Consider the 2-stage stochastic LP with simple recourse in which only the right-hand-side is random.

Cf. Stochastic Programming, by Willem K. Klein Haneveld and Maarten H. van der Vlerk, Dept. of Econometrics & OR, University of Groningen, Netherlands
P: \[
\begin{align*}
\text{Minimize} & \quad cx + E_\omega \left[ \sum_{i=1}^{m_2} \tilde{Q}_i(y_i) \right] \\
\text{subject to} & \quad Ax \geq b \\
& \quad Tx - y(\omega) = h(\omega) \\
& \quad x \geq 0 \\
& \quad y_i = y_i^+ - y_i^{-}, \quad y_i^+ \geq 0, \quad y_i^- \geq 0 \quad \forall i = 1, \ldots m_2
\end{align*}
\]

(The first-stage constraints might be instead "=" or "\leq".)

The right-hand-side \( h(\omega) \) may be interpreted as the random demand for a set of outputs, with expected value \( h_\omega \).
The **second-stage variables**

\[ y(\omega) = Tx - h(\omega) \]

represent *surplus* (if positive) or *shortage* (if negative) of the outputs.

For example,

- \( y_i^+ \) = quantity of demand in excess of output (shortage of output) which must be acquired (at a cost \( q_i^+ \) per unit),
- \( y_i^- \) = shortage of demand (excess of output which must be disposed of) (at a cost \( q_i^- \) per unit),

where it is assumed that \( q_i^+ + q_i^- > 0 \).

**Warning:** *the terminology & notation is confusing!*
The expected second-stage cost is

\[ Q_i(z) = E_\omega \left[ \min_y \left\{ q_i^+ y_i^+ + q_i^- y_i^- : y_i^+ - y_i^- = \omega - z, y_i^+ \geq 0, y_i^- \geq 0 \right\} \right] \]

\[ = q_i^+ G_i(z) + q_i^- H_i(z) \]

where \( G_i(z) \) is the expected surplus of demand (shortage of output):

\[ G_i(z) = \int_{-\infty}^{+\infty} (t - z)^+ F_i(t) \, dt = \int_{z}^{+\infty} (1 - F_i(t)) \, dt \]

and \( H_i(z) \) is the expected shortage of demand (surplus of output):

\[ H_i(z) = \int_{-\infty}^{+\infty} (z - t)^+ F_i(t) \, dt = \int_{-\infty}^{z} F_i(t) \, dt \]

If demand is random and a supply \( z \) is made available, \( G_i(z) \) is the expected demand in excess of the supply, i.e., the expected deficit in the supply.

Note the danger of confusion in the terminology!
The stochastic LP may therefore be restated as

\[
\begin{align*}
\text{Minimize} & \quad cx + \sum_{i=1}^{m_2} Q_i(z_i) \\
\text{subject to} & \quad Ax = b \\
& \quad Tx - z = 0 \\
& \quad x \geq 0
\end{align*}
\]

If the probability distributions are discrete, then \( Q_i(z) \) is a \textit{piecewise-linear convex} function.

This optimization problem can then be solved by an extension of LP usually called \textit{"separable programming"}.

If the probability distributions are continuous, then a piecewise-linear approximation of each \( Q_i(y_i) \) can be constructed.
Suppose that for each output $i$, a set of $J_i$ grid points is given,

$$\{\left\{\hat{z}_j^i \right\}\}_{j \in J_i}.$$ 

Represent each second-stage variable $z_i$ as a *convex combination* of these grid points:

$$z_i = \sum_{j \in J_i} \lambda_j^i \hat{z}_j^i \quad \text{where} \quad \sum_{j \in J_i} \lambda_j^i = 1, \quad \lambda_j^i \geq 0$$

and $Q_i(z_i)$ as the corresponding convex combination of function values:

$$Q_i(z_i) \approx \sum_{j \in J_i} \lambda_j^i \hat{q}_j^i, \quad \text{where} \quad \hat{q}_j^i \equiv Q_i\left(\hat{z}_j^i\right)$$
The inner linearization of the original nonlinear problem P is the LP:

\[
\begin{align*}
\text{Minimize} & \quad cx + \sum_{i=1}^{m_2} \sum_{j \in J_i} \hat{q}_i \lambda_i^j \\
\text{subject to} & \quad Ax \geq b, \\
& \quad \sum_{j=1}^{n_i} T_{ij} x_j - \sum_{j \in J_i} \lambda_i^j \hat{z}_i^j = 0, \quad i = 1, 2, \ldots m_2 \\
& \quad \sum_{j \in J_i} \lambda_i^j = 1, \quad i = 1, \ldots m_2 \\
& \quad x \geq 0, \quad \lambda_i^j \geq 0 \quad \forall i = 1, \ldots m_2 \text{ & } j \in J_i
\end{align*}
\]

Note that the variables of this problem are \( x \) and \( \lambda \).
The $m_2$ convexity constraints are of type "**GUB**" (*Generalized Upper Bounds*), which are handled by many LP-solvers without increasing the size of the basis matrix. Hence, when GUB facility is available, the *number of constraints* in the tableau is *identical* to that of the *expected value* problem (i.e., with random variables replaced by their expected values)!

The computational effort should therefore be of the same order of magnitude as that of the expected value problem!
This is sometimes referred to as the "Lambda" separable programming formulation, with the new variables associated with the grid points and convexity (GUB) constraints added.

An alternative formulation is the "Delta" separable formulation, with a new variable associated with each of the intervals between grid points, and simple upper bounds (SUB) constraints added.

