Obtaining a Feasible Geometric Programming Primal Solution, Given a Near-Optimal Dual Solution

DENNIS L. BRICKER

and

JAE CHUL CHOI

Dept. of Industrial Engineering
The University of Iowa

June 1994

A dual algorithm applied to a posynomial geometric programming problem generally terminates with a dual feasible solution which is only approximately optimal. The usual methods for recovering the primal solution then yields solutions with corresponding slight infeasibilities in the primal constraints. This paper demonstrates a linear programming problem for computation of a primal solution which is strictly feasible.

KEYWORDS: geometric programming, duality,

1. INTRODUCTION

The geometric programming dual problem offers several computational advantages over the primal, most notably the fact that the objective (after a logarithmic transformation) is separable, and the constraints are linear equalities. This comes not without computational obstacles, including the nondifferentiability of the objective function when one or more dual variables are zero (e.g., when a primal constraint is inactive.) These obstacles and suggestions for overcoming them are discussed in a large body of literature[1] and will not be discussed here. Another difficulty inherent in dual algorithms is in the recovery of a primal feasible solution. This is sometimes true even if the exact dual optimum has been found[2].

More importantly, though, is the need to find a primal feasible (near-optimal) solution when only a near-optimal dual solution has been found. Such is the case if the dual algorithm is terminated before it has converged to the optimum, the usual circumstance in practice. In this case, the corresponding primal solution found by commonly-used procedures will either violate one or more of the inequality constraints or be non-optimal--generally the former. The following describes a procedure for computing a primal feasible solution, given estimates of the dual optimal solution.

2. THE POSYNOMIAL GEOMETRIC PROGRAMMING PROBLEM

The primal geometric programming problem is to

$$\text{Minimize } y_0(x) = \sum_{t=1}^{T_0} c_{0t} \prod_{n=1}^{N} x_n^{a_{0tn}} \tag{2.1}$$

$$\text{subject to } y_m(x) = \sum_{t=1}^{T_m} c_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \leq 1, \; m=1,\cdots M \tag{2.2}$$

$$x_n > 0, \; n=1, \ldots N$$

(Note that the primal variables are constrained to be strictly positive, so that the feasible region is not, in general, closed.) The dual problem is to

$$\text{Maximize } v(\delta,\lambda) = \prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{c_{mt}\lambda_m}{\delta_{mt}} \right)^{\delta_{mt}} \tag{2.3}$$

or, equivalently (since the logarithm function is monotonically increasing),

$$\text{Maximize } \ln v(\delta,\lambda) = \sum_{m=0}^{M} \sum_{t=1}^{T_m} \left(\ln \frac{c_{mt}\lambda_m}{\delta_{mt}} \right) \delta_{mt} \tag{2.4}$$

$$\text{subject to } \quad \sum_{m=0}^{M} \sum_{t=1}^{T_m} a_{mtn}\delta_{mt} = 0, \; n=1,\cdots N \tag{2.5}$$

$$\lambda_m = \sum_{t=1}^{T_m} \delta_{mt},\ m=0,\ 1, \cdots M \tag{2.6}$$

$$\lambda_0 = 1 \tag{2.7}$$

$$\delta_{mt} \geq 0,\ t=1, \cdots T_m,\ m=0,\ 1, \cdots M$$

Let T= $\Sigma_m T_m$ and denote the N×T coefficient matrix of equations (2.5) by A. Then x^* and (δ^*,λ^*) are the primal and dual optimal solutions, respectively, if and only if the *invariance conditions*

$$\delta^*_{mt}\ y_m(x^*) = \lambda^*_m\ c_{mt} \prod_{n=1}^{N} x^{*a_{mtn}}_n\ ,\ t=1, \cdots T_m,\ m=0,\ 1, \cdots M \tag{2.8}$$

are satisfied.

If a dual solution (δ^*,λ^*) is known, then these relationships may perhaps be used to compute a primal solution x*. Note that, from these relationships, one may obtain a system of equations linear in the logarithms of the optimal values of the primal variables (where y_m^* , for m>0, is $y_m(x^*)=1$ if $\lambda_m \neq 0$, and $y_0(x^*) = d(\delta^*,\lambda^*)$ for m=0):

$$\sum_{n=1}^{N} a_{mtn}\ \ln x_n = \ln\left(\frac{\delta^*_{mt} y^*_m}{\lambda^*_m c_{mt}}\right),\ t=1, \cdots T_m \tag{2.9}$$

for m=0, 1, ...M such that $\lambda^*_m \neq 0$.

