Transportation Problem

Find the least-cost shipments which will satisfy the given requirements, using the available supplies.

Supplies

\[ \begin{array}{c|ccc}
  & \text{10} & \text{10} & \text{20} \\
  \hline
  \text{1} & \text{3} & \text{5} & \text{10} \\
  \text{2} & \text{2} & \text{5} & \text{5} \\
  \text{3} & \text{3} & \text{3} & \text{3} \\
\end{array} \]

Quantities Required

\[ \begin{array}{c|ccc}
  & \text{10} & \text{10} & \text{20} \\
  \hline
  \text{1} & \text{5} & \text{6} & \text{1} \\
  \text{2} & \text{3} & \text{3} & \text{5} \\
  \text{3} & \text{1} & \text{1} & \text{3} \\
\end{array} \]

LP Formulation of the Transportation Problem

Let \( i \) index the sources, and \( j \) the destinations.

\( m \) - \# of sources, \( n \) - \# of destinations

Given:
- \( S_i \) = quantity of goods available at source \( i \)
- \( D_j \) = quantity of goods required at destination \( j \)
- \( C_{ij} \) = unit cost of shipping goods from source \( i \) to destination \( j \)

Find:
- \( X_{ij} \) = quantity of goods to be shipped from source \( i \) to destination \( j \)

Minimize
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}
\]

subject to
\[
\sum_{j=1}^{n} X_{ij} \leq S_i \quad \text{for } i=1, \ldots, m
\]
\[
\sum_{i=1}^{m} X_{ij} \geq D_j \quad \text{for } j=1, \ldots, n
\]
\[
X_{ij} \geq 0, \text{ all } i, j
\]

This is an LP with: \( m \times n \) variables, \( m \times n \) constraints (not including nonnegativity).

The standard, "balanced", transportation problem has

\[
\text{total supply } = \text{total demand} \Rightarrow \sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j
\]

so that all constraints will be "tight" at a feasible solution, i.e.,

\[
\sum_{i=1}^{m} X_{ij} = S_i \quad \text{for } i=1, \ldots, m
\]
\[
\sum_{j=1}^{n} X_{ij} = D_j \quad \text{for } j=1, \ldots, n
\]

Conversion to Standard Form

When total supply exceeds total demand: \( \sum_{i=1}^{m} S_i > \sum_{j=1}^{n} D_j \)

Create a "dummy" destination \((n+1)\) whose "demand" is equal to the surplus supply:

\[
D_{n+1} = \sum_{i=1}^{m} S_i - \sum_{j=1}^{n} D_j
\]

and let the cost of "shipping" to this destination be \( C_{i,n+1} = 0 \)

\((X_{ij} \text{ will equal the unshipped supply at source } i)\)

Conversion to Standard Form

When total demand exceeds total supply: \( \sum_{i=1}^{m} S_i < \sum_{j=1}^{n} D_j \)

In this case, the problem is infeasible, i.e., not all demand can be satisfied.

One can create a "dummy" source \((m+1)\) whose available supply is the shortfall, i.e.,

\[
S_{m+1} = \sum_{j=1}^{n} D_j - \sum_{i=1}^{m} S_i
\]

and define the cost of "shipping" to be \( C_{m+1,j} = \text{unit shortage cost at destination } j \)
The LP tableau

\[\begin{array}{cccc|c|c}
\text{Dest.} & \text{5C} & \text{5S} & \text{5O} & \text{5N} & \text{MIN} \\
\hline
\text{S1} & 1 & 1 & 1 & 1 & 12 \\
\text{S2} & 1 & 1 & 1 & 1 & 7 \\
\text{S3} & 1 & 1 & 1 & 1 & 15 \\
\end{array}\]

Even though the transportation problem is an LP problem and is solved by the Simplex Method for LP, we do not use the usual LP tableau.

