M/M/1/N Queue with Removable Server

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The arrival of items for processing at a manufacturing center is a Poisson process with rate $\lambda = 1$/day.

The processing time of an item has exponential distribution, with mean $1/\mu = 0.5$ day.

When no items await processing, the mfg. center is shut down. Cost to restart the center is $125. Holding cost for items awaiting processing is $1/day.

There is a capacity of 10 items at the mfg. center-- if at item arrives while at capacity, the arrival process is interrupted and a penalty of $1000/day is incurred.

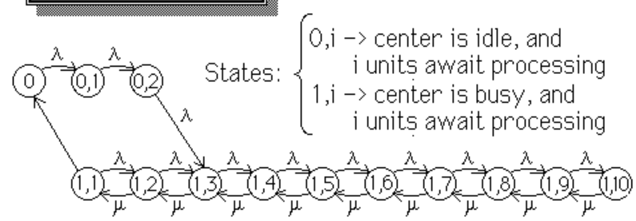
Let Q denote the number of waiting items which will trigger the start-up of the center.

What is the optimal value of Q?

small Q ⟶ frequent start-up costs
large Q ⟶ higher holding costs & risk of overflow.

Continuous-Time Markov Chain

Example: Q = 3



Not a birth-death process!

Q=3

Λ =

	0	0