Chapter 5 Mass, Momentum, and Energy Equations

Flow Rate and Conservation of Mass

1. cross-sectional area oriented normal to velocity vector (simple case where $V \perp A$)

$U = \text{constant}: \quad Q = \text{volume flux} = UA \ [m/s \times m^2 = m^3/s]$

$U \neq \text{constant}: \quad Q = \int_A UdA$

Similarly the mass flux = $\dot{m} = \int_A \rho UdA$

2. general case

$Q = \int_{CS} V \cdot ndA$

$= \int_{CS} |V| \cos \theta dA$

$\dot{m} = \int_{CS} \rho (V \cdot n) dA$
average velocity: \( \bar{V} = \frac{Q}{A} \)

Example:
At low velocities the flow through a long circular tube, i.e. pipe, has a parabolic velocity distribution (actually paraboloid of revolution).

\[
u = u_{\text{max}} \left(1 - \left(\frac{r}{R}\right)^2\right)
\]

i.e., centerline velocity

a) find \( Q \) and \( \bar{V} \)

\[
Q = \int_{A} V \cdot ndA = \int_{A} udA
\]

\[
\int_{A} udA = \int_{0}^{2\pi} \int_{0}^{R} u(r)rd\theta dr
\]

\[
= 2\pi \int_{0}^{R} u(r)r dr
\]

\[
dA = 2\pi rdr
\]

\[
u = u(r) \text{ and not } \theta \therefore \int_{0}^{2\pi} d\theta = 2\pi \]
\[ Q = 2\pi \int_0^R u_{\text{max}} \left( 1 - \left( \frac{r}{R} \right)^2 \right) r \, dr = \frac{1}{2} u_{\text{max}} \pi R^2 \]

\[ \bar{V} = \frac{1}{2} u_{\text{max}} \]

**Continuity Equation**

RTT can be used to obtain an integral relationship expressing conservation of mass by defining the extensive property \( B = M \) such that \( \beta = 1 \).

\[
B = M = \text{mass} \\
\beta = \frac{dB}{dM} = 1
\]

**General Form of Continuity Equation**

\[
\frac{dM}{dt} = 0 = \frac{d}{dt_{CV}} \int \rho dV + \int \rho \bar{V} \cdot dA
\]

or

\[
\int_{CS} \rho \bar{V} \cdot dA = -\frac{d}{dt_{CV}} \int \rho dV
\]

- net rate of outflow
- rate of decrease of
- of mass across CS
- mass within CV

**Simplifications:**

1. **Steady flow:** \(- \frac{d}{dt_{CV}} \int \rho dV = 0\)
2. \( \mathbf{V} = \text{constant over discrete } dA \) (flow sections):

\[
\int_{CS} \rho \mathbf{V} \cdot dA = \sum_{CS} \rho \mathbf{V} \cdot A
\]

3. Incompressible fluid (\( \rho = \text{constant} \))

\[
\int_{CS} \mathbf{V} \cdot dA = -\frac{d}{dt} \int_{CV} dV
\]

conservation of volume

4. Steady One-Dimensional Flow in a Conduit:

\[
\sum_{CS} \rho \mathbf{V} \cdot A = 0
\]

\[-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0\]

for \( \rho = \text{constant} \) \( \quad Q_1 = Q_2 \)

Some useful definitions:

Mass flux \( \dot{m} = \int_{A} \rho \mathbf{V} \cdot dA \)

Volume flux \( Q = \int_{A} \mathbf{V} \cdot dA \)

Average Velocity \( \bar{V} = \frac{Q}{A} \)

Average Density \( \bar{\rho} = \frac{1}{A} \int \bar{\rho} dA \)

Note: \( \dot{m} \neq \bar{\rho} Q \) unless \( \rho = \text{constant} \)
Example

*Steady flow

*V_{1,2,3} = 50 \text{ fps}

*@ V varies linearly from zero at wall to \( V_{\text{max}} \) at pipe center

*find \( \dot{m}_4, Q_4, V_{\text{max}} \)

*water, \( \rho_w = 1.94 \text{ slug/ft}^3 \)

\[
\int \rho V \cdot dA = 0 = -\frac{d}{dt} \int \rho dV
\]

i.e., \(-\rho_1 V_1 A_1 - \rho_2 V_2 A_2 + \rho_3 V_3 A_3 + \rho \int \frac{V_4 dA_4}{A_4} = 0\)

\( \rho = \text{const.} = 1.94 \text{ lb-s}^2 /\text{ft}^4 = 1.94 \text{ slug/ft}^3 \)

\[
\dot{m}_4 = \rho \int V_4 dA_4 = \rho V (A_1 + A_2 - A_3) \quad V_1 = V_2 = V_3 = V = 50 \text{ f/s}
\]

\[
= \frac{1.94}{144} \times 50 \times \frac{\pi}{4} \left( l^2 + 2^2 - 1.5^2 \right)
\]

\[= 1.45 \text{ slugs/s} \]
\[ Q_4 = \frac{m_4}{\rho} = 0.75 \text{ ft}^3/\text{s} \]

\[ = \int V_4 \, dA_4 \]

\[ Q_4 = \int_{0}^{r_o} \int_{0}^{2\pi} V_{\text{max}} \left( 1 - \frac{r}{r_o} \right) r \, d\theta \, dr \]