Denote by A̲ the transpose of the coefficient matrix of equations (2.9), i.e., the submatrix of A in which are deleted the columns for which the corresponding variable λ_m is zero. Equations (2.9) provide necessary, but not sufficient conditions for primal optimality; if rank $\underline{A}^T < N$, then the optimal primal variables are not uniquely determined by (2.9), requiring the solution of a subsidiary problem[3]. In this new problem, all terms determined by equation (2.8), i.e., terms in posynomial m for which $\lambda_m \neq 0$, are fixed and at least one additional posynomial constraint is tightened:

Minimize x_0

subject to $$c_{0t} \prod_{n=1}^{N} x_n^{a_{0tn}} \le \delta_{0t}^{*} y_0^{*}, \; t=1,\cdots T_0 \tag{2.10}$$

$$c_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \le \frac{\delta_{mt}^{*}}{\lambda_m^{*}}, \; t=1,\cdots T_m \;\text{ if } \lambda_m^{*} \ne 0, \; m=1,\cdots M \tag{2.11}$$

$$y_m(x) = \sum_{t=1}^{T_m} c_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \le x_0, \;\text{ if } \lambda_m^{*} = 0, \; m=1,\cdots M \tag{2.12}$$

$x_n > 0$, n=1, ...N

(If, after solution of this subsidiary problem, equations (2.9) again do not determine the optimal primal variables, then still another subsidiary problem must be solved.)

An often more convenient method of recovering the primal solution when one has available the optimal Lagrange multipliers u^*_n, n=1,... N, for the orthogonality constraints in the GP dual problem (with the logarithm of (2.3) as the objective) is to use the relationship[3]

$$x_n^* = \exp(u_n^*), \;\; n = 1, \ldots N \tag{2.13}$$

In the aforementioned situation where equations (2.9) fail to determine the primal solution, however, the Lagrange multipliers are nonunique, and equation (2.13) also fails to determine x^*_n.

Relationships (2.9) and (2.13) apply to the *optimal* pair of primal-dual solutions. If, as is generally the case in practice, the algorithm for solving the GP dual is terminated before an exact dual optimum is achieved, the use of equation (2.13) for computing x_n will yield a nonoptimal primal solution, which generally violates to some degree one or more of the primal inequality constraints, if any, while equations (2.9) will generally constitute an overdetermined system of equations with no solution. Of course, one might solve any M linearly independent equations from (2.9), or find a least-squares-error solution to (2.9), in order to find a primal solution, but such a solution will generally be neither optimal nor feasible. Depending upon the circumstances, and the degree of the infeasibility, the primal solution thus obtained may be acceptable. For example, if the constraint coefficients are only estimates, the constraint need not be considered a "hard" constraint, and small infeasibilities may be of no consequence[4]. In other circumstances, however, the constraint may not permit any infeasibility, as might be the case in which the constraint embodies some physical law. In the section that follows, we introduce a subsidiary problem, similar to (2.10-12), to ensure that x_n is truly feasible.

3. ENSURING PRIMAL CONSTRAINT FEASIBILITY

Suppose that $(\underline{\delta}, \underline{\lambda})$ is a feasible, near-optimal, solution to the geometric programming dual problem. For each positive $\underline{\delta}_{mt}$, define the ratio $\underline{\rho}_{mt} = \underline{\delta}_{mt} / \underline{\lambda}_m$ (where $\underline{\lambda}_0$=1). Note that , for each posynomial m for which $\underline{\lambda}_m > 0$,

$$\sum_{t=1}^{T_m} \underline{\rho}_{mt} = 1 \tag{3.1}$$

(The algorithm in Bricker and Rajgopal[5] provides the ratios $\underline{\rho}_{mt}$ directly for *all* terms, even when the corresponding constraint is loose and $\underline{\lambda}_m$=0.)