*Computational efforts of the two formulations should be comparable, and results will be equivalent.*
**Example:**

Stochastic Transportation Problem with Simple Recourse

Consider the small example with

- two sources, each with supply = 10, and
- three destinations, each with random demand.

<table>
<thead>
<tr>
<th></th>
<th>Dstn #1</th>
<th>Dstn #2</th>
<th>Dstn #3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source #1</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Source #2</td>
<td>2</td>
<td>4</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Dstn #1</th>
<th>Dstn #2</th>
<th>Dstn #3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^+$</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$q^-$</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
Discrete Probability Distributions

Random demand #1, Mean = 7, # points = 3

\begin{align*}
\text{id: } & 1 & 2 & 3 \\
\text{d: } & 5 & 7 & 9 \\
\text{p: } & 0.25 & 0.5 & 0.25
\end{align*}

The piecewise-linear function $Q_1(z)$, with $q_1^+ = 6$ & $q_1^- = 3$:

\[
Q_1(z) = q_1^+ \sum_j p_j (d_j - z)^+ + q_1^- \sum_j p_j (z - d_j)^+ \\
= \sum_j p_j \left[ q_1^+ (d_j - z)^+ + q_1^- (z - d_j)^+ \right]
\]
$$= 0.25 \left[ 6(5 - z)^+ + 3(z - 5)^+ \right]$$
$$+ 0.5 \left[ 6(7 - z)^+ + 3(z - 7)^+ \right]$$
$$+ 0.25 \left[ 6(9 - z)^+ + 3(z - 9)^+ \right]$$

That is,
$$Q_1(5) = 0.25[0 + 0] + 0.5[6(7 - 5) + 0] + 0.25[6(9 - 5) + 0] = 12$$
$$Q_1(7) = 0.25[0 + 3(7 - 5)] + 0.5[0 + 0] + 0.25[6(9 - 7) + 0] = 4.5$$
$$Q_1(9) = 0.25[0 + 3(9 - 5)] + 0.5[0 + 3(9 - 7)] + 0.25[0 + 0] = 6$$

The piecewise-linear curve joins the points (5,12), (7,4.5), and (9,6), with slopes $-q_1^+ = -6$ on the left and $+q_1^- = +3$ on the right.
Random demand #2, Mean = 6, # points = 2

\begin{align*}
i: & \quad 1 \quad 2 \\
d: & \quad 4 \quad 8 \\
p: & \quad 0.5 \quad 0.5
\end{align*}

The Piecewise-Linear function $Q_2(z)$ with $q_2^+ = 7 \ & q_2^- = 3$: 

![Graph showing the Piecewise-Linear function $Q_2(z)$ with $q_2^+ = 7$ and $q_2^- = 3$.]
Random demand #3, Mean = 7, # points = 4

\[
\begin{array}{cccc}
  i & 1 & 2 & 3 & 4 \\
  d & 4 & 6 & 8 & 10 \\
  p & 0.1 & 0.4 & 0.4 & 0.1 \\
\end{array}
\]

The piecewise-linear function $Q_3(z)$ with $q^+_3 = 3$ & $q^-_3 = 7$:
Compare the size of this tableau with that of the LP with the second-stage variables ($y_k$) for each scenario!
Solution of LP:  **Objective**: 96.3

**First stage**: nonzero variables

<table>
<thead>
<tr>
<th>i</th>
<th>variable</th>
<th>value</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>X11</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>X13</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>X21</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>X22</td>
<td>4</td>
</tr>
</tbody>
</table>

**Multipliers** in convex combinations

<table>
<thead>
<tr>
<th>i</th>
<th>Grid #</th>
<th>Grid pt</th>
<th>Multipliers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

**Second stage** primal & dual solutions:

<table>
<thead>
<tr>
<th>i</th>
<th>output</th>
<th>value</th>
<th>v</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>AAA</td>
<td>7</td>
<td>3</td>
<td>25.5</td>
</tr>
<tr>
<td>2</td>
<td>BBB</td>
<td>4</td>
<td>5</td>
<td>34.0</td>
</tr>
<tr>
<td>3</td>
<td>CCC</td>
<td>6</td>
<td>6</td>
<td>46.8</td>
</tr>
</tbody>
</table>

v & w are dual variables for 2nd-stage and convexity rows, respectively.
**Optimal LP Tableau**

**First-Stage Recourse**

| rhs | -z | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  |
|-----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |    | 96.3| 1   | 0   | 0   | 0   | 0   | 0   | 2   | 0   | 1   | 16.5| 1.5 | 0   | 7.5 | 61.5| 8   | 0   | 12  | 76  | 9.2 | 1.2 | 0   | 10  |
| 3   | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1   | -7  | -2  | 0   | 2   | 11  | -4  | 0   | 4   | 12  | -6  | -2  | 0   | 2   |
| 1   | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 6   | 0  | 0   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 4   | 0   | 4   | 12  | 0   | 0   | 0   | 0   |
| 4   | 0  | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 4   | 0   | -4  | -12 | 0   | 0   | 0   | 0   | 0   | 0   |
| 6   | 0  | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 6   | 2   | 0   | -2  |     |
| 1   | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1   | 0  | 1   | 1   | 0   | 0   | 0   | -1  | 1   | 0   | -1  | 7   | 2   | 0   | -2  | -11 | 4   | 0   | -4  | -12 | 0   | 0   | 0   | 0   | 0   |
| 1   | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |