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57:022 Principles of Design II
Midterm Exam Solutions - Spring 2000

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Part:	I	II	III	Total
Possible Pts:	13	15	24	52
Your score:	_____	_____	_____	_____

✳✳✳✳✳✳ PART I ✳✳✳✳✳✳

Choose the answers to the questions below from the list of distributions, or the table of probabilities. (State "NONE" if the answer does not appear in either place!

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The foreman of a casting section in a certain factory finds that, on the average, 8 castings are made each day, and 1 in every 5 castings made is defective.

__g___1. What is the *probability* that 2 or fewer defective castings are made in one day? *Choose nearest value*:

a. ≤ 20%	b. 30%	c. 40%
d. 50%	e. 60%	f. 70%
g. 80%	h. 90%	i. 100%

Note: Let N_t = # defective castings in interval [0,t].

This random variable has *Poisson* distribution with $\lambda = 0.2 \times 8$/day = 1.6/day, so if t=1 day, the expected value is 1.6 and $P\{N_1 \leq 2\}$ = 78.33% can be read from the table below.

Bernouilli 2. What's the *name* of the probability dist'n of the quality of casting #5 (either defective or OK)?

__b___3. If the section makes 8 castings on Monday, what is the *probability* that *exactly* 2 of these will be defective? *Choose nearest value*:

a. ≤ 20%	b. 30%	c. 40%
d. 50%	e. 60%	f. 70%
g. 80%	h. 90%	i. 100%

Note: Let X = # of defective castings among 8 produced. X has binomial distribution with n=8, p=0.2, and $P\{X=2\}=0.2936$ (from table below).

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Advertising states that, for a certain lottery ticket, "every *fifth* ticket carries a prize". If you buy *eight* tickets, what is...

__g___4. the *probability* that you get *at least* one winning ticket among the eight? *Choose nearest value*:

a. ≤ 20%	b. 30%	c. 40%
d. 50%	e. 60%	f. 70%
g. 80%	h. 90%	i. 100%

Note: Let Y = # winning tickets among 8 purchased. Then Y has binomial distribution with n=8, p=0.2, and $P\{Y \geq 1\} = P\{Y>0\} = 0.8322$ (from table below).

__b___5. the *probability* that you get *exactly* one winning ticket? *Choose nearest value*:

a. ≤ 20%	b. 30%	c. 40%
d. 50%	e. 60%	f. 70%
g. 80%	h. 90%	i. 100%

Note: $P\{Y=1\} = 0.3355$.

If, instead of deciding in advance how many tickets to buy, you continue buying tickets until you have a winning ticket...

Geometric 6. what's the *name* of the probability distribution of the number of tickets you buy ?

If you continue buying tickets until you have *two* winning tickets...

Pascal 7. what's the *name* of the probability distribution of the number of tickets you buy ?

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The arrival of parts to be processed by a machine is a Poisson process, with the rate 4/hour. What is...

Poisson 8. the *name* of the probability distribution of the number of parts which arrive during the first hour?

Exponential 9. the *name* of the probability distribution of the time between arrivals of parts?

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What is the name of …

Gumbel 10. The probability distribution of the maximum of a large number of failure times, none of which has an upper bound.

Weibull 11. The probability distribution of the minimum of a large number of failure times (all of which are nonnegative).

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Some common probability distributions:

A. Bernouilli	F. Exponential	K. Uniform
B. Normal	G. Beta	L. Poisson
C. Lambda	H. Erlang	M Pascal
D Binomial	I. Geometric	N. Random
E. Chi-square	J. Weibull	O. Gumbel

Binomial distribution (n= 8, p= 0.2)

x	P{x}	P{X ≤ x}	P{X > x}
0	0.16777216	0.16777216	0.83222784
1	0.33554432	0.50331648	0.49668352
2	0.29360128	0.79691776	0.20308224
3	0.14680064	0.94371840	0.05628160
4	0.04587520	0.98959360	0.01040640
5	0.00917504	0.99876864	0.00123136
6	0.00114688	0.99991552	0.00008448
7	0.00008192	0.99999744	0.00000256
8	0.00000256	1.00000000	0.00000000

