Consider the LP:

\[
\begin{align*}
\text{Max} \quad & 2x_1 + 4x_2 + 5x_3 + 3x_4 \\
\text{subject to} \quad & x_1 + 3x_2 + 6x_3 + 2x_4 \leq 24 \\
& 5x_2 + 4x_3 + 4x_4 \leq 20 \\
& \begin{cases} 
0 \leq x_1 \leq 3 \\
0 \leq x_2 \leq 4 \\
0 \leq x_3 \leq 3 \\
0 \leq x_4 \leq 3 
\end{cases}
\end{align*}
\]

Simple upper bounds

When using the Upper Bounding Technique,

- **nonbasic variable** may be at either
  - lower bound
  - upper bound
- a variable enters the basis by
  - increasing if it is at its lower bound
  - decreasing if it is at its upper bound

When using the Upper Bounding Technique,

- **choice of the pivot column**:
  
  reduced cost \(
  \begin{cases} 
  < 0 \text{ if at lower bound} \\
  > 0 \text{ if at upper bound}
  \end{cases}
  \)

  for minimization problem

  relative profit \(
  \begin{cases} 
  > 0 \text{ if at lower bound} \\
  < 0 \text{ if at upper bound}
  \end{cases}
  \)

  for maximization problem

Consider the LP:

\[
\begin{align*}
\text{Maximize} \quad & c^T x \\
\text{subject to} \quad & A x = b \\
& L_i \leq x_i \leq U_i
\end{align*}
\]

\(L_i\) might be, but need not be, zero!

Define a basis ("working basis") \(B\) such that

\[
[A_B^{-1}] \quad \text{exists, i.e.,} \quad \det(A_B^{-1}) = 0
\]

and partition the non-basic variables into subsets

\(L = \{ x_i : L_i \} \) and \(U = \{ x_i : U_i \} \)

The current basic solution is

\[
\begin{align*}
& x_B = (A_B^{-1})^T b - (A_B^{-1}) A_L^T x_L - (A_B^{-1})^T A_U^T x_U \\
& x_U = U_i \\
& x_L = L_i
\end{align*}
\]
A nonbasic variable may be in either set L (at lower bound) or set U (at upper bound).

The "relative profit" ("reduced cost" if minimizing) specifies the change in the objective function per unit increase in the nonbasic variable.

<table>
<thead>
<tr>
<th>Nonbasic Variable</th>
<th>Substitution Rate in Row i ( c_{ij} )</th>
<th>Effect on Basic Variable ( x_i ) in Row i</th>
<th>Blocking Value</th>
<th>Change in ( x_j ) if entering basis</th>
<th>sign of ( c_j - x_j ) ( A^t )</th>
<th>change in objective</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>positive</td>
<td>decrease</td>
<td>( (X_k - L_k)/c_{ij} )</td>
<td>decrease</td>
<td>positive</td>
<td>decrease</td>
</tr>
<tr>
<td>( U )</td>
<td>negative</td>
<td>increase</td>
<td>( (U_k - X_k)/c_{ij} )</td>
<td>increase</td>
<td>negative</td>
<td>increase</td>
</tr>
</tbody>
</table>

Suppose that a nonbasic variable \( x_j \) were to be selected to enter the basis.

When the nonbasic variable \( x_i \) is increasing from its LOWER BOUND:

\[
\theta_0 = U_j - L_j \\
\theta_1 = \min \left\{ \frac{X_k - L_k}{c_{ij}} \right\} \\
\theta_2 = \min \left\{ \frac{U_k - X_k}{c_{ij}} \right\}
\]

The bound on the increase is \( \theta \), where:

\[
\theta = \min \{ \theta_0, \theta_1, \theta_2 \}
\]

When the nonbasic variable \( x_i \) is decreasing from its LOWER BOUND:

\[
\theta_0 = U_j - L_j \\
\theta_1 = \min \left\{ \frac{X_k - L_k}{c_{ij}} \right\} \\
\theta_2 = \min \left\{ \frac{U_k - X_k}{c_{ij}} \right\}
\]

The bound on the decrease is \( \theta \), where:

\[
\theta = \min \{ \theta_0, \theta_1, \theta_2 \}
\]

Examples, with output from APL workspace UBT

**Minimize** 18\( x_1 \) + 25\( x_2 \)

subject to

\[
\begin{align*}
30 \leq 5x_1 + 4x_2 & \leq 55 \\
-12 \leq 4x_1 - 3x_2 & \leq 4 \\
2 \leq x_1 & \leq 5 \\
4 \leq x_2 & \leq 8
\end{align*}
\]