Then a solution to the following "subsidiary" problem will be primal feasible:

$$\text{Minimize } x_0 \tag{3.2}$$

$$\text{subject to } c_{0t} \prod_{n=1}^{N} x_n^{a_{0tn}} \leq \underline{\rho}_{0t} x_0 \text{ , } t=1,\cdots T_0 \tag{3.3}$$

$$c_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \leq \underline{\rho}_{mt}, \; t=1,\cdots T_m \text{ if } \underline{\lambda}_m \neq 0, \; m=1,\cdots M \tag{3.4}$$

$$x_n > 0, \; n=1, \ldots N$$

This is essentially a geometric programming problem, with each posynomial consisting of a single term, and it is well-known[6] that the constraints are linear under a logarithmic transformation. Letting $u_n = \ln x_n$, we obtain the linear programming problem

$$\text{Minimize } u_0 \tag{3.5}$$

$$\text{subject to} \quad \ln c_{0t} + \sum_{n=1}^{N} a_{0tn} u_n \le \ln \underline{\rho}_{0t} + u_0, \ \ t = 1, \ldots T_0 \tag{3.6}$$

$$\ln c_{mt} + \sum_{n=1}^{N} a_{mtn} u_n \le \ln \underline{\rho}_{mt}, \ t=1,\cdots T_m \text{ if } \underline{\lambda}_m \ne 0, \ m=1,\cdots M \tag{3.7}$$

The dual of this LP problem is

$$\text{Maximize} \sum_{m=0}^{M} \sum_{t=1}^{T_m} \left(\ln \frac{c_{mt}}{\underline{\rho}_{mt}}\right) \delta_{mt} \tag{3.8}$$

$$\text{subject to} \sum_{t=1}^{T_0} \delta_{0t} = 1 \tag{3.9}$$

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} a_{mtn} \delta_{mt} = 0, \ n=1,\cdots N \tag{3.10}$$

$$\delta_{mt} \ge 0, \ t=1,\cdots T_m, \ m=0, 1,\cdots M$$

It is interesting to compare this LP with the GP dual problem and to note that the constraints are identical, and that to each term

$$\left(\ln \frac{c_{mt}}{\underline{\rho}_{mt}}\right) \delta_{mt}, \text{ i.e.,} \left(\ln \frac{c_{mt}\underline{\lambda}_m}{\underline{\delta}_{mt}}\right) \delta_{mt} \tag{3.11}$$

in the LP objective there corresponds the term

$$\left(\ln \frac{c_{mt}\lambda_m}{\delta_{mt}}\right) \delta_{mt} \tag{3.12}$$

in the (logarithm) of the GP dual problem. Thus we see that the LP problem above may be considered a linearization of the GP dual problem. To obtain a primal feasible (near-optimal) solution to a GP problem, then, we may apply a dual algorithm to the problem, obtaining a dual-feasible near-optimal solution $(\underline{\delta},\underline{\lambda})$, replace the dual objective terms (3.12) with linear terms (3.11), optimally solve the resulting LP, and, finally, exponentiate the optimal simplex multipliers.

4. EXAMPLE

Let us consider the following posynomial geometric programming problem of Rijckaert and Martens[2].

Minimize x_1^{-1}

subject to $x_1 x_2^{-1} + 0.5 x_3^{-1} \le 1$

$0.01\, x_3 x_4^{-1} + 0.01\, x_2 + 0.0005\, x_2\, x_4^{-1} \le 1$

$x_n > 0,\ n=1,2,3,4$

(Note that there is apparently a typographical error in the source cited above.) The dual of this problem is to

$$\text{Maximize } \left(\ln \frac{\lambda_0}{\delta_{01}}\right)\delta_{01} + \left(\ln \frac{\lambda_1}{\delta_{11}}\right)\delta_{11} + \left(\ln \frac{0.5\lambda_1}{\delta_{12}}\right)\delta_{12}$$
$$+ \left(\ln \frac{0.01\lambda_2}{\delta_{21}}\right)\delta_{21} + \left(\ln \frac{0.01\lambda_2}{\delta_{22}}\right)\delta_{22} + \left(\ln \frac{0.0005\lambda_2}{\delta_{23}}\right)\delta_{23}$$

subject to

$$-\delta_{01} + \delta_{11} = 0$$
$$-\delta_{11} + \delta_{22} + \delta_{23} = 0$$
$$-\delta_{12} + \delta_{21} = 0$$
$$-\delta_{21} - \delta_{23} = 0$$
$$\delta_{01} = 1$$
$$\delta_{11} + \delta_{12} = \lambda_1$$
$$\delta_{21} + \delta_{22} + \delta_{23} = \lambda_2$$