(Exponential distribution (Lambda = 0.5/minute)

t	P{T ≤ t}	Δp	P{T ≥ t}
0	0.00000000	0.00000000	1.00000000
0.5	0.22119922	0.22119922	0.77880078
1	0.39346934	0.17227012	0.60653066
1.5	0.52763345	0.13416411	0.47236655
2	0.63212056	0.10448711	0.36787944
2.5	0.71349520	0.08137464	0.28650480
3	0.77686984	0.06337464	0.22313016
3.5	0.82622606	0.04935622	0.17377394
4	0.86466472	0.03843866	0.13533528
4.5	0.89460078	0.02993606	0.10539922
5	0.91791500	0.02331423	0.08208500
5.5	0.93607214	0.01815714	0.06392786
6	0.95021293	0.01414079	0.04978707
6.5	0.96122579	0.01101286	0.03877421
7	0.96980262	0.00857682	0.03019738
7.5	0.97648225	0.00667964	0.02351775
8	0.98168436	0.00520211	0.01831564
8.5	0.98573577	0.00405140	0.01426423
9	0.98889100	0.00315524	0.01110900
9.5	0.99134830	0.00245730	0.00865170
10	0.99326205	0.00191375	0.00673795

Poisson Cumulative Distribution Function, expected value 1.6

x	P{X ≤ x}	P{X > x}
0	0.20189652	0.79810348
1	0.52493095	0.47506905
2	0.78335849	0.21664151
3	0.92118651	0.07881349
4	0.97631772	0.02368228
5	0.99395971	0.00604029
6	0.99866424	0.00133576
7	0.99973956	0.00026044
8	0.99995462	0.00004538
9	0.99999286	0.00000714

Multiple Choice:

_c_ 12. In simulating the arrival process in (16) & (17) above, an inter-arrival time T can be randomly generated by obtaining a uniformly-generated random variable X and computing

a. $T = -\dfrac{\ln(1-X)}{4}$ d. $T = -\dfrac{\ln X}{4}$

b. $T = 1 - e^{-4X}$ e. $T = e^{-4X}$

c. Both (a) & (d) are correct f. Both (b) & (e) are correct

_a_ 13. The CDF of the distribution in (12) above, i.e., the inter-arrival times, is F(t) =

a. $1 - e^{-4X}$ b. $4\,e^{-4X}$

c. $1 - 4\,e^{-4X}$ d. $4\,e^{-4X}$

e. e^{-4X} f. *None of the above*

✱✱✱✱✱✱ PART II ✱✱✱✱✱✱

The time between arrivals of cars are measured for 3 hours. It is expected that these observations have an exponential distribution with mean of 4 minutes (although the *actual* average value of the observations was 3.68 minutes). We wish to decide whether the discrepancy between the assumed arrival rate (one every 4 minutes) and the observed arrival rate (one every 3.68 minutes) is so large as to disqualify our assumption. The number of observations O_i falling within each of several intervals is shown in the table below. We wish to test the "goodness of fit" of the exponential distribution with mean 4 minutes, and have prepared the table below:

i	interval	O_i	p_i	E_i	$\dfrac{(O_i - E_i)^2}{E_i}$
1	[0, 1]	12	0.221199	11.06	0.0798984
2	[1, 2]	11	0.17227	8.61351	0.661212
3	[2, 3]	6	0.134164	6.70821	0.0747674
4	[3, 5]	6	0.185862	9.29309	1.16693
5	[5, 9]	10	0.181106	9.05528	0.098561
6	[9, +∞]	5	0.105399	5.26996	0.0138291

Notes: (i) the total number of observations was not determined in advance, but happens to be 50.

(ii) the sum of the last column is D=2.0952. A portion of a table of the chi-square distribution is given below:

deg.of freedom	Chi-square Dist'n $P\{D \geq \chi^2\}$ 99%	95%	90%	10%	5%	1%
2	0.0201	0.103	0.211	4.605	5.991	9.210
3	0.115	0.352	0.584	6.251	7.815	11.341
4	0.297	0.711	1.064	7.779	9.488	13.277
5	0.554	1.145	1.610	9.236	11.070	15.086
6	0.872	1.635	2.204	10.645	12.592	16.812

7	1.239	2.167	2.833	12.017	14.067	18.475

Note: The # degrees of freedom is equal to 6, the number of cells. The total # of observations was not pre-determined, nor was the parameter λ estimated from the data.

