A paper manufacturer must build a wastewater treatment plant for the removal of pulp and other byproducts. The quality of the effluent is measured in units of % 5-day BOD (biological oxygen demand) removed.

1 lb. 5-day BOD = quantity of organic waste which will consume 1 pound of oxygen during 5 days of decomposition.

Each process \( i \) may be designed to remove any specified fraction of BOD from its input.

\[
\begin{align*}
& C_i \leq a_i X_i, \quad a_i < 0 \\
& X_i = \% \text{ of BOD input to process } i \text{ which remains in the output of that process}
\end{align*}
\]

The design problem is one of choosing the combination of processes and appropriate process levels to

- minimize the sum of total annual costs
- attain a required effluent quality \( K \) = maximum 5-day BOD as % of raw waste BOD

It would be very expensive to use a single process to remove the entire required amount of BOD.

The primary process will remove a relatively large amount of BOD very cheaply... then a secondary process may bring effluent to required levels.
Since the individual processes act in series, their
effect is multiplicative:

\[ X_i X_j = \% \text{ of original BOD remaining} \]

For a design involving processes \( i = 1, 2, \ldots, N \)
the minimum cost is found by

\[
\begin{align*}
\text{Min:} & \quad C_1X_1^{-1.47} + C_2X_2^{-1.66} + \cdots + C_NX_N^{-0.33} + 120X_3^{-0.33} \\
\text{subject to:} & \quad X_1 X_2 \cdots X_N \leq K \\
& \quad X_1 > 0, X_2 > 0, \ldots, X_N > 0
\end{align*}
\]

\[ T = \# \text{terms} = N+1 \]
\[ \# \text{degrees of difficulty} = T - (N+1) = 0 \]

Example

Design #1

Uses combination of 4 processes in series:

* #1: Primary Clarifier
* #2: Trickling Filter
* #3: Activated Sludge
* #4: Carbon Adsorption

Suppose that 97.1% of the BOD must be removed,
\[ \frac{1}{K} = \frac{1}{0.029} \approx 34.5 \]

\[
\begin{align*}
-1.47\delta_1 + \delta_5 & = 0 \Rightarrow \delta_1 = \frac{\delta_5}{1.47} \\
-1.66\delta_2 + \delta_5 & = 0 \Rightarrow \delta_2 = \frac{\delta_5}{1.66} \\
-0.3\delta_3 + \delta_5 & = 0 \Rightarrow \delta_3 = \frac{\delta_5}{0.3} \\
-0.33\delta_4 + \delta_5 & = 0 \Rightarrow \delta_4 = \frac{\delta_5}{0.33} \\
\delta_1 + \delta_2 + \delta_3 + \delta_4 & = 1 \Rightarrow \delta_5 = \frac{1}{1.47 + 1.66 + 0.3 + 0.33} = 0.13078
\end{align*}
\]

Optimal cost is

\[
\begin{bmatrix} 19.4 \delta_1 \\ 16.8 \delta_2 \\ 19.5 \delta_3 \\ 120 \delta_4 \\ 34.5 \delta_5 \end{bmatrix} \lambda_1 = 387.439
\]

**optimal dual value - optimal primal cost!**

Solving for the primal variables:

\[ C_i X_i^{-\text{opt}} = \delta_i V^* \Rightarrow X_i = \left[ \frac{\delta_i V^*}{C_i} \right] \]

E.g., \( 19.4X_1^{-1.47} = 0.0889673 \times 387.439 \Rightarrow X_1 = 0.676367 \)

\[
\begin{bmatrix} X_1 = 0.676367 \\ X_2 = 0.69787 \\ X_3 = 0.129609 \\ X_5 = 0.473792 \end{bmatrix} \]

\[ \Rightarrow \quad \text{i.e., process #1 should remove all but 67.6367\% of the BOD, etc.} \]
In general, for any $K$ the optimal cost for design 1 is

$$V(K) = \begin{bmatrix} 19.4 \delta_1 & 16.8 \delta_2 & 91.5 \delta_3 & 120 \delta_4 & 1 \delta_5 \end{bmatrix} \begin{bmatrix} \lambda_1 \end{bmatrix}$$

$$= 243.829 K - 0.130782$$

By enumerating all of the possible combinations of processes, the least-cost design may be determined. (choice depends upon $K$)

For each design $t$ (t=1, 2, 3, ..., 10)
the optimal cost may easily be computed:

$$V_t^*(K) = C_t K^{-A_t}$$

where $C_t$ and $A_t$ are given on the following screen.

### Design Processes C A V(0.029)

<table>
<thead>
<tr>
<th>Design</th>
<th>Processes</th>
<th>C</th>
<th>A</th>
<th>V(0.029)</th>
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<td>387.414</td>
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