$\lambda_m \ge 0$ and $\delta_{mt} \ge 0$, for all m and t

(The primal solution given is not feasible[2], violating the constraints by the quantities 0.002548762 and 0.000003288, respectively.) Suppose that we have solved this problem, using a dual algorithm, e.g. that of Bricker and Rajgopal[5], specifying a stopping tolerance of 10^{-3} (i.e., the maximum relative objective gap and constraint violation) and have obtained the dual feasible solution $\underline{\rho}_{01}=1$, $\underline{\rho}_{11}= \underline{\delta}_{11}/\underline{\lambda}_1 = 0.943320$, $\underline{\rho}_{12}= \underline{\delta}_{12}/\underline{\lambda}_1 = 0.056680$, $\underline{\rho}_{21}= \underline{\delta}_{21}/\underline{\lambda}_2 = 0.058948$, $\underline{\rho}_{22}= \underline{\delta}_{22}/\underline{\lambda}_2 =0.875606$, $\underline{\rho}_{23}= \underline{\delta}_{23}/\underline{\lambda}_2 = 0.065446$, $\underline{\lambda}_1= \underline{\lambda}_2= 1.066262$, with objective value (ln 0.012093786). (Because the termination criterion of the algorithm uses a positive tolerance, this is not in fact optimal, but only nearly so.)

Using (2.13) with the Lagrange multipliers for the orthogonality constraints (2.6), we obtain the primal solution $x_1=82.68709$, $x_2=87.58691$, $x_3=8.81448$, $x_4=1.49486$, having the same primal objective value, but for which the two primal constraints are violated by the quantities 0.00078246 and 0.00029945, respectively. Instead, to obtain a primal solution which is feasible, we replace the nonlinear term

$$\left(\ln \frac{c_{mt}\lambda_m}{\delta_{mt}}\right) \delta_{mt}$$

by the linear term

$$\left(\ln \frac{c_{mt}}{\underline{\rho}_{mt}}\right) \delta_{mt}$$

in the objective function of the dual problem. For example, the first term

$$\left(\ln \frac{\lambda_1}{\delta_{11}}\right) \delta_{11}$$

is replaced by the linear term

$$\left(\ln \frac{1}{0.943320}\right) \delta_{11} = 0.0583497\ \delta_{11}$$

In this way, we obtain the LP problem with the objective

$$\text{Maximize } 0.0583497\delta_{11} + 2.87033\delta_{12} - 1.77407\delta_{21} - 4.47233\delta_{22} - 4.87437\delta_{23}$$

and the constraints of the dual GP problem above. Using (2.13) with the optimal simplex multipliers of this LP, we obtain the primal GP solution $x_1=82.50842$, $x_2=87.46604$,

x_3=8.82138, x_4=1.49648, for which the objective is 1/x_1 = 0.0121998 and primal constraint functions $y_1(x)$=1 and $y_2(x)$= 0.999053. That is, we have a feasible primal solution, for which the objective function is guaranteed to be within (0.012119975-0.012093786)/0.012093786= 0.21656% of the optimal cost.

5. COMPUTATIONAL EXPERIANCE

To further illustrate the results obtained by the procedure proposed above, we have selected five problems from the literature, with characteristics displayed in Table 1. Problems RM1, RM2, RM3, and RM5 appear in the collection compiled by Rijckaert and Martens[2], while B&E1 appears in the collection compiled by Beck and Ecker[7].

Table 1 : Problem Characteristics

Problem	Charecteristics				
	No. of variables	No. of constraints	Total no. of terms	Optimal value (tol=1.0E-10)	No. of tight constraints
RM 1	4	2	6	0.0121031862	2
RM 2	3	1	9	6299.8424	1
RM 3	4	1	12	126303.1780	1
B&E1	3	2	6	11.77813352	1
RM 5	4	3	8	623249.8761	3

The algorithm used to solve the GP dual problems is that of Bricker and Rajgopal[5], which is comparable to the GGP algorithm of Dembo[8], both yield dual feasible solutions and employ as a stopping criterion the maximum permissible primal constraint infeasibilities and objective gap.

For each of these five problems, and each of four tolerances (ranging from 10^{-1} to 10^{-4}), we performed the following computations:

i. "solution" of the geometric programming dual problem with the given tolerances employed in the stopping criterion, obtaining four feasible near-optimal dual solutions,

ii. exponentiation of the Lagrange multipliers of the orthogonality constraints as in equation (2.13) to obtain a near-feasible primal solution,

iii. use of the dual "solution" to form a linear approximation (3.8) to the dual objective,

iv. solution of the linear program (3.8)-(3.10) with this linear objective and the dual GP linear constraints,

v. exponentiation of the simplex multipliers of the orthogonality constraints (3.10) as in (2.13) to obtain a primal feasible near-optimal solution, and

vi. computation of the relative gap between the primal objective evaluated at the infeasible solution found in step (ii.) and those in step (v.).