Indicate whether true or false, using "**+**" for true, "**o**" for false.

__+_ 1. The CDF of the inter-arrival time distribution is $F(t) = P\{T \le t\}$

__o_ 2. The parameter λ of the exponential distribution was assumed to be 4 minutes.

__o_ 3. If T actually does have an exponential distribution with mean 4 minutes, then (based upon the table above) the probability that D exceeds 2.0952 is less than 10%.
Note: $P\{D \ge 10.645\}$=10%, and $P\{D \ge 2.0952\}$ is larger than 10% (nearly 90%, according to the table).

__+_ 4. The quantity $\sum_{i=1}^{6} \frac{(O_i - E_i)^2}{E_i}$ is assumed to have a "chi-square" distribution.

__+_ 5. The smaller the value of D, the better the fit for the distribution being tested.

__+_ 6. The quantity E_i is the expected number of observations in interval # i, if the assumption is true.

__o_ 7. The probability p_i that a car arrives in interval #4, i.e., [3,4], is $F(3) - F(4)$, where F(t) is the CDF of the interarrival times.

__+_ 8. If the gender of the car's driver is recorded also, with X_n=1 if the driver of car #n is female (0 otherwise), then the sequence $\{X_1, X_2, X_3, X_4, ...\}$ forms a *Bernouilli* process.

__+_ 9. If the assumption above (that the times between arrivals have exponential distribution) is correct, the arrivals of the cars forms a Poisson process.

__+_ 10. The number of "degrees of freedom" of the chi-square distribution for this test will be 6, the number of cells in the histogram.

__o_ 11. Based upon these observations, the exponential distribution with mean 4 minutes should be rejected as a model for the inter-arrival times of the vehicles.

__o_ 12. The chi-square distribution for this goodness-of-fit test will have 4 degrees of freedom.

__o_ 13. The number of observations O_i in interval # i is a random variable with approximately binomial distribution with n =4 and probability of "success" $p = p_i$.

__o_ 14. The quantity E_i is a random variable with approximately a Poisson distribution.

__o_ 15. The quantity D is assumed to have approximately a Normal distribution.

❊❊❊❊❊❊ PART III ❊❊❊❊❊❊

Suppose that 500 light bulbs are tested by simultaneously lighting them and recording the number of failures every 100 hours. The test is interrupted at the end of 1000 hours, when 291 bulbs have failed. As in your homework assignment, a Weibull probability model is then "fit" to the data.

For each statement, indicate "+" for true, "o" for false:

_+_ 1. The quantity R_t is the fraction of the 500 bulbs which are surviving at time t .

_+_ 2. The Weibull distribution is usually appropriate for the minimum of a large number of nonnegative random variables.

_o_ 3. If the failure rate is known to be decreasing, it may be more appropriate to use the Gumbel distribution than the Weibull distribution.

_o_ 4. We assume that the number of survivors at time t , $N_s(t)$, has a Weibull distribution. *Note*: the time of failure of a bulb is being assumed to have the Weibull distribution.

_+_ 5. The Weibull CDF, i.e., F(t), gives, for each bulb, the probability that at time *t* it has already failed.

_o_ 6. The method used to estimate the Weibull parameters u & k requires that you compute the mean and standard deviation of the 291 bulbs which have failed.

_o_ 7. The Minitab program fits a line which minimizes the maximum error.

_+_ 8. Regression analysis (as performed by Minitab, for example) can be used to fit a function of the form $Y= a\, b^X$.

_+_ 9. The sum of the CDF (cumulative distribution function) F(t) and the Reliability function R(t) is always equal to 1 for *every* probability distribution.

_+_ 10. The exponential distribution is a special case of the Weibull distribution, with constant failure rate.

_+_ 11. A positive value of *ln k* indicates an increasing failure rate, and negative *ln k* indicates a decreasing failure rate. *Note*: $\ln k > 0 \Rightarrow k > 1 \Rightarrow$ increasing failure rate.

_+_ 12. If each bulb's lifetime has an exponential distribution, the time of the 10^{th} failure has Erlang-10 distribution.

_o_ 13. If 6 bulbs are installed in an office's light fixtures, the number still functioning after 1000 hours has a Weibull distribution. *Note*: this number will have binomial distribution, with parameters n=6 and p=R(1000), where R(t) is the reliability function for the bulbs.