velocity profile

\[ V_4 \neq V_4(\theta) \]

\[ = 2\pi \int_{0}^{r_o} V_{\text{max}} \left( 1 - \frac{r}{r_o} \right) r \, dr \]

\[ = 2\pi V_{\text{max}} \int_{0}^{r_o} \left[ r - \frac{r^2}{r_o} \right] \, dr \]

\[ = 2\pi V_{\text{max}} \left[ \frac{r^2}{2} \bigg|_{0}^{r_o} - \frac{r^3}{3r_o} \bigg|_{0}^{r_o} \right] \]

\[ = 2\pi V_{\text{max}} r_o^2 \left[ \frac{1}{2} - \frac{1}{3} \right] = \frac{1}{3} \pi r_o^2 V_{\text{max}} \]

\[ V_{\text{max}} = \frac{Q_4}{\frac{1}{3} \pi r_o^2} = 2.86 \text{ fps} \]
Momentum Equation

RTT with $B = M\overrightarrow{V}$ and $\beta = \overrightarrow{V}$

$$\sum [F_S + F_B] = \frac{d}{dt} \int \rho \overrightarrow{V} d\overrightarrow{V} + \int \rho \overrightarrow{V}_R \cdot dA$$

$\overrightarrow{V}$ = velocity referenced to an inertial frame (non-accelerating)
$\overrightarrow{V}_R$ = velocity referenced to control volume
$F_S$ = surface forces + reaction forces (due to pressure and viscous normal and shear stresses)
$F_B$ = body force (due to gravity)

Applications of the Momentum Equation

Initial Setup and Signs
1. Jet deflected by a plate or a vane
2. Flow through a nozzle
3. Forces on bends
4. Problems involving non-uniform velocity distribution
5. Motion of a rocket
6. Force on rectangular sluice gate
7. Water hammer

Derivation of the Basic Equation

Recall RTT:

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \int \beta \rho d\overrightarrow{V} + \int \beta \rho \overrightarrow{V}_R \cdot dA$$

General form for moving but non-accelerating reference frame

$\overrightarrow{V}_R$ = velocity relative to CS = $\overrightarrow{V} - \overrightarrow{V}_S$ = absolute – velocity CS

Subscript not shown in text but implied!

i.e., referenced to CV

Let, $B = M\overrightarrow{V}$ = linear momentum

$\beta = \overrightarrow{V}$

$\overrightarrow{V}$ must be referenced to inertial reference frame
\[
\frac{d(MV)}{dt} = \sum \vec{F} = \int_{CV} \rho \vec{V} \cdot d\vec{A} + \int_{CS} \rho \vec{V}_R \cdot d\vec{A} \]

Newton's 2nd law

where \( \sum \vec{F} \) = vector sum of all forces acting on the control volume including both surface and body forces

\[
\sum F = \sum F_S + \sum F_B
\]

\( \sum F_S \) = sum of all external surface forces acting at the CS, i.e., pressure forces, forces transmitted through solids, shear forces, etc.

\( \sum F_B \) = sum of all external body forces, i.e., gravity force

\[
\begin{align*}
\sum F_x &= p_1 A_1 - p_2 A_2 + R_x \\
\sum F_y &= -W + R_y 
\end{align*}
\]

\( R \) = resultant force on fluid in CV due to \( p_w \) and \( \tau_w \), i.e., reaction force on fluid

Important Features (to be remembered)

1) Vector equation to get component in any direction must use dot product

\[
\sum F_x = \frac{d}{dt} \int_{CV} \rho u dV + \int_{CS} \rho u \vec{V}_R \cdot dA
\]

Carefully define coordinate system with forces positive in positive direction of coordinate axes
y equation
\[ \sum F_y = \frac{d}{dt} \int \rho v dV + \int \rho v V_R \cdot dA \]

z equation
\[ \sum F_z = \frac{d}{dt} \int \rho w dV + \int \rho w V_R \cdot dA \]

2) Carefully define control volume and be sure to include all external body and surface faces acting on it. For example,

3) Velocity \( V \) must be referenced to a non-accelerating inertial reference frame. Sometimes it is advantageous to use a moving (at constant velocity) reference frame. Note \( V_R = V - V_s \) is always relative to CS.

\[(R_x, R_y) = \text{reaction force on fluid}\]
\[(R_x, R_y) = \text{reaction force on nozzle}\]

i.e., in these cases \( V \) used for B also referenced to CV (i.e., \( \vec{V} = \vec{V}_B \))
4) Steady vs. Unsteady Flow

\[ \text{Steady flow} \Rightarrow \frac{d}{dt} \int_{CV} \rho V dV = 0 \]

5) Uniform vs. Nonuniform Flow

\[ \int_{CS} V \rho V_R \cdot dA = \text{change in flow of momentum across } CS = \sum V \rho V_R \cdot A \text{ uniform flow across } A \]

6) \( F_{\text{pres}} = -\int p \cdot dA \quad \int \nabla f dV = \int_{\Sigma} f nds \quad f = \text{constant, } \nabla f = 0 \)

= 0 for p = constant and for a closed surface

i.e., always use gage pressure

7) Pressure condition at a jet exit

at an exit into the atmosphere jet pressure must be \( p_a \)

