"On Geometric Programming Problems Having NegativeDegree of Difficulty"

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Abstract: The dual of a geometric programming problem with negative degree of difficulty is often infeasible. It has been suggested that such problems be solved by finding a dual "approximate" solution which minimizes a measure of the infeasibility, e.g., the summed squares of the infeasibilities in the dual constraints. We point out the shortcomings in that approach, and suggest a simple technique to ensure dual feasibility, namely the addition of a constant term to the primal objective.
1. Introduction

The degree of difficulty of a geometric programming problem is the number of dual variables minus the number of dual equality constraints. If zero (and assuming the system of linear equality constraints of the dual has full rank), then there is a unique dual feasible solution. If the degree of difficulty is positive, then the dual feasible region must be searched to maximize the dual objective, while if the degree of difficulty is negative, the dual constraints may be inconsistent. Unlike the analogous situation in linear programming, dual infeasibility does not imply that the primal objective is unbounded (since every posynomial is bounded below by zero). Rather, in the case of GP dual infeasibility, the primal minimum is not attained, but instead the objective approaches the infimum as one or more primal variables approach either 0 (which is outside the primal feasible region) or infinity. (Cf. [4].) (Fortunately, this type of problem is pathological and very rarely arises in a practical situation if the problem is well formulated.)

In an earlier paper, Sinha, Biswas, and Biswal [5] discuss posynomial geometric programming problems with negative degree of difficulty, i.e., problems with no more terms than variables. Sinha et al. propose methods for computation of a so-called "approximate" dual solution, namely, values of the dual variables which minimize either the summed square of the errors or, by an LP model, the maximum absolute error in the dual equality constraints. They then suggest the computation, using the primal-dual optimality relationship, of values for the primal variables. However, their approach, while perhaps superficially appealing, ignores the character of the primal solution.

Two simple examples serve to illustrate this behavior.

Example 1: Minimize $x$ subject to $x > 0$

Example 2: Minimize $x^{-1}$ subject to $x > 0$

In neither example is the minimum attained. The objective approaches the infima, namely zero in both cases, as $x$ approaches 0 and infinity, respectively. Both these example problems have degree of difficulty -1, and the dual problem possesses
no feasible solution. For these two particular examples, it happens that whichever approach of reference [5] we use, we obtain the same "approximate dual solution" $w_{01}=0.5$, which minimizes both the summed squares of the errors and the maximum absolute error. The corresponding value of the dual objective is $\sqrt{2}$. Using $w_{01}=0.5$ to compute $x$ from the primal-dual relationship yields $x=1/\sqrt{2}$ for example 1, and $x=\sqrt{2}$ for example 2, with primal objective value $1/\sqrt{2}$ for both examples.

A problem with negative degree of difficulty need not, of course, be dual infeasible. Consider, for instance

**Example 3:**

Minimize $\frac{x}{y}$ subject to $\frac{y}{x} \leq 1$, $x>0$, $y>0$

The dual of this problem has degree of difficulty -1, but in this case the dual constraints have a unique feasible solution, and the primal optimal value is 1, attained at each point of the solution set $\{(x,y): x=y, x>0, y>0\}$.

2. Ensuring Dual Feasibility

Let us suppose that a dual geometric programming problem is infeasible. It is a rather simple matter to ensure dual feasibility. Clearly the orthogonality constraints alone are always consistent, being satisfied by the zero vector. If a positive constant term, say $\alpha>0$, is added to the primal objective function, the dual problem is expanded by the addition of a variable, $w_{0,T_0+1}$ (following the notation of [5]). This new dual variable appears in no orthogonality constraint, but only in the normality constraint, since in this new term, the exponent of each primal variable is zero. Since the zero vector is always feasible in equations, the zero vector extended by one element, $w_{0,T_0+1}=1$, is feasible in this (expanded) dual problem. Furthermore, if the rank of the constraint coefficient matrix of this problem is $T+1$, i.e., full column rank, then this solution is unique.

It is shown in reference [1] that, if the primal is consistent and has a positive infimum (in our case, $\alpha$), then a term is bounded away from zero in the primal optimal
set if and only if its corresponding dual variable is positive for some dual feasible solution. Once a solution is found for the (expanded) dual problem, then, an attempt may be made to determine the primal solution, using the primal-dual relationship, where a primal term whose corresponding dual variable is zero is set equal to a parameter $\theta_{m_i} > 0$ approaching zero. The result is a characterization of the behavior of each primal variable as the infimum is approached. The following example serves to illustrate this approach.