**Max** 2\( x_1 \) + 4\( x_2 \) + 5\( x_3 \) + 3\( x_4 \)

subject to

\[
\begin{align*}
x_1 + 3x_2 + 6x_3 + 2x_4 & \leq 24 \\
5x_1 + 4x_2 + 4x_4 & \leq 20 \\
x_1 + 3, x_2 & \leq 4 \\
x_3 + 3, x_4 & \leq 3 \\
x_1 \geq 0 \quad \forall \end{align*}
\]
Example

Minimize \(18X_1 + 25X_2\)
subject to
\[
\begin{align*}
30 &\leq 5X_1 + 4X_2 \leq 55 \\
-12 &\leq 4X_1 - 3X_2 \leq 4 \\
2 &\leq X_1 \leq 5, \quad 4 \leq X_2 \leq 8
\end{align*}
\]

The ordinary simplex or revised simplex method would require a tableau with 8 constraints, and 8 slack \&/or surplus variables (in addition to \(X_1\) and \(X_2\)). That is, an 8x8 basis matrix is required.

Add slack variables to the \(\leq\) constraints to create equalities. Then put upper bounds on these slack variables:

Maximize \(18X_1 + 25X_2 + 0X_3 + 0X_4\)
subject to
\[
\begin{align*}
5X_1 + 4X_2 + X_3 &\leq 55 \\
4X_1 - 3X_2 + X_4 &\leq 4
\end{align*}
\]

upper & lower bounds
\[
\begin{align*}
2 \leq X_1 \leq 5 \\
4 \leq X_2 \leq 8 \\
0 \leq X_3 \leq 25 \\
0 \leq X_4 \leq 16
\end{align*}
\]

Using UBT, only a 2x2 basis matrix is required, i.e. a reduction of nearly 94% in the number of elements in the inverse matrix!

Current partition:
\(B = \{4\} / L = \{3\} / U = \{1\}\)

Basis inverse matrix = 
\[
\begin{bmatrix}
0.25 & 0 \\
0.75 & 1
\end{bmatrix}
\]

Basic solution: \(X = [5, 7.5, 0, 8.5, 0, 0, 0, 0]^{T}\) with \(Z = 277.5\)

\(X_4 = 16\)

Iteration 1

\[X_4 = 16\]

\[\begin{align*}
\text{initial point} &\quad B = \{4\} \\
L = \{3\} \\
U = \{1\}
\end{align*}\]

Entering variable is \(X_1\) from set \(L\)

Substitution rates: 0.25, 0.75

The substitution rates indicate that for each unit increase by \(X_3\), the first basic variable \(X_4\) will be reduced by 0.25 and the second \(X_3\) will be reduced by 0.75.

\(X_2\) is currently 7.5, and its lower bound is 4, so that it must leave the basis when it is decreased by 3.5, i.e., when \(X_3\) is increased by \(14\). Likewise, \(X_4\) can decrease by only 6.5 before it must leave the basis, i.e., \(X\) can increase by only \(\frac{6.5}{0.75} = 8.67\).
Entering variable is $X_3$: from set $L$.

Substitution Rates: 0.25 0.75

The substitution rates indicate that for each unit increase by $X_3$, the first basic variable ($X_4$) will be reduced by 0.25 and the second ($X_2$) will be reduced by 0.75.

Decreasing variables:
- Block at value: 2
- Block at value: 4

Block at $X_1$ at value 0.66667 ← the minimum ratio

$X_4 = 16$

Initial partition
- $X_4 = 0$
- $X_2 = 0$
- $X_3 = 0$
- $X_1 = 0$

As $X_3$ is increased (while $X_1$ is kept at its upper bound & nonbasic), both $X_2$ and $X_4$ will decrease.

Since we are minimizing, we would choose to decrease either $X_1$ or $X_4$.

However, $X_4$ is already at its lower bound, and so we choose to enter $X_1$ into the basis (from set $U$).

Entering variable is $X_1$: from set $U$.

Substitution Rates: -1.33333 10.3333

The negative substitution rate indicates that the first basic variable ($X_4$) will also decrease as $X_1$ is decreased, while the positive substitution rate indicates that the second basic variable ($X_3$) will increase as $X_1$ is decreased.