Application of the Momentum Equation
1. Jet deflected by a plate or vane

Consider a jet of water turned through a horizontal angle
x-equation: \[ \sum F_x = F_x = \frac{d}{dt} \int \rho u dV + \sum_{CS} \rho u \cdot dA \]
\[ F_x = \sum_{CS} \rho u \cdot A \] steady flow
\[ = \rho V_{1x} (-V_1 A_1) + \rho V_{2x} (V_2 A_2) \]

continuity equation: \[ \rho A_1 V_1 = \rho A_2 V_2 = \rho Q \] for \( A_1 = A_2 \)
\[ V_1 = V_2 \]
\[ F_x = \rho Q (V_{2x} - V_{1x}) \]

y-equation: \[ \sum F_y = F_y = \sum_{CS} \rho v \cdot A \]
\[ F_y = \rho V_{1y} (-A_1 V_1) + \rho V_{2y} (-A_2 V_2) \]
\[ = \rho Q (V_{2y} - V_{1y}) \] for above geometry only

where: \[ V_{1x} = V_1 \quad V_{2x} = -V_2 \cos \theta \quad V_{2y} = -V_2 \sin \theta \quad V_{1y} = 0 \]

note: \[ F_x \text{ and } F_y \text{ are force on fluid} \]
\[ -F_x \text{ and } -F_y \text{ are force on vane due to fluid} \]

If the vane is moving with velocity \( V_v \), then it is convenient to choose \( CV \) moving with the vane.
i.e., \( V_R = V - V_v \) and \( V \) used for B also moving with vane

**x-equation:**

\[
F_x = \int \rho u V_R \cdot dA
\]

\[
F_x = \rho V_{1x}[-(V - V_v)A_1] + \rho V_{2x}[-(V - V_v)A_2]
\]

**Continuity:**

\[
0 = \int \rho V_R \cdot dA
\]

i.e., \( \rho (V - V_v)A_1 = \rho (V - V_v)A_2 = \rho (V - V_v)A \)

\[
F_x = \rho (V - V_v)A [V_{2x} - V_{1x}]
\]

on fluid

\[
V_{2x} = (V - V_v)_{2x} \]

\[
V_{1x} = (V - V_v)_{1x}
\]

**Power =** \(-F_x V_v\)  

i.e., = 0 for \( V_v = 0 \)

\[
F_y = \rho Q_{rel} (V_{2y} - V_{1y})
\]

2. Flow through a nozzle

Consider a nozzle at the end of a pipe (or hose). What force is required to hold the nozzle in place?

CV = nozzle and fluid

\( (R_x, R_y) = \) force required to hold nozzle in place
Assume either the pipe velocity or pressure is known. Then, the unknown (velocity or pressure) and the exit velocity \( V_2 \) can be obtained from combined use of the continuity and Bernoulli equations.

**Bernoulli:**

\[
p_1 + \gamma z_1 + \frac{1}{2} \rho V_1^2 = p_2 + \gamma z_2 + \frac{1}{2} \rho V_2^2 \quad z_1 = z_2
\]

\[
p_1 + \frac{1}{2} \rho V_1^2 = \frac{1}{2} \rho V_2^2
\]

**Continuity:**

\[
A_1 V_1 = A_2 V_2 = Q
\]

\[
V_2 = \frac{A_1}{A_2} V_1 = \left( \frac{D}{d} \right)^2 V_1
\]

\[
p_1 + \frac{1}{2} \rho V_1^2 \left( 1 - \left( \frac{D}{d} \right)^4 \right) = 0
\]

Say \( p_1 \) known:

\[
V_1 = \left[ \frac{-2p_1}{\rho \left( 1 - \left( \frac{D}{d} \right)^4 \right)} \right]^{1/2}
\]

To obtain the reaction force \( R_x \) apply momentum equation in x-direction

\[
\sum F_x = \frac{d}{dt} \int_{CV} u \rho dV + \int_{CS} \rho u V \cdot dA
\]

\[
= \sum_{CS} \rho u V \cdot A \quad \text{steady flow and uniform flow over CS}
\]

\[
R_x + p_1 A_1 - p_2 A_2 = \rho V_1 (-V_1 A_1) + \rho V_2 (V_2 A_2)
\]

\[
= \rho Q (V_2 - V_1)
\]
\[ R_x = \rho Q(V_2 - V_1) - p_1 A_1 \]

To obtain the reaction force \( R_y \) apply momentum equation in y-direction

\[ \sum F_y = \sum \rho v \mathbf{V} \cdot \mathbf{A} = 0 \quad \text{since no flow in y-direction} \]

\[ R_y - W_f - W_N = 0 \quad \text{i.e., } R_y = W_f + W_N \]

Numerical Example: Oil with \( S = .85 \) flows in pipe under pressure of 100 psi. Pipe diameter is 3” and nozzle tip diameter is 1”

\[
\begin{align*}
\rho &= \frac{S \gamma}{g} = 1.65 \\
V_1 &= 14.59 \text{ ft/s} \\
V_2 &= 131.3 \text{ ft/s} \\
\frac{\pi}{4} \left( \frac{1}{12} \right)^2 V_2 &= .716 \text{ ft}^3/\text{s} \\
\end{align*}
\]

\[
\begin{align*}
R_x &= 141.48 - 706.86 = -569 \text{ lbf} \\
R_z &= 10 \text{ lbf} \\
\end{align*}
\]

