Example 1

Consider the steady, two-dimensional velocity field given by $\vec{V} = (u, v) = (1.3 + 2.8x)\vec{i} + (1.5 - 2.8y)\vec{j}$. Verify that this flow field is incompressible.

Solution of Example 1:

Check if $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$?

$u=1.3+2.8x$, $\frac{\partial u}{\partial x}=2.8$

$v=1.5-2.8y$, $\frac{\partial v}{\partial y}=-2.8$

Since $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$, the given flow field is incompressible.

Example 2

The u velocity component of a steady, two-dimensional, incompressible flow field is $u = ax^2 - bxy$, where a and b are constants. Velocity component v is unknown. Generate an expression for v as a function of x and y.

Solution of Example 2:

Since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

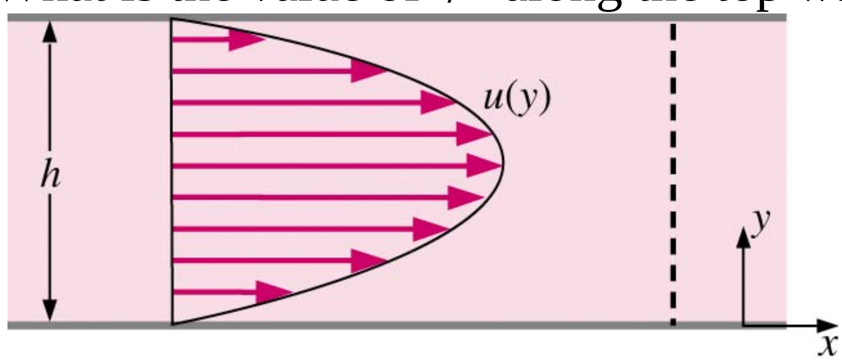
$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2ax + by$$

Integrate with respect to y:

$$v = -2axy + \frac{1}{2}by^2 + f(x)$$

Example 3

Consider fully developed, two-dimensional channel flow—flow between two infinite parallel plates separated by distance h, with both the top plate and bottom plate stationary, and a forced pressure gradient dP/dx driving the flow as illustrated in the figure. (dP/dx is constant and negative.) The flow is steady, incompressible, and two-dimensional in the xy-plane. The velocity components are given by $u = (1/2\mu)(dP/dx)(y^2 - hy)$ and $v = 0$, where $\mu$ is the fluid's viscosity. Generate an expression for stream function $\psi$ along the vertical dashed line in the Figure. For convenience, let $\psi = 0$ along the bottom wall of the channel. What is the value of $\psi$ along the top wall?



Solution of Example 3:

Start with the definition of stream function:

$$\frac{\partial \psi}{\partial y} = u = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)$$

Integrate with respect to y:

$$\psi = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right) + g\left(x\right)$$

Use the definition of stream function again:

$$0 = v = -\frac{\partial \psi}{\partial x} = -g'\left(x\right)$$

$$g\left(x\right) = C$$

So the stream function is

$$\psi = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right) + C$$

Use the BC for $\psi$ to determine the constant $C$

$\psi = 0$ along $y = 0$: $C = 0$

So the stream function is

$$\psi = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right)$$

Stream function along the top wall $y = h$:

$$\psi_{top} = -\frac{1}{12\mu}\frac{dP}{dx}h^3$$

Example 4

As a follow-up to Example 3, calculate the volume flow rate per unit width into the page of Figure 3 from first principles (integration of the velocity field). Compare your result to that obtained directly from the stream function. Discuss.

Solution of Example 4:

Integrate the velocity along y direction to obtain the volume flow rate:

$$Q = \int_A u\,dA = \int_{y=0}^{y=h} \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right) W dy \,.$$

$$= \left[\frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right)W\right]_{y=0}^{y=h} = \frac{1}{2\mu}\frac{dP}{dx}\left(-\frac{h^3}{6}\right)W$$

$$= -\frac{1}{12\mu}\frac{dP}{dx}h^3 W$$

Where W is the width of the channel into the page. So the volume flow rate per unit width

$$\frac{Q}{W} = -\frac{1}{12\mu}\frac{dP}{dx}h^3$$

The volume flow rate per unit width obtained directly from the stream function according to Example 3:

$$\frac{Q}{W} = \psi_{top} - \psi_{bottom} = -\frac{1}{12\mu}\frac{dP}{dx}h^3 - 0 = -\frac{1}{12\mu}\frac{dP}{dx}h^3$$

Results from both approaches agree.