces
and when these surfaces are in contact with a boundary

4) **Mach number**
\[ \text{Ma} = \frac{V}{\sqrt{a}} \]
Paramount importance in high speed flow \((V \geq c)\)

5) **Pressure Coefficient**
\( \frac{\Delta p}{\rho V^2} \) (Euler Number)
\[ \text{C}_p \]

- \( F_t = M a = \rho V^2 L \)
- \( f_i = \rho V^2 / L \)
- \( f_v = \mu V / L^2 \)
- \( f_g = \gamma L^2 \)
- \( f_o = \sigma / L^2 \)
Nondimensionalization of the Basic Equation

It is very useful and instructive to nondimensionalize the basic equations and boundary conditions. Consider the situation for \( \rho \) and \( \mu \) constant and for flow with a free surface.

Continuity: \[ \nabla \cdot \mathbf{V} = 0 \]

Momentum: \[ \rho \frac{D\mathbf{V}}{Dt} = -\nabla(p + \gamma z) + \mu \nabla^2 \mathbf{V} \]

\( \rho g = \) specific weight

Boundary Conditions:
1) fixed solid surface: \[ \mathbf{V} = 0 \]
2) inlet or outlet: \[ \mathbf{V} = \mathbf{V}_o \quad p = p_o \]
3) free surface: \[ w = \frac{\partial \eta}{\partial t} \quad p = p_a - \gamma \left( R_x^{-1} + R_y^{-1} \right) \]

\( (z = \eta) \) **surface tension**

All variables are now nondimensionalized in terms of \( \rho \) and \( U = \) reference velocity

\( L = \) reference length

\[ V^* = \frac{V}{U} \quad t^* = \frac{tU}{L} \]

\[ x^* = \frac{x}{L} \quad p^* = \frac{p + \rho gz}{\rho U^2} \]
All equations can be put in nondimensional form by making the substitution
\[ \mathbf{V} = \mathbf{V}^* \mathbf{U} \]

\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t^*} \frac{t^*}{t} = \frac{U}{L} \frac{\partial}{\partial t^*} \]

\[ \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \]

\[ = \frac{\partial}{\partial x^*} \frac{\partial x^*}{\partial x} \mathbf{i} + \frac{\partial}{\partial y^*} \frac{\partial y^*}{\partial y} \mathbf{j} + \frac{\partial}{\partial z^*} \frac{\partial z^*}{\partial z} \mathbf{k} \]

\[ = \frac{1}{L} \nabla^* \]

and \[ \frac{\partial u}{\partial x} = \frac{1}{L} \frac{\partial}{\partial x^*} \left( U u^* \right) = \frac{U}{L} \frac{\partial u^*}{\partial x^*} \] etc.

Result:
\[ \nabla^* \cdot \mathbf{V}^* = 0 \]

\[ \frac{D\mathbf{V}^*}{Dt} = -\nabla^* p^* + \frac{\mu}{\rho V L} \nabla^* V^* \]

1) \[ \mathbf{V}^* = 0 \]

2) \[ \mathbf{V}^* = \frac{V_o}{U} \quad p^* = \frac{p_o}{\rho V^2} \]

3) \[ w^* = \frac{\partial \eta^*}{\partial t^*} \quad p^* = \frac{p_o}{\rho V^2} + \frac{gL}{U^2} z^* + \frac{\gamma}{\rho V^2 L} \left( R_x^* + R_y^* \right) \]

pressure coefficient \[ \frac{Fr^2}{We^{-1}} \]

\[ V = U \]
**Similarity and Model Testing**

Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for model and prototype

\[ \Pi_{i\text{ model}} = \Pi_{i\text{ prototype}} \quad i = 1, r = n - \hat{m} \ (m) \]

Enables extrapolation from model to full scale

However, complete similarity usually not possible

Therefore, often it is necessary to use Re, or Fr, or Ma scaling, i.e., select most important \( \Pi \) and accommodate others as best possible

Types of Similarity:

1) Geometric Similarity (similar length scales):
A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratios

\[ \alpha = \frac{L_m}{L_p} \quad (\alpha < 1) \]