**Example 4:**

Minimize $xyz$ subject to

\[
\frac{y}{z^2} + \frac{z}{xy} \leq 1, \ x > 0, \ y > 0, \ z > 0
\]

This problem has degree of difficulty equal to -1, and the dual constraints are infeasible:

Normality:

\[
w_{01} = 1
\]

Orthogonality:

\[
w_{01} - w_{12} = 0 \\
w_{01} + w_{11} - w_{12} = 0 \\
w_{01} - 2w_{11} - w_{12} = 0
\]

If we apply the first approach of reference [5], that of finding the least-squares-error solution of the dual constraints, we obtain $w_{01} = w_{11} = 0.6$, $w_{12} = 0.8$, from which we compute the "approximate solution" $x = 3.34908$, $y = -0.637091$, and $z = 1.21924$, with cost 2.60146. On the other hand, using an LP model to minimize the maximum absolute error yields $w_{01} = w_{11} = 2/3$, $w_{12} = 1$. In this case, the primal-dual relationship then yields the "approximate" solution $x = 3.07003$, $y = 0.736806$, and $z = 1.35721$, at which the primal objective value is 3.07003. However, as we have pointed out, neither of these "solutions" indicates the character of the primal solution.

The approach which we have suggested above is to add a positive quantity $\alpha$ to the primal objective function. The orthogonality constraints are unchanged, but the normality constraint includes an additional dual variable $w_{02}$:

\[
w_{01} + w_{02} = 1
\]

The dual constraints now have a unique (and therefore dual optimal) solution $w_{01} = w_{11} = w_{12} = 0$, $w_{02} = 1$. This indicates that the (first term of the) objective function
and both constraint terms converge to zero as \((x,y,z)\) approach the infimum. Letting \(\theta_{01}, \theta_{11}, \text{ and } \theta_{12}\) be three parameters converging to zero, corresponding to these three terms, we use the primal-dual relationship (in log-linear form) to determine a path by which \((x,y,z)\) converges to the infimum:

\[
\begin{align*}
\ln x + \ln y + \ln z &= \ln \theta_{01} \\
\ln y - 2 \ln z &= \ln \left( \frac{\theta_{11}}{\theta_{11} + \theta_{12}} \right) \\
-\ln x - \ln y + \ln z &= \ln \left( \frac{\theta_{12}}{\theta_{11} + \theta_{12}} \right)
\end{align*}
\]

which is a linear system in the logarithms of the primal variables, having the solution:

\[
\begin{align*}
x &= \frac{(\theta_{11} + \theta_{12})^{2.5}}{\theta_{01}^{0.5} (\theta_{11} + \theta_{12})^{1.5}} \\
y &= \frac{\theta_{01} \theta_{11} \theta_{12}^{0.5}}{(\theta_{11} + \theta_{12})^{2}} \\
z &= \frac{\theta_{01} \theta_{12}^{0.5}}{(\theta_{11} + \theta_{12})^{0.5}}
\end{align*}
\]

Each path for the parameters \(\theta=(\theta_{01}, \theta_{11}, \theta_{12}) \to 0\) then defines a path for \((x,y,z)\). In particular, if \(\theta_{01}=\theta_{11}=\theta_{12} \to 0\) then we see that on the path for \((x,y,z)\),

\[
x = 2^{2.5} \theta_{01}^{0.5}, \quad y = \theta_{01}^{0.5}, \quad z = \theta_{01}^{0.5}
\]

and so the infimum is approached as \(x \to +\infty, y \to 0, \text{ and } z \to 0\).

3. Summary

If a posynomial geometric programming problem has at least as many variables as terms, then the dual problem possesses "negative degree of difficulty", with more linear equality constraints than dual variables. Depending upon the rank of the dual linear equality constraints, the dual solution may be "overdetermined", i.e. the dual problem may be infeasible. The method suggested by Sinha et al. ([5]) for obtaining an "approximate solution" to an infeasible dual problem, from which an "approximate primal solution" is then computed, is poorly motivated and ignores the nature of the primal optimal solution, in which the minimum is not attained and primal variables may either converge to zero or diverge. We have instead assured dual feasibility by adding a
constant term to the primal objective function and, from the dual feasible solution of this expanded problem, determined the limiting behavior of the primal variables.

References