The TP tableau

<table>
<thead>
<tr>
<th>Data</th>
<th>ATLANTA</th>
<th>L.A.</th>
<th>DALLAS</th>
<th>CHGO</th>
<th>N.Y.</th>
<th>EXCESS</th>
<th>CAP.</th>
<th>supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>HOME CITY</td>
<td>0.95</td>
<td>0.05</td>
<td>0.80</td>
<td>0.15</td>
<td>0.00</td>
<td>12</td>
<td>12</td>
<td>7</td>
</tr>
<tr>
<td>BRANCH #1</td>
<td>0.35</td>
<td>1.00</td>
<td>1.40</td>
<td>0.80</td>
<td>0.30</td>
<td>7</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>BRANCH #2</td>
<td>0.90</td>
<td>1.80</td>
<td>1.60</td>
<td>0.70</td>
<td>0.85</td>
<td>15</td>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>demand:</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>11</td>
<td>8</td>
<td>2</td>
<td>34</td>
<td>15</td>
</tr>
</tbody>
</table>

Shipment will be routed here

To perform the Simplex Method, we need to:

1. Obtain an initial basic feasible solution
2. "Price" the nonbasic variables & select an entering variable
3. Select the basic variable which will leave the basis

How are these steps performed using the transportation tableau?

The Simplex method requires a **BASIC FEASIBLE SOLUTION** (bfs) to begin.

- **3 commonly used methods:**
  - **Northwest Corner Method**
  - **Least-Cost Method**
  - **Vogel's Approximation Method**

**Obtaining an initial bfs:**

1. **Northwest Corner Rule**

   Step 1: Assign to the upper left corner of the TP tableau the minimum of the supply in that row & the demand in that column:
   
   \[X_{ij} = \min\{s_i, d_j\}\]

   Step 2: Reduce the supply & demand for that row & column by \(X_{ij}\)

   Step 3: Delete any row &/or column with zero supply or demand, and return to step 1.

**Northwest Corner Rule**

<table>
<thead>
<tr>
<th>Data</th>
<th>ATLANTA</th>
<th>L.A.</th>
<th>DALLAS</th>
<th>CHGO</th>
<th>N.Y.</th>
<th>EXCESS</th>
<th>CAP.</th>
<th>supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>HOME CITY</td>
<td>0.95</td>
<td>0.05</td>
<td>0.80</td>
<td>0.15</td>
<td>0.00</td>
<td>12</td>
<td>12</td>
<td>7</td>
</tr>
<tr>
<td>BRANCH #1</td>
<td>0.35</td>
<td>1.00</td>
<td>1.40</td>
<td>0.80</td>
<td>0.30</td>
<td>7</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>BRANCH #2</td>
<td>0.90</td>
<td>1.80</td>
<td>1.60</td>
<td>0.70</td>
<td>0.85</td>
<td>15</td>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>demand:</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>11</td>
<td>8</td>
<td>2</td>
<td>34</td>
<td>15</td>
</tr>
</tbody>
</table>

Starting in the upper-left ("northwest") corner, i.e., the shipping route from HOME CITY to ATLANTA, we assign \(X = \min\{12, 5\} = 5\) to the route, and reduce the supply at HOME CITY, and the demand at ATLANTA each by 5.
### Northwest Corner Rule

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#### Initial Feasible Solution

<table>
<thead>
<tr>
<th></th>
<th>ATLANTA</th>
<th>L.A.</th>
<th>DALLAS</th>
<th>CHGO</th>
<th>N.Y.</th>
<th>EXCESS CAP.</th>
<th>SUPPLY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>HOME CITY</strong></td>
<td>.95</td>
<td>.05</td>
<td>.95</td>
<td>.05</td>
<td>.95</td>
<td>.05</td>
<td>0</td>
</tr>
<tr>
<td><strong>BRANCH #1</strong></td>
<td>.95</td>
<td>.05</td>
<td>.95</td>
<td>.05</td>
<td>.95</td>
<td>.05</td>
<td>0</td>
</tr>
<tr>
<td><strong>BRANCH #2</strong></td>
<td>.95</td>
<td>.05</td>
<td>.95</td>
<td>.05</td>
<td>.95</td>
<td>.05</td>
<td>0</td>
</tr>
</tbody>
</table>

**Demand:** 8 4 4 2

Assign $X_{13} = \min (3, 4) = 3$ to the shipping route from **HOME CITY** to DALLAS.
Reduce supply at **HOME CITY** & demand at DALLAS by 3.

**Demand:** 8 4 4 0

Assign $X_{23} = \min (7, 1) = 1$ to the shipping route from **BRANCH #1** to DALLAS.
Reduce supply at **BRANCH #1** & demand at DALLAS by 1.

**Demand:** 5 4 4 2

Assign $X_{34} = \min (6, 11) = 6$ to the shipping route from **BRANCH #1** to CHGO.
Reduce supply at **BRANCH #1** & demand at CHGO by 6.