This is force on nozzle

3. Forces on Bends

Consider the flow through a bend in a pipe. The flow is considered steady and uniform across the inlet and outlet sections. Of primary concern is the force required to hold the bend in place, i.e., the reaction forces \( R_x \) and \( R_y \) which can be determined by application of the momentum equation.
Continuity: \[ 0 = \sum \rho V \cdot A = -\rho V_1 A_1 + \rho V_2 A_2 \]
i.e., \( Q = \) constant = \( V_1 A_1 = V_2 A_2 \)

x-momentum: \[ \sum F_x = \sum \rho u V \cdot A \]
\[ p_1 A_1 - p_2 A_2 \cos \theta + R_x = \rho V_{1x} (-V_1 A_1) + \rho V_{2x} (V_2 A_2) \]
\[ = \rho Q (V_{2x} - V_{1x}) \]

y-momentum: \[ \sum F_y = \sum \rho v V \cdot A \]
\[ p_2 A_2 \sin \theta + R_y - w_f - w_b = \rho V_{1y} (-V_1 A_1) + \rho V_{2y} (V_2 A_2) \]
\[ = \rho Q (V_{2y} - V_{1y}) \]

4. Problems involving Nonuniform Velocity Distribution
See text pp. 215–216
5. Force on a rectangular sluice gate
The force on the fluid due to the gate is calculated from the x-momentum equation:

\[ \sum F_x = \sum \rho u V \cdot A \]

\[ F_1 + F_{GW} - F_{visc} - F_2 = \rho V_1 (-V_1 A_1) + \rho V_2 (V_2 A_2) \]

\[ F_{GW} = F_2 - F_1 + \rho Q (V_2 - V_1) + F_{visc} \]

usually can be neglected

\[ F_{GW} = \frac{1}{2} b \gamma \left( y_2^2 - y_1^2 \right) + \rho Q (V_2 - V_1) \]

\[ V_1 = \frac{Q}{y_1 b} \]

\[ V_2 = \frac{Q}{y_2 b} \]

Moment of Momentum Equation
See text pp. 221 – 229
**Energy Equations**

**Derivation of the Energy Equation**

The First Law of Thermodynamics
The difference between the heat added to a system and the work done by a system depends only on the initial and final states of the system; that is, depends only on the change in energy \( E \): principle of conservation of energy

\[
\Delta E = Q - W
\]

\( \Delta E \) = change in energy
\( Q \) = heat added to the system
\( W \) = work done by the system

\[ E = E_u + E_k + E_p = \text{total energy of the system} \]

**Internal energy due to molecular motion**

The differential form of the first law of thermodynamics expresses the rate of change of \( E \) with respect to time

\[
\frac{dE}{dt} = \dot{Q} - \dot{W}
\]

\( \frac{dE}{dt} \) = rate of work being done by system
\( \dot{Q} - \dot{W} \) = rate of heat transfer to system
Energy Equation for Fluid Flow

The energy equation for fluid flow is derived from Reynolds transport theorem with

\[ B_{\text{system}} = E = \text{total energy of the system (extensive property)} \]

\[ \beta = \frac{E}{\text{mass}} = e = \text{energy per unit mass (intensive property)} \]

\[ = \hat{u} + e_k + e_p \]

\[ \frac{dE}{dt} = \frac{d}{dt} \int_{CV} \rho e dV + \int_{CS} \rho eV \cdot dA \]

\[ \dot{Q} - \dot{W} = \frac{d}{dt} \int_{CV} \rho (\hat{u} + e_k + e_p) dV + \int_{CS} \rho (\hat{u} + e_k + e_p) V \cdot dA \]

This can be put in a more useable form by noting the following:

\[ e_k = \frac{\text{Total KE of mass with velocity } V}{\text{mass}} = \frac{\Delta MV^2}{2} = \frac{V^2}{2} \]

\[ e_p = \frac{\frac{E_p}{\Delta M}}{\frac{\rho \Delta V}{\rho \Delta V}} = g z \]  

(for \( E_p \) due to gravity only)

\[ \dot{Q} - \dot{W} = \frac{d}{dt} \int_{CV} \rho \left( \frac{V^2}{2} + gz + \hat{u} \right) dV + \int_{CS} \rho \left( \frac{V^2}{2} + gz + \hat{u} \right) V \cdot dA \]

rate of work done by system  
rate of change of energy in CV  
flux of energy out of CV (ie, across CS)  
rate of heat transfer to system
Rate of Work Components: \( \dot{W} = \dot{W}_s + \dot{W}_f \)

For convenience of analysis, work is divided into shaft work \( W_s \) and flow work \( W_f \)

\( W_f = \) net work done on the surroundings as a result of normal and tangential stresses acting at the control surfaces

\[= W_{f\text{pressure}} + W_{f\text{shear}} \]

\( W_s = \) any other work transferred to the surroundings usually in the form of a shaft which either takes energy out of the system (turbine) or puts energy into the system (pump)

Flow work due to pressure forces \( W_{f_p} \) (for system)

Note: here \( \mathbf{V} \) uniform over \( A \)

\( W_{f_p} = pA \times V \Delta t \) (on surroundings)

rate of work \( \Rightarrow \dot{W}_{f_p} = p_2 A_2 V_2 = p_2 V_2 \cdot A_2 \)

\( \dot{W}_1 = -p_1 A_1 \times V_1 \Delta t \)