Increasing variables:
- Block at value: 3

Decreasing variables:
- Block at value: 2

Block at $X_2$ at value 1

$X_4 = 16$

Partition
- $X_4 = 0$
- $X_3 = 0$

As $X_1$ is decreased, $X_2$ is also decreased, while $X_3$ is increased (while $X_4$ is kept at its lower bound, zero).

$X_3 = 25$
Upper Bounding Technique 7/23/98 page 5

As $X_4$ is increased, first $X_3$ reaches its upper bound, and then $X_1$ reaches its lower bound.

Since no change in the nonbasic variables will yield an improved solution, the current solution is optimal!

Example:

Max $2x_1 + 4x_2 + 5x_3 + 3x_4$

subject to

$\begin{align*}
x_1 + 3x_2 + 6x_3 + 2x_4 & \leq 24 \\
5x_1 + 4x_2 & \leq 20
\end{align*}$

Simple upper bounds

Current partition:

$B = \{1, 4\}$

$L = \{2, 3\}, U = \{3\}$

Basis inverse matrix:

$\begin{bmatrix}
0.2 & 0 \\
0.8 & 1
\end{bmatrix}$

Basic solution: $2.6 \ 4 \ 25 \ 4.8$ with $Z = 150.4$

Simplex multipliers: $3.6 \ 0$

Reduced costs:

The positive reduced cost indicates that lowering $X_3$ would improve the solution, but $X_3$ is already at its lower bound.

The negative reduced cost indicates that increasing $X_2$ would improve the solution, but $X_2$ is already at its upper bound.

Optimal Solution!

Iterations:

Iteration 1:

Current partition:

$B = \{1, 4\}$

$L = \{2, 3\}, U = \{3\}$

Basis inverse matrix:

$\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}$

Basic solution: $0 \ 0 \ 0 \ 0 \ 24 \ 20$ with $Z = 0$

Simplex multipliers: $0 \ 0$

Relative profits: $2 \ 4 \ 5 \ 3 \ 0 \ 0$

Entering variable is $X_{13}$ from set $L$

Substitution rates: $6 \ 0$

Decreasing variables:

Variable does not enter basis, but moves to opposite bound

Iteration 2:

Current partition:

$B = \{5 \ 6\}$

$L = \{1 \ 2 \ 3 \ 4\}, U = \{3\}$

Basis inverse matrix:

$\begin{bmatrix}
0.2 \ 3333333 & 0 \\
1\ 5333333 & 1
\end{bmatrix}$

Basic solution: $0 \ 0 \ 0 \ 0 \ 24 \ 20$ with $Z = 0$

Simplex multipliers: $0 \ 0$

Relative profits: $2 \ 4 \ 5 \ 3 \ 0 \ 0$

Entering variable is $X_{13}$ from set $L$

Substitution rates: $6 \ 0$

Decreasing variables:

Variable 2 replaces $X_5$, and $X_5$ enters $L$

Current partition:

$B = \{2 \ 6\}$

$L = \{1 \ 4 \ 5\}, U = \{3\}$

Basis inverse matrix:

$\begin{bmatrix}
0.5333333 \ 0.1533333 \\
0 \ 1
\end{bmatrix}$

Basic solution: $0 \ 2 \ 3 \ 0 \ 0 \ 12$ with $Z = 23$

Simplex multipliers: $1 \ 3333333$

Relative profits: $0.6666667 \ 0 \ 0.3333333 \ 0$

Entering variable is $X_{13}$ from set $U$

Substitution rates: $2 \ 8$

Increasing variables:

Variable 2 is replaced by variable 3 in $B$, and variable 2 enters set $U$
Current partition: B = \{3, 6\}, L = \{1, 5, 6\}, U = \{2\}

Basis inverse matrix:

\[
\begin{bmatrix}
0.166667 & 0 \\
0 & 1
\end{bmatrix}
\]

Basic solution: 0 4 2 0 0 4 with Z = 26

Simplex multipliers: 0.833333 0

Relative profits: 1.16667 1.5 0 1.33333 -0.833333 0

Entering variable is \(x(4)\) from set L

Substitution rates: 0.333333 4

Decreasing variables: 3 6

Block at value: 6.000 1.000

Block at \(x(5)\) at value 1

Variable 6 is replaced in B by variable 4, and enters set L

Optimal partition: B = \{3, 4\}, L = \{1, 5, 6\}, U = \{2\}

Optimal Solution:

\[
\begin{bmatrix}
0.000 \\
4.000 \\
1.833 \\
1.000 \\
0.000 \\
0.000
\end{bmatrix}
\]

Objective Z = 27.33333