**Demand:** 5 4 3 0

Continuing, we get $X_{54} = 5$, $X_{55} = 8$, and $X_{56} = 2$.

#### Least-Cost Rule

**Obtaining an initial b.i.s.**

Whereas the NW-corner rule ignored costs completely, this rule selects the least-cost shipping route for the next assignment.
(Otherwise, similar to the NW-corner rule.)

**Total cost of this shipping plan is \$27.85**

**Assign minimum (7, 8) = 7 to the shipping route from **BRANCH #1** to N.Y., and reduce the supply & demand for this route.**

**Assigning the EXCESS CAPACITY destination, the least-cost shipment is from **HOME CITY** to CHGO.**

**Assign min(12, 11) to this shipping route, and reduce the supply & demand.**
Assign minimum $(15, 1) = 1$ to the shipping route from BRANCH #2 to N.Y., and reduce supply & demand.

Next, we assign minimum $(14, 5) = 5$ to the shipping route from BRANCH #2 to ATLANTA, and reduce supply & demand.

These 8 shipments are feasible, and a basic solution, with cost $21.90$

For each row, compute a "penalty" equal to the difference between the two smallest costs in that row.

(Vogel's Approximation Method (VAM))

Find the maximum penalty (which may be on either a row or column), and the least-cost cell within that row or column.

As in NW-corner Method, assign as great a shipment as possible to this cell, reduce the supply & demand for the row & column, and repeat (recomputing the penalties)
The maximum penalty is that for L.A.
So we select the least-cost cell for L.A. (HOME CITY - L.A.)

Again we update the penalties, and choose the largest penalty,
that of ATLANTA, and the least-cost cell in that column
(BRANCH#1 - ATLANTA)

Since only one source remains, we can complete the solution!

The total shipping cost for this solution is $21.35
Suppose that we ship one unit from HOME CITY to ATLANTA.
Change in total cost = +0.95 - 0.80 + 1.60 - 0.90 - 0.85
(Reduced Cost)

Then the reduced cost, as in the revised simplex method, is computed by:

\[
C_q = C_q - \left[ u_i + v_j \right] \text{ cost of } X_{ij} \text{ Column of Tableau for } x_i
\]

which is much simpler than identifying the cycles!
Let's arbitrarily set \( u_2 = 0 \).
Then complementary slackness implies that
\[
\begin{align*}
u_1 + v_3 &= 0.80 \\
\Rightarrow v_3 &= 0.80
\end{align*}
\]

and
\[
\begin{align*}
u_1 + v_4 &= 0.15 \\
\Rightarrow v_4 &= 0.15
\end{align*}
\]

Now we can use complementary slackness to obtain \( v_1, v_2, v_3, \) and \( v_0 \):
\[
\begin{align*}
u_1 + v_3 &= 0.90 \\
\Rightarrow v_3 &= 0.10
\end{align*}
\]

\[
\begin{align*}
u_1 + v_2 &= 1.00 \\
\Rightarrow v_2 &= 1.00
\end{align*}
\]

\[
\begin{align*}
u_4 + v_3 &= 0.85 \\
\Rightarrow v_3 &= 0.05 \\
\Rightarrow v_0 &= -0.85
\end{align*}
\]

Now let's use the simplex multipliers to compute the reduced costs, using the formula: \( C_{ij} = C_{ij} - (u_i + v_j) \)

\[
\begin{align*}
C_{11} &= 0.95 - (0.80 + 0.10) = -0.05 \\
C_{31} &= 0.80 - (0.25 + 0.15) = 0.40 \\
C_{41} &= 0.70 - (0.80 + 0.15) = -0.25
\end{align*}
\]

These are in agreement with the earlier computations!

Selecting variable to leave the basis

Once we have selected the variable to enter the basis, we must select the variable to leave the basis.

(In the simplex method, this is usually decided by the "Minimum Ratio Test")

Since each unit shipped along the \( \text{BRANCH}^2\text{-CHGO} \) route reduces our cost by \$0.25, so we wish to ship as much as possible.

What is the upper limit on \( \theta \)?

As soon as \( \theta = 3 \), the shipment from \( \text{BRANCH}^2\text{-DALLAS} \) becomes zero, preventing any further increase in \( \theta \).

