[Posynomial] Geometric Programming

The general primal problem of geometric programming (GP) [Duffin et al.] is to

\[
\begin{align*}
\text{Minimize} \quad & g_0(x) \\
\text{subject to} \quad & g_i(x) \leq 1, \quad i=1, 2, \ldots, m \\
\text{x > 0} 
\end{align*}
\]  

(1)

where the functions \( g_i \) are \textit{posynomials}, i.e.,

\[
g_i(x) = \sum_{j=1}^{T_i} c_{ij} \prod_{n=1}^{N} x_n^{a_{ijn}}
\]

(2)

The exponents \( a_{ijn} \) are arbitrary real numbers, but the coefficients \( c_{ij} \) are assumed to be positive constants and the decision variables \( x_n \) are required to be strictly positive. The corresponding posynomial GP dual problem is to

\[
\begin{align*}
\text{Maximize} \quad & v(\delta, \lambda) = \prod_{i=0}^{m} \prod_{j=1}^{T_i} \left( \frac{c_{ij} \lambda_i}{\delta_{ij}} \right)^{\delta_{ij}} \\
\text{subject to} \quad & \sum_{i=0}^{m} \sum_{j=1}^{T_i} a_{ijn} \delta_{ij} = 0, \quad n=1, 2, \ldots, N \\
& \lambda_i = \sum_{j=1}^{T_i} \delta_{ij}, \quad i=0, 2, \ldots, m \\
& \lambda_0 = 1 \\
& \delta_{ij} \geq 0, \quad j=1,2,\ldots,T_i, \quad i=0,1,\ldots,m
\end{align*}
\]

(3)

(4)

(5)

(6)

(7)

This dual problem offers several computational advantages: after using (5) to eliminate \( \lambda \), the logarithm of the objective (3) is a concave function to be maximized over a linear system. This linear system has \( T \) variables, where \( T=T_0 + T_1 + \ldots + T_m \) and \( N+1 \) equations, and hence \( T-(N+1) \) is referred to as its \textit{degree of difficulty}. If an optimal dual solution \((\delta^*, \lambda^*)\) is known, then the following relationships may be used to compute a primal solution \( x^* \) in nonpathological cases:

\[
\delta_{ij}^* g_i(x^*) = \lambda_i^* c_{ij} \prod_{n=1}^{N} x_n^{a_{ijn}}, \quad j=1, 2, \ldots, T_i, \quad i=0, 1, \ldots, m
\]

(8)

where \( g_0(x^*)=v(\delta^*, \lambda^*) \) and, for \( i > 0 \), \( g_i(x^*)=1 \) if \( \lambda_i \geq 0 \). Note that, from these relationships, one may obtain a system of equations linear in the logarithms of the optimal values of the primal variables:
\[ \sum_{n=1}^{N} a_{ijn} \ln x_n = \ln \left( \frac{\delta_{ij}^* g_i(x^*)}{\lambda_i^* c_{ij}} \right), \quad j=1, 2, ..., T \] for each \( i = 0, 1, ..., m \) such that \( \lambda_i^* \neq 0 \).

Typically, but not always, the system of linear equations (9) uniquely determines the optimal \( x^* \). (Cf. [Dembo] for a discussion of the recovery of primal solutions from the dual solution in general.)

**References**