\( \dot{W}_1 = p_1 \mathbf{V}_1 \cdot \mathbf{A}_1 \)

neg. sign since pressure force on surrounding fluid acts in a direction opposite to the motion of the system boundary
In general,

\[ \dot{W}_{fp} = pV \cdot A \]

for more than one control surface and \( V \) not necessarily uniform over \( A \):

\[ \dot{W}_{fp} = \int_{CS} pV \cdot dA = \int_{CS} \rho \left( \frac{p}{\rho} \right) V \cdot dA \]

\[ \dot{W}_f = \dot{W}_{fp} + \dot{W}_{f,\text{shear}} \]

Basic form of energy equation

\[ \dot{Q} - \dot{W}_s - \dot{W}_{f,\text{shear}} - \int_{CS} \rho \left( \frac{p}{\rho} \right) V \cdot dA = \frac{d}{dt} \int_{CV} \rho \left( \frac{V^2}{2} + gz + \dot{h} \right) dV + \int_{CS} \rho \left( \frac{V^2}{2} + gz + \dot{h} \right) V \cdot dA \]

\[ \dot{Q} - \dot{W}_s - \dot{W}_{f,\text{shear}} = \frac{d}{dt} \int_{CV} \rho \left( \frac{V^2}{2} + gz + \dot{h} \right) dV + \int_{CS} \rho \left( \frac{V^2}{2} + gz + \dot{h} + \frac{p}{\rho} \right) V \cdot dA \]

Usually this term can be eliminated by proper choice of \( CV \), i.e. CS normal to flow lines. Also, at fixed boundaries the velocity is zero (no slip condition) and no shear stress flow work is done. Not included or discussed in text!
**Simplified Forms of the Energy Equation**

Energy Equation for Steady One-Dimensional Pipe Flow

Consider flow through the pipe system as shown

\[
\dot{Q} - \dot{W}_s = \int_{CS} \rho \left( \frac{V^2}{2} + gz + \frac{p}{\rho} + \hat{u} \right) V \cdot dA
\]

\[
\dot{Q} - \dot{W}_s + \int_{A_1} \left( \frac{p_1}{\rho} + gz_1 + \hat{u}_1 \right) \rho_1 V_1 A_1 + \int_{A_1} \frac{\rho_1 V_1^3}{2} dA_1
\]

\[
= \int_{A_2} \left( \frac{p_2}{\rho} + gz_2 + \hat{u}_2 \right) \rho_2 V_2 A_2 + \int_{A_2} \frac{\rho_2 V_2^3}{2} dA_2
\]

*Although the velocity varies across the flow sections the streamlines are assumed to be straight and parallel; consequently, there is no acceleration normal to the streamlines and the pressure is hydrostatically distributed, i.e., \( \frac{p}{\rho} + gz = \) constant.*
*Furthermore, the internal energy \( u \) can be considered as constant across the flow sections, i.e. \( T = \text{constant} \). These quantities can then be taken outside the integral sign to yield

\[
\dot{Q} - \dot{W}_s + \left( \frac{p_1}{\rho} + gz_1 + \dot{u}_1 \right) \rho \int_{A_1} V_1 dA_1 + \rho \int_{A_1} \frac{V_1^3}{2} dA_1
\]

\[
= \left( \frac{p_2}{\rho} + gz_2 + \dot{u}_2 \right) \rho \int_{A_2} V_2 dA_2 + \rho \int_{A_2} \frac{V_2^3}{2} dA_2
\]

Recall that \( Q = \int V \cdot dA = \overline{V} A \)

So that \( \rho \int V \cdot dA = \rho \overline{V} A = \dot{m} \) mass flow rate

Define:

\[
\rho \int_{A} \frac{V^3}{2} dA = \alpha \frac{\rho \overline{V}^3 A}{2} = \alpha \frac{\overline{V}^2}{2} \dot{m}
\]

K.E. flux for \( V = \overline{V} = \text{constant across pipe} \)

\[\alpha = 1 \text{ if } V \text{ is constant across the flow section} \]

\[\alpha > 1 \text{ if } V \text{ is nonuniform} \]

\[
\frac{1}{\dot{m}} \left( \dot{Q} - \dot{W} \right) + \frac{p_1}{\rho} + gz_1 + \dot{u}_1 + \alpha_1 \frac{\overline{V}_1^2}{2} = \frac{p_2}{\rho} + gz_2 + \dot{u}_2 + \alpha_2 \frac{\overline{V}_2^2}{2}
\]

\[\text{Note that: } \alpha = 1 \text{ if } V \text{ is constant across the flow section} \]

\[\alpha > 1 \text{ if } V \text{ is nonuniform} \]

laminar flow \( \alpha = 2 \) turbulent flow \( \alpha = 1.05 \sim 1 \) may be used
Shaft Work

Shaft work is usually the result of a turbine or a pump in the flow system. When a fluid passes through a turbine, the fluid is doing shaft work on the surroundings; on the other hand, a pump does work on the fluid

\[ \dot{W}_s = \dot{W}_t - \dot{W}_p \]

where \( \dot{W}_t \) and \( \dot{W}_p \) are magnitudes of power \( \left( \frac{\text{work}}{\text{time}} \right) \).