_o_ 14. Regression analysis (as performed by *Minitab*, for example) can be used to fit a function of the form $Y = a + X^b$.

_+_ 15. If k=1, then $\Gamma\left(1+\frac{1}{k}\right) = 1$. *Note*: $\Gamma(1+1/1) = \Gamma(2) = 1!$

_o_ 16. If 10 bulbs are installed in an office's light fixtures, the number still functioning after 1000 hours has a Poisson distribution.

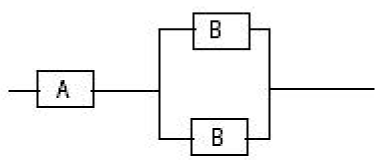
_+_ 17. Regression analysis (as performed by *Minitab*, for example) can be used to fit a function of the form $Y = aX^b$.

_+_ 18. Given only the coefficient of variation for the Weibull distribution (the ratio σ/μ but not μ & σ), the Weibull shape parameter *k* but *not* the scale parameter *u* can be computed.

_o_ 19. Regression analysis (as performed by *Minitab*, for example) can be used to fit a function of the form $Y = a + b^X$.

_o_ 20. The "gamma" function has the property $\Gamma(x) = x!$ for all nonnegative integer values of x.
Note: $\Gamma(x) = (x-1)!$ for integer values of x.

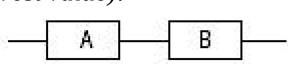
_e_ 21. If components A and B are both 60% reliable, the reliability of the system represented below is *(choose nearest value):*



a. ≤ 10%	b. 20%	c. 30%
d. 40%	e. 50%	f. 60%
g. 70%	h. 80%	i. 90%

Note: Probability that both B's fail is $0.40 \times 0.40 = 0.16$, so that the subsystem has 84% reliability. The probability that both component A and the B subsystem survive is $0.60 \times 0.84 = 50.4\%$.

_d_ 22. If component A is 70% reliable and B is 60% reliable, the reliability of the system below is *(choose nearest value):*



a. ≤ 10%	b. 20%	c. 30%
d. 40%	e. 50%	f. 60%
g. 70%	h. 80%	i. 90%

Note: the probability that both A and B survive is $0.70 \times 0.60 = 42\%$

_a_ 23. If F(t) is the CDF of the lifetime of each of 10 devices in a system (and f(t) the density function), then the CDF of time of the final unit to fail is

a. $[F(t)]^{10}$ b. $[1 - F(t)]^{10}$ c. $[f(t)]^{10}$ d. $[1 - f(t)]^{10}$ e. *None of the above*

Note: Let T_i = failure time of device #i, with CDF $F(t) = P\{T_i \le t\}$. Then probability that all 10 units have failed is therefore $P\{[\max_i T_i] \le t\}$ $P\{T_1 \le t \ \& \ T_2 \le t \ \& \ \ldots \ T_{10} \le t\} = P\{T_1 \le t\} \times P\{T_2 \le t\} \times \ldots \times P\{T_{10} \le t\} = [F(t)]^{10}$

_d_ 24. Suppose that the time between arriving cars has exponential distribution, with average of 15 seconds, and a pedestrian requires 30 seconds between cars in order to cross the highway. Then the probability that the pedestrian is still waiting after 5 cars have arrived is

a. $2e^{-2(5)}$ b. $\left(e^{-2}\right)^5$ c. $1 - e^{-2(5)}$ d. $\left(1 - e^{-2}\right)^5$ e. *None of the above*

Note: Let T_i = time between arrivals of car # i–1 and #i have exponential distribution with rate λ = 4/minute. This random variable has CDF $F(t) = 1 - e^{-\lambda t}$. Then the pedestrian is still waiting after five cars have arrived if $T_1 \le 0.5$ & $T_2 \le 0.5$ & … $T_5 \le 0.5$ which has probability $P\{T_1 \le 0.5 \ \& \ T_2 \le 0.5 \ \& \ \ldots \ T_5 \le 0.5\} = P\{T_1 \le 0.5\} \times P\{T_2 \le 0.5\} \times \ldots \times P\{T_5 \le 0.5\} = F(0.5)^5$ where $F(0.5) = 1 - e^{-4(0.5)}$