The new solution has a total shipping cost of \$21.15, a savings of \$0.75 (\( \theta = 3 \times 0.25 \)).
To proceed with the next iteration, we first compute the dual variables. For example, start with \( u_4 = 0 \):
\[
\begin{align*}
\nu_4 &= \frac{1}{a_{24}} = \frac{1}{0.5} = 0.8 \\
\nu_1 &= \frac{1}{a_{11}} = \frac{1}{0.35} = 1.25 \\
\nu_2 &= \frac{1}{a_{12}} = \frac{1}{0.3} = 3.33 \\
\nu_3 &= \frac{1}{a_{13}} = \frac{1}{0.3} = 3.33
\end{align*}
\]
\( u_3 = 0.55 \Rightarrow \nu_3 = 0 \)

We identify the cycle formed by adding the new shipment, and determine the adjustments required. The maximum allowed increase in \( \theta \) is \( \min(8, 4) = 4 \).

By first assigning \( u_3 = 0 \), the dual variables shown above are computed.

Because one nonbasic variable has a zero reduced cost, there is an alternate optimal solution:

\[
\begin{array}{cccccccc}
0.75 & +0.50 & 0.75 & 0.75 & +0.50 & 0.50 & 0.75 & +0.25 \\
0.75 & +0.75 & +0.50 & 0.75 & +0.50 & 0.75 & +0.25 & 0.50
\end{array}
\]

Increasing \( \lambda_{21} \) by \( \theta \) results in no change in the total cost, since the reduced cost is zero. Any increase up to 5 will be feasible, and therefore optimal.
For example, an increase of 3 is optimal (although this gives us 9 positive shipments, which exceeds the number of basic variables, and is therefore optimal but not basic)

If $\theta = 5$, then $X_{31}$ becomes zero and can leave the basis, giving the basic optimal solution shown above.

A Complication: Degeneracy

A degenerate feasible solution is one in which a basic variable is zero.

When this occurs, the next basis change may not result in an improvement in the total cost!

As $\theta$ is increased, two of the basic variables reach zero simultaneously!

Only one basic variable can be replaced by $X_{41}$ while the other remains in the basis, even though it's value is zero.

New basis is degenerate

As we try to increase $X_{13}$, we see that we are immediately blocked at $\theta = 0$

If $\theta > 0$, then $X_{41}$ becomes negative (the solution is infeasible).

Even though we cannot increase $X_{13}$, we do change the basis.

Reduced costs:

$C_{i1} = 5(0+9) = 0$
$C_{i2} = 14(-3+4) > 0$
$C_{i3} = 4(-3+3) > 0$

Only $X_{32}$ has a negative reduced cost, so we will enter it into the basis.
A Production Planning Problem

- demands for next 4 weeks (which must be satisfied) are: 300, 700, 900, and 800
- regular production capacity is 700/week
- overtime is available in the second & third weeks, adding 200 to the production capacity
- production costs are $10/unit during weeks 1 & 2, increasing to $15/unit during weeks 3 & 4; overtime adds $5/unit to the cost.
- excess production may be stored at a cost of $3/unit per week.

How should production be scheduled to minimize costs?

Units which are produced in week 1 and which satisfy demand in week 1 are modeled as a flow from the source node to the destination node:

The cost of "transportation" is the production cost.

Flows in this model do not represent changes in geographical location!

A TRANSPORTATION model of production planning:

For each week, represent each of regular and overtime capacities as a source:

Likewise, for each week represent each demand as a destination.

Units which are produced in week 1 and which are used to satisfy the demand in week 2 are modeled by a flow from the week 1 source to the week 2 destination:

10 + 3.4 = the sum of production & storage costs.

Transportation Tableau:

<table>
<thead>
<tr>
<th>Week 1</th>
<th>Week 2</th>
<th>Week 3</th>
<th>Week 4</th>
<th>Unused</th>
</tr>
</thead>
<tbody>
<tr>
<td>RT</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OT</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Demand</td>
<td>10</td>
<td>15</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td></td>
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<tr>
<td>15</td>
<td>21</td>
<td>15</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

Note that flows above are never "backward" in time.
What meaning could a shipment backward in time have?

Suppose we produce a unit in week 2 with which to satisfy week 1's demand:

That is, week 1 demand has been "backordered". The cost of such a "shipment" should include backorder costs.