Using this result in the energy equation and dividing by \( g \) results in

\[
\frac{\dot{W}_p}{mg} + \frac{p_1}{\gamma} + z_1 + \alpha_1 \frac{V_1^2}{2} = \frac{\dot{W}_t}{mg} + \frac{p_2}{\gamma} + z_2 + \alpha_2 \frac{V_2^2}{2} + \frac{\dot{u}_2 - \dot{u}_1}{g} - \frac{\dot{Q}}{mg}
\]

\[
\begin{align*}
\text{mechanical part} & \quad \text{thermal part}
\end{align*}
\]

Note: each term has dimensions of length

Define the following:

\[
\begin{align*}
\dot{h}_p &= \frac{\dot{W}_p}{mg} = \frac{\dot{W}_p}{\rho Q g} = \frac{\dot{W}_p}{\gamma Q} \\
\dot{h}_t &= \frac{\dot{W}_t}{mg} \\
\dot{h}_L &= \frac{\dot{u}_2 - \dot{u}_1}{g} - \frac{\dot{Q}}{mg} = \text{head loss}
\end{align*}
\]
Head Loss
In a general fluid system a certain amount of mechanical energy is converted to thermal energy due to viscous action. This effect results in an increase in the fluid internal energy. Also, some heat will be generated through energy dissipation and be lost (i.e. \(-\dot{Q}\)). Therefore the term

from 2\textsuperscript{nd} law

\[ h_L = \frac{\dot{u}_2 - \dot{u}_1}{g} - \frac{\dot{Q}}{g\dot{m}} > 0 \]

represents a loss in mechanical energy due to viscous stresses.

Note that adding \(\dot{Q}\) to system will not make \(h_L = 0\) since this also increases \(\Delta u\). It can be shown from 2\textsuperscript{nd} law of thermodynamics that \(h_L > 0\).

Drop — over \(\bar{V}\) and understand that \(V\) in energy equation refers to average velocity.

Using the above definitions in the energy equation results in (steady 1-D incompressible flow)

\[
\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L
\]

form of energy equation used for this course!
Comparison of Energy Equation and Bernoulli Equation

Apply energy equation to a stream tube without any shaft work

\[
\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L
\]

- If \( h_L = 0 \) (i.e., \( \mu = 0 \)) we get Bernoulli equation and conservation of mechanical energy along a streamline

- Therefore, energy equation for steady 1-D pipe flow can be interpreted as a modified Bernoulli equation to include viscous effects (\( h_L \)) and shaft work (\( h_p \) or \( h_t \))

Summary of the Energy Equation

The energy equation is derived from RTT with

\[ B = E = \text{total energy of the system} \]

\[ \beta = \epsilon = \frac{E}{M} = \text{energy per unit mass} \]
heat added

\[ \dot{W} = \dot{W}_s + \dot{W}_p + \dot{W}_v \]

shaft work done on or by system (pump or turbine)

pressure work done on CS

Viscous stress work on CS

Neglected in text presentation

\[ \dot{W}_p = \int p V \cdot dA = \int \rho \left( \frac{p}{\rho} \right) V \cdot dA \]

\[ \dot{W}_s = \dot{W}_t - \dot{W}_p \]

\[ \dot{Q} - \dot{W}_t + \dot{W}_p = \frac{d}{dt} \int p e dV + \int \rho (e + p/e) V \cdot dA \]

\[ e = \dot{u} + \frac{1}{2} V^2 + gz \]

For steady 1-D pipe flow (one inlet and one outlet):

1) Streamlines are straight and parallel

\[ \Rightarrow p/\rho + gz = \text{constant across CS} \]
2) \( T = \text{constant} \Rightarrow u = \text{constant across CS} \)

3) define \( \alpha = \frac{1}{A_{CS}} \int A \left( \frac{V}{V} \right)^3 dA = \text{KE correction factor} \)

\[ \Rightarrow \frac{\rho}{2} \int V^3 dA = \alpha \frac{\rho V^3}{2} A = \alpha \frac{V^2}{2} \bar{m} \]

\[
\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_L
\]

\( h_p = \dot{W}_p / mg \) \hspace{1cm} \text{Note: each term has units of length}

\( h_t = \dot{W}_t / mg \)

\( h_L = \dot{u} - \dot{u}_1 - \frac{\dot{Q}}{mg} = \text{head loss} \)

\( > 0 \) represents loss in mechanical energy due to viscosity
Concept of Hydraulic and Energy Grade Lines

\[ \frac{p_1}{\gamma} + \left(1 + \alpha_1 \right) \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \left(1 + \alpha_2 \right) \frac{V_2^2}{2g} + z_2 + h_t + h_L \]

Define \( HGL = \frac{p}{\gamma} + z \)  
\[ EGL = \frac{p}{\gamma} + z + \alpha \frac{V^2}{2g} \]

Point-by-point application is graphically displayed

HGL corresponds to pressure tap measurement + \( z \)  
EGL corresponds to stagnation tube measurement + \( z \)

\[ EGL_1 = EGL_2 + h_L \]  
for \( h_p = h_t = 0 \)

\[ h_L = f \frac{L \ V^2}{D \ 2g} \]  
i.e., linear variation in \( L \) for \( D, V, \) and \( f \) constant

\[ f = \text{friction factor} \]  
\( f = f(Re) \)

Pressure tap: \( \frac{p_2}{\gamma} = h \)  
Stagnation tube: \( \frac{p_2}{\gamma} + \alpha \frac{V^2}{2g} = h \)  

\[ EGL_1 + h_p = EGL_2 + h_t + h_L \]  
\[ EGL_2 = EGL_1 + h_p - h_t - h_L \]  

