Facility Location Problems in the Plane

Suppose that we wish to select the location of a single facility, anywhere in the plane, to serve a set of demand points.

**Given**
- for each of demand points \(j=1,2,...,n\):
  - \((x_j,y_j)\) coordinates of the point
  - \(\beta_j\) cost per unit volume per unit distance
  - \(w_j\) volume of shipments per unit time

**Find**
- coordinates of the source facility, \((x,y)\), which will minimize the total shipping cost per unit time:

\[
C(x,y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}
\]

assuming "straight-line" Euclidean distance.

**Webber's Problem**

Tie together in a knot ("X") \(n\) strings of equal length \(L\)

**Theorem**

The function:

\[
C(x,y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}
\]

is convex in \((x,y)\)

A necessary condition for \((X^*,Y^*)\) to minimize

\[
C(x,y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}
\]

is

\[
\begin{align*}
\frac{\partial}{\partial x} C(X^*,Y^*) &= 0 \\
\frac{\partial}{\partial y} C(X^*,Y^*) &= 0
\end{align*}
\]

That is, \((X^*,Y^*)\) should be a "stationary point" of the function \(C\).

This condition yields the equations

\[
\begin{align*}
\sum_{j=1}^{n} \frac{\beta_j w_j (X^*-x_j)}{\sqrt{(X^*-x_j)^2 + (Y^*-y_j)^2}} &= 0 \\
\sum_{j=1}^{n} \frac{\beta_j w_j (Y^*-y_j)}{\sqrt{(X^*-x_j)^2 + (Y^*-y_j)^2}} &= 0
\end{align*}
\]

which, unfortunately, we cannot solve analytically for the values of \(X^*\) and \(Y^*\)!
Rearrange terms:

\[
\begin{align*}
X^* &= \sum_{j=1}^{n} \frac{\beta_j W_j x_j}{d_j(x^*, y^*)} \\
Y^* &= \sum_{j=1}^{n} \frac{\beta_j W_j y_j}{d_j(x^*, y^*)}
\end{align*}
\]

We will use a "successive substitution" method using these equations to find \( X^* \) & \( Y^* \)

\[
\begin{align*}
X^* &= \sum_{j=1}^{n} \frac{\beta_j W_j x_j}{d_j(x^*, y^*)} \\
Y^* &= \sum_{j=1}^{n} \frac{\beta_j W_j y_j}{d_j(x^*, y^*)}
\end{align*}
\]

Suppose, at iteration \( k \), we have an approximate solution \((X^k, Y^k)\).
We obtain an improved approximate solution \((X^{k+1}, Y^{k+1})\) by

\[
\begin{align*}
X^{k+1} &= \sum_{j=1}^{n} \frac{\beta_j W_j x_j}{d_j(x^{k+1}, y^k)} \\
Y^{k+1} &= \sum_{j=1}^{n} \frac{\beta_j W_j y_j}{d_j(x^{k+1}, y^k)}
\end{align*}
\]

**Weiszfeld Algorithm**

Starting with an initial "guess" \((X^0, Y^0)\), we will generate a sequence of approximate solutions, \((X^1, Y^1)\), \((X^2, Y^2)\), \((X^3, Y^3)\), ..., which converge to the optimal facility location \((X^*, Y^*)\).

We terminate the method when two successive approximate solutions are "close enough", i.e.,

\[|X^{k+1} - X^k| + |Y^{k+1} - Y^k| < \epsilon \approx 0\]

**Example**

<table>
<thead>
<tr>
<th>Customer</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location</td>
<td>(0.3)</td>
<td>(2.4)</td>
<td>(4.3)</td>
<td>(1.0)</td>
</tr>
<tr>
<td>Rqmt. / (Ton/yr)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

¿Cost per ton-mile is same for all customers

Where should a supply facility be located so that total shipping cost per week is minimized?

**Starting Point**
A good starting point is the centroid, i.e., the weighted average of the customer coordinates:

\[
\begin{align*}
X^0 &= \frac{1}{n} \sum_{j=1}^{n} \beta_j W_j x_j \\
Y^0 &= \frac{1}{n} \sum_{j=1}^{n} \beta_j W_j y_j
\end{align*}
\]

\[
\begin{align*}
X^0 &= \frac{1}{n} \sum_{j=1}^{n} \beta_j W_j x_j = \frac{1 \times 2 + 2 \times 2 + 3 \times 4 + 2 \times 1}{1 + 2 + 3 + 2} = 2.25 \\
Y^0 &= \frac{1}{n} \sum_{j=1}^{n} \beta_j W_j y_j = \frac{1 \times 3 + 2 \times 2 + 3 \times 4 + 2 \times 1}{1 + 2 + 3 + 2} = 2.5
\end{align*}
\]

\[\beta_j = 1 \forall j\]
Now compute distance from \((x^0, y^0)\) to each customer:

\[
\begin{align*}
    d_1 &= \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{-3}{2}\right)^2} = 2.305 \\
    d_2 &= \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = 1.521 \\
    d_3 &= \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{-3}{2}\right)^2} = 1.820 \\
    d_4 &= \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = 2.795
\end{align*}
\]

Apply our successive substitution method to (we hope!) obtain a better approximate solution:

\[
\begin{align*}
    x' &= \frac{\sum w_j x_j d_j}{\sum w_j d_j} = \frac{1 \times 0 + 2 \times 2 \times 3 \times 4 + 2 \times 1}{\sum w_j d_j} = \frac{10.373}{4.113} = 2.522 \\
    y' &= \frac{\sum w_j y_j d_j}{\sum w_j d_j} = \frac{1 \times 3 + 2 \times 4 + 3 \times 3 + 2 \times 0}{\sum w_j d_j} = \frac{11.506}{4.113} = 2.796
\end{align*}
\]

Distance between initial and improved solution:

0.421

First guess

Improved solution

Perform additional iterations, until distance moved is "sufficiently small"
Facility location at \( x = 2.09607, y = 2.06078 \)

Distance to demand point:

\[
\begin{array}{c|c|c|c|c}
1 & 2 & 3 & 4 \\
\hline
2.6577 & 4.7796 & 1.44794 & 3.2664 \\
\end{array}
\]

Total cost is 15.9088

New location is at \( x = 2.09607, y = 2.06078 \)

Rectilinear distance moved is 0.0004

< 0.01 (stopping criterion)