Table 2 displays the following results for each of these twenty cases:

- the primal objective value evaluated at the primal infeasible solution found in step (ii) above
- the primal objective value evaluated at the primal feasible solution found in step (v) above
- the "gap" between the previous two values, expressed as a fraction of the former, i.e., the "degradation" of the objective function which resulted when primal feasibility was enforced
- the number of primal constraints violated by the solution found in step (ii) above
- the sum of the violations of the primal constraints by the solution found in step (ii) above

Table 2: Computational Results

Problem	Stopping tolerance	Objective value at termination	No. of infeasible constraints	Sum of infeasi-bilities	Objective value of feasible solution	Gap (%)
	10^{-1}	0.0115639242	2	0.060223266	0.0130195647	11.1804
RM1	10^{-2}	0.0119342099	2	0.013750256	0.0122657862	2.7033
	10^{-3}	0.0120937860	2	0.001081912	0.0121199750	0.2166
	10^{-4}	0.0121028663	2	0.000107195	0.0121054613	0.0214
	10^{-1}	6479.2910	1	0.05737484	6625.7211	2.2600
RM2	10^{-2}	6309.8208	1	0.00403163	6318.3694	0.1355
	10^{-3}	6300.5979	1	0.00023193	6301.1223	0.0083
	10^{-4}	6299.7106	1	0.00005909	6299.8452	0.0021
	10^{-1}	126116.05	1	0.097989755	133365.77	5.7485
RM3	10^{-2}	126281.45	1	0.009143721	126974.14	0.5485
	10^{-3}	126277.37	1	0.00843411	126341.81	0.0510
	10^{-4}	126300.38	1	0.000069778	126305.71	0.0042
	10^{-1}	11.781512	1	0.052043228	11.982448	1.7055
B&E1	10^{-2}	11.796952	1	0.003092963	11.809102	0.1030
	10^{-3}	11.779655	1	0.000771352	11.782683	0.0257
	10^{-4}	11.778348	1	0.000048173	11.778537	0.0016
	10^{-1}	566305.11	1	0.0819200	656133.37	15.8622
RM5	10^{-2}	622678.40	1	0.0008108	623594.29	0.1471
	10^{-3}	622751.18	1	0.0005423	623362.14	0.0981
	10^{-4}	623294.33	1	0.0000453	623345.42	0.0082

6. SUMMARY

For geometric programming problems where a dual algorithm has been used, the termination criterion in general results in a dual feasible near-optimal solution. The usual methods for recovering the primal solution then produces slight infeasibilities in the primal constraints. In many situations, these slight infeasibilities are tolerable. When, however, absolutely no primal feasibility can be tolerated, we have demonstrated how a primal feasible solution can be obtained through solution of a linear programming problem whose objective coefficients are determined by the dual near-optimal solution.

7. REFERENCES

1. Ecker, J. G., Gochet, W. and Smeers, Y. (1978) Computational Aspects of Geometric Programming 3: Some Primal and Dual Algorithms for Posynomial and Signomial Geometric Programs, *Engineering Optimization* , 3, pp. 147-160.

2. Rijckaert, M. J. and Martens, X. M. (1978) Comparison of generalized geometric programming algorithms, *Journal of Optimization Theory and Applications* , 26, pp. 205-242.

3. Dembo, R. S. (1980) Primal to dual conversion in geometric programming, in: Avriel, M. (ed.), *Advances in Geometric Programming*, Plenum Press, New York, pp. 333-342.

4. Rosenberg, E. (1981) On Solving a Primal Geometric Program by Partial Dual Optimization, *Mathematical Programming* , 21, pp. 319-330.

5. Bricker, D. L. and Rajgopal, J. (1982) Yet Another Geometric Programming Dual Algorithm, *Operations Research Letters*, 2, pp. 177-180.

6. Duffin, R. J., Peterson, E. L. and Zener, C. M. (1967) *Geometric Programming*, John Wiley & Sons, New York, pp. 265-267.

7. Beck, P. A. and Ecker, J. G. (1972) *Some Computational Experience with a Modified Convex Simplex Algorithm in Geometric Programming*, Rensselaer Polytechnic Institute, Report AFSC.

8. Avriel, M., Dembo, R. and Passy, U. (1975) Solution of Generalized Geometric Programs, *International Journal of Numerical Methods in Engineering*, 9, pp. 149-168.