Example 1
Consider the steady, two-dimensional velocity field given by \( \vec{V} = (u, v) = (1.3 + 2.8x) \vec{i} + (1.5 - 2.8y) \vec{j} \). Verify that this flow field is incompressible.
Solution of Example 1:

Check if \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \) ?

\[ u = 1.3 + 2.8x, \quad \frac{\partial u}{\partial x} = 2.8 \]

\[ v = 1.5 - 2.8y, \quad \frac{\partial v}{\partial y} = -2.8 \]

Since \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \), the given flow field is incompressible.
Example 2
The $u$ velocity component of a steady, two-dimensional, incompressible flow field is $u = ax^2 - bxy$, where $a$ and $b$ are constants. Velocity component $v$ is unknown. Generate an expression for $v$ as a function of $x$ and $y$. 
Solution of Example 2:

Since \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \)

\[ \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2ax + by \]

Integrate with respect to \( y \):

\[ v = -2axy + \frac{1}{2}by^2 + f(x) \]
Example 3
Consider fully developed, two-dimensional channel flow—flow between two infinite parallel plates separated by distance $h$, with both the top plate and bottom plate stationary, and a forced pressure gradient $\frac{dP}{dx}$ driving the flow as illustrated in the figure. ($\frac{dP}{dx}$ is constant and negative.) The flow is steady, incompressible, and two-dimensional in the xy-plane. The velocity components are given by $u = \frac{1}{2\mu} \left( \frac{dP}{dx} \right) (y^2 - hy)$ and $v = 0$, where $\mu$ is the fluid’s viscosity. Generate an expression for stream function $\psi$ along the vertical dashed line in the Figure. For convenience, let $\psi = 0$ along the bottom wall of the channel. What is the value of $\psi$ along the top wall?
Solution of Example 3:

Start with the definition of stream function:
\[ \frac{\partial \psi}{\partial y} = u = \frac{1}{2\mu} \frac{dP}{dx} \left( y^2 - hy \right) \]

Integrate with respect to y:
\[ \psi = \frac{1}{2\mu} \frac{dP}{dx} \left( \frac{y^3}{3} - h\frac{y^2}{2} \right) + g(x) \]

Use the definition of stream function again:
\[ 0 = v = -\frac{\partial \psi}{\partial x} = -g'(x) \]
\[ g(x) = C \]
So the stream function is
\[ \psi = \frac{1}{2\mu} \frac{dP}{dx} \left( \frac{y^3}{3} - h\frac{y^2}{2} \right) + C \]

Use the BC for \( \psi \) to determine the constant \( C \)
\[ \psi = 0 \text{ along } y = 0: \quad C = 0 \]
So the stream function is
\[ \psi = \frac{1}{2\mu} \frac{dP}{dx} \left( \frac{y^3}{3} - h\frac{y^2}{2} \right) \]

Stream function along the top wall \( y = h \):
\[ \psi_{top} = -\frac{1}{12\mu} \frac{dP}{dx} h^3 \]
Example 4
As a follow-up to Example 3, calculate the volume flow rate per unit width into the page of Figure 3 from first principles (integration of the velocity field). Compare your result to that obtained directly from the stream function. Discuss.
Solution of Example 4:

Integrate the velocity along y direction to obtain the volume flow rate:

\[ Q = \int_A u dA = \int_{y=0}^{y=h} \frac{1}{2\mu} \frac{dP}{dx} \left( y^2 - hy \right) W dy. \]

\[ = \left[ \frac{1}{2\mu} \frac{dP}{dx} \left( \frac{y^3}{3} - \frac{h y^2}{2} \right) W \right]_{y=0}^{y=h} = \frac{1}{2\mu} \frac{dP}{dx} \left( -\frac{h^3}{6} \right) W \]

\[ = -\frac{1}{12\mu} \frac{dP}{dx} h^3 W \]

Where \( W \) is the width of the channel into the page. So the volume flow rate per unit width

\[ \frac{Q}{W} = -\frac{1}{12\mu} \frac{dP}{dx} h^3 \]

The volume flow rate per unit width obtained directly from the stream function according to Example 3:

\[ \frac{Q}{W} = \psi_{top} - \psi_{bottom} = -\frac{1}{12\mu} \frac{dP}{dx} h^3 - 0 = -\frac{1}{12\mu} \frac{dP}{dx} h^3 \]

Results from both approaches agree.