1/10 or 1/50

2) Kinematic Similarity (similar length and time scales):
The motions of two systems are kinematically similar if homologous (same relative position) particles lie at homologous points at homologous times
3) Dynamic Similarity (similar length, time and force (or mass) scales):
in addition to the requirements for kinematic similarity the model and prototype forces must be in a constant ratio

Model Testing in Water (with a free surface)

\[ F(D, L, V, g, \rho, v) = 0 \]

\[ \Rightarrow n = 6 \text{ and } m = 3 \text{ thus } r = n - m = 3 \text{ pi terms} \]

In a dimensionless form,

\[ f(C_D, Fr, Re) = 0 \]

or \[ C_D = f(Fr, Re) \]

where

\[ C_D = \frac{D}{\frac{1}{2} \rho V^2 L^2} \]
\[ Fr = \frac{V}{\sqrt{gL}} \]
\[ Re = \frac{V_L}{v} \]

If \[ Fr_m = Fr_p \] or \[ \frac{V_m}{\sqrt{gL}_m} = \frac{V_p}{\sqrt{gL}_p} \]

\[ V_m = \sqrt{\alpha} V_p \] Froude scaling
and \( \text{Re}_m = \text{Re}_p \) or \( \frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \)

\[ \frac{\nu_m}{\nu_p} = \frac{V_m L_m}{V_p L_p} = \alpha^{3/2} \]

Then,

\[ C_{D_m} = C_{D_p} \text{ or } \frac{D_m}{\rho_m \nu_m^2 L_m^2} = \frac{D_p}{\rho_p \nu_p^2 L_p^2} \]

However, impossible to achieve, since

if \( \alpha = 1/10 \), \( \nu_m = 3.1 \times 10^{-8} \ m^2/s < 1.2 \times 10^{-7} \ m^2/s \)

For mercury \( \nu = 1.2 \times 10^{-7} \ m^2/s \)

Alternatively one could maintain \( \text{Re} \) similarity and obtain

\[ V_m = V_p / \alpha \]

But, if \( \alpha = 1/10 \), \( V_m = 10V_p \),

High speed testing is difficult and expensive.

\[ \frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p} \]

\[ \frac{g_m}{g_p} = \frac{V_m^2}{V_p^2} \frac{L_p}{L_m} \]
\[
\frac{g_m}{g_p} = \frac{V_m^2}{V_p^2} \frac{L_p}{L_m}
\]

But if \( \alpha = 1/10 \), \( g_m = 1000 g_p \)
Impossible to achieve

Model Testing in Air

\[ F(D, L, V, \rho, \nu, a) = 0 \]

\[ \Rightarrow n = 6 \text{ and } m = 3 \text{ thus } r = n - m = 3 \pi \text{ terms} \]

In a dimensionless form,

\[ f(CD, Re, Ma) = 0 \]

or

\[ C_D = f(Re, Ma) \]

where

\[ C_D = \frac{D}{\frac{1}{2} \rho \nu^2 L^2} \]

\[ Re = \frac{\nu L}{\nu} \]

\[ Ma = \frac{\nu}{a} \]
If \[ \frac{V_m L_m}{v_m} = \frac{V_p L_p}{v_p} \]

and \[ \frac{V_m}{a_m} = \frac{V_p}{a_p} \]

Then, \[ C_{Dm} = C_{Dp} \text{ or } \frac{D_m}{\rho_m v_m^2 L_m^2} = \frac{D_p}{\rho_p v_p^2 L_p^2} \]

However, \[ \frac{v_m}{v_p} = \frac{L_m}{L_p} \left[ \frac{a_m}{a_p} \right] = \alpha \]

again not possible

Therefore, in wind tunnel testing Re scaling is also violated
Model Studies w/o free surface
\[ c_p = \frac{\Delta p}{\frac{1}{2} \rho V^2} \]
High Re

Model Studies with free surface

In hydraulics model studies, Fr scaling used, but lack of We similarity can cause problems. Therefore, often models are distorted, i.e. vertical scale is increased by 10 or more compared to horizontal scale

Ship model testing:

\[ C_T = f(Re, Fr) = C_w(Fr) + C_v(Re) \]

\[ V_m \text{ determined for Fr scaling} \]

\[ C_{wm} = C_{Tm} - C_v \]
\[ C_{Ts} = C_{wm} + C_v \]

Based on flat plate of same surface area