\( h = \text{height of fluid in tap/tube} \)
Helpful hints for drawing HGL and EGL

1. \[ \text{EGL} = \text{HGL} + \alpha V^2/2g = \text{HGL for } V = 0 \]

2. & 3. \[ h_L = f \frac{L V^2}{D 2g} \] in pipe means EGL and HGL will slope downward, except for abrupt changes due to \( h_t \) or \( h_p \)

\[ \text{FIGURE 7.5} \]
Rise in EGL and HGL due to pump.

\[ \frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L \]

\[ \text{HGL}_2 = \text{EGL}_1 - h_L \]

\[ h_L = \frac{V^2}{2g} \] for abrupt expansion

\[ \text{Drop in EGL and HGL due to turbine.} \]

\[ \text{Gradual expansion of conduit allows kinetic energy to be converted to pressure head with much smaller } h_p \text{ at the outlet; hence the HGL approaches the EGL.} \]
4. \( p = 0 \Rightarrow HGL = z \)

5. for \( h_L = f \frac{L V^2}{D 2g} = \text{constant} \times L \)  
   
   EGL/HGL slope downward  
   
   i.e., linearly increased for increasing \( L \) with slope \( \frac{f V^2}{D 2g} \)

6. for change in \( D \) \( \Rightarrow \) change in \( V \)  

   i.e.  
   
   \[ \begin{align*} 
   V_1 A_1 &= V_2 A_2 \\
   V_1 \frac{\pi D_1^2}{4} &= V_2 \frac{\pi D_2^2}{4} \\
   V_1 D_1^2 &= V_1 D_2^2 
   \end{align*} \]  

   change in distance between HGL & EGL and slope change due to change in \( h_L \)
7. If $HGL < z$ then $p/\gamma < 0$ i.e., cavitation possible

Condition for cavitation:

$$p = p_{va} = 2000 \frac{N}{m^2}$$

Gage pressure

$$p_{va,g} = p_A - p_{atm} \approx -p_{atm} = -100,000 \frac{N}{m^2}$$

$$\frac{p_{va,g}}{\gamma} \approx -10 m$$

$9810 \text{ N/m}^3$
4.15 METHOD OF SOLUTION OF FLOW PROBLEMS

For the solutions of problems of liquid flow there are two fundamental equations: the equation of continuity (3.10) and the energy equation in one of the forms from Eqs. (4.5) to (4.10). The following procedure may be employed:

1. Choose a datum plane through any convenient point.
2. Note at what sections the velocity is known or is to be assumed. If at any point the section area is great compared with its value elsewhere, the velocity head is so small that it may be disregarded.
3. Note at what points the pressure is known or is to be assumed. In a body of liquid at rest with a free surface the pressure is known at every point within the body. The pressure in a jet is the same as that of the medium surrounding the jet.
4. Note whether or not there is any point where all three terms, pressure, elevation, and velocity, are known.
5. Note whether or not there is any point where there is only one unknown quantity.

It is generally possible to write an energy equation that will fulfill conditions 4 and 5. If there are two unknowns in the equation, then the continuity equation must be used also. The application of these principles is shown in the following illustrative examples.

Illustrative Example 4.7 A pipeline with a pump leads to a nozzle as shown in the accompanying figure. Find the flow rate when the pump develops a head of 80 ft. Assume that the head loss in the 5-in-diameter pipe may be expressed by \( h_2 = 5V_2^2/2g \), while the head loss in the 4-in-diameter pipe is \( h_4 = 12V_4^2/2g \). Sketch the energy line and hydraulic grade line, and find the pressure head at the suction side of the pump.

Select the datum as the elevation of the water surface in the reservoir. Note from continuity that

\[ V_4 = \left( \frac{5}{4} \right)^{1/2} V_5 = 0.25 V_5 \quad \text{and} \quad V_6 = \left( \frac{2}{3} \right)^{1/2} V_5 = 0.563 V_5 \]

where \( V_j \) is the jet velocity. Writing an energy equation from the surface of the reservoir to the jet,

\[ \left( z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \right) - h_{4f} + h_f = z_3 + \frac{p_3}{\gamma} + \frac{V_3^2}{2g} \]

\[ 0 + 0 + 0 - 5 \frac{V_2^2}{2g} + 80 - 12 \frac{V_4^2}{2g} = 10 + 0 + \frac{V_3^2}{2g} \]

Express all velocities in terms of \( V_2^2 \):

\[ \frac{5(0.25 V_2)^2}{2g} + 80 - 12 \left( \frac{0.563 V_2}{2g} \right)^2 = 10 + \frac{V_3^2}{2g} \]

\[ V_2^2 = 29.7 \text{ fps} \]

\[ Q = A_3 V_3 = \frac{\pi}{4} \left( \frac{3}{2} \right)^2 \times 29.7 = 1.45 \text{ cfs} \]
4.15 Method of Solution of Flow Problems

Head loss in suction pipe:

\[ h_L = \frac{\frac{\frac{V_1^2}{2g}}{2g} \times 3(0.25V_1)^2}{2g} = 0.312\frac{V_1^2}{2g} \]

\[ = 4.3 \text{ ft} \]

Head loss in discharge pipe:

\[ h_L = \frac{\frac{\frac{V_2^2}{2g}}{2g} \times 12(0.563V_2)^2}{2g} = 52.1 \text{ ft} \]

\[ \frac{V_2^2}{2g} = 13.7 \text{ ft} \]

\[ \frac{V_1^2}{2g} = 4.3 \text{ ft} \]

\[ \frac{V_2^2}{2g} = 0.86 \text{ ft} = 0.9 \text{ ft} \]

The energy line and hydraulic grade line are drawn on the figure to scale. Inspection of the figure shows that the pressure head on the suction side of the pump is \( p_h = 14.8 \text{ ft} \). Likewise, the pressure head at any point in the pipe may be found if the figure is to scale.

Illustrative Example 4.7

Illustrative Example 4.8 Given the two-dimensional flow as shown in the accompanying figure. Determine the flow rate. Assume no head loss.
Application of the Energy, Momentum, and Continuity Equations in Combination

In general, when solving fluid mechanics problems, one should use all available equations in order to derive as much information as possible about the flow. For example, consistent with the approximation of the energy equation we can also apply the momentum and continuity equations

Energy:
\[ \frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_L \]

Momentum:
\[ \sum F_s = \rho V_2^2 A_2 - \rho V_1^2 A_1 = \rho Q (V_2 - V_1) \]
\{ one inlet and one outlet \}
\[ \rho = \text{constant} \]

Continuity:
\[ A_1 V_1 = A_2 V_2 = Q = \text{constant} \]
Abrupt Expansion

Consider the flow from a small pipe to a larger pipe. Would like to know \( h_L = h_L(V_1, V_2) \). Analytic solution to exact problem is extremely difficult due to the occurrence of flow separations and turbulence. However, if the assumption is made that the pressure in the separation region remains approximately constant and at the value at the point of separation, i.e., \( p_1 \), an approximate solution for \( h_L \) is possible:

Apply Energy Eq from 1-2 \((\alpha_1 = \alpha_2 = 1)\)

\[
\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L
\]

Momentum eq. For CV shown (shear stress neglected)

\[
\Sigma F_s = p_1 A_2 - p_2 A_2 - \underbrace{W \sin \alpha}_{\gamma A_2 L \Delta z} = \Sigma \rho u \cdot A
\]

\[
= \rho V_1 (-V_1 A_1) + \rho V_2 (V_2 A_2)
\]

\[
= \rho V_2^2 A_2 - \rho V_1^2 A_1
\]

next divide momentum equation by \( \gamma A_2 \)
\[ \frac{p_1 - p_2}{\gamma} - (z_1 - z_2) = \frac{V_2^2}{g} - \frac{V_1^2}{g} \frac{A_1}{A_2} = \frac{V_1^2}{g} \frac{A_1}{A_2} \left( \frac{A_1}{A_2} - 1 \right) \]

from energy equation

\[ \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + h_L = \frac{V_2^2}{g} - \frac{V_1^2}{g} \frac{A_1}{A_2} \]

\[ h_L = \frac{V_2^2}{2g} + \frac{V_1^2}{2g} \left( 1 - \frac{2A_1}{A_2} \right) \]

\[ h_L = \frac{1}{2g} \left[ V_2^2 + V_1^2 - 2V_1^2 \frac{A_1}{A_2} \right] \]

continuity eq.

\[ V_1 A_1 = V_2 A_2 \]

\[ \frac{A_1}{A_2} = \frac{V_2}{V_1} \]

\[ h_L = \frac{1}{2g} [V_2 - V_1]^2 \]

If \( V_2 \ll V_1 \),

\[ h_L = \frac{1}{2g} V_1^2 \]
Forces on Transitions

Example 7-6

\[ Q = 0.707 \text{ m}^3/\text{s} \]

head loss = \[ 0.1 \frac{V_2^2}{2g} \]

(empirical equation)

Fluid = water
\[ p_1 = 250 \text{ kPa} \]
\[ D_1 = 30 \text{ cm} \]
\[ D_2 = 20 \text{ cm} \]
\[ F_x = ? \]

First apply momentum theorem

\[ \sum F_x = \sum \rho u V \cdot A \]

\[ F_x + p_1 A_1 - p_2 A_2 = \rho V_1 (-V_1 A_1) + \rho V_2 (V_2 A_2) \]

\[ F_x = \rho Q (V_2 - V_1) - p_1 A_1 + p_2 A_2 \]

force required to hold transition in place
The only unknown in this equation is $p_2$, which can be obtained from the energy equation.

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + h_L$$

Note: $z_1 = z_2$ and $\alpha = 1$

$$p_2 = p_1 - \gamma \left[ \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + h_L \right]$$

Drop in pressure

$$\Rightarrow F_x = \rho Q (V_2 - V_1) + A_2 \left[ p_1 - \gamma \left( \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + h_L \right) \right] - p_1 A_1$$

$p_2$ (note: if $p_2 = 0$ same as nozzle)

In this equation, continuity $A_1 V_1 = A_2 V_2$

$V_1 = Q/A_1 = 10 \text{ m/s}$
$V_2 = Q/A_2 = 22.5 \text{ m/s}$

$h_L = 0.1 \frac{V_2^2}{2g} = 2.58 \text{ m}$

$F_x = -8.15 \text{ kN}$ is negative $x$ direction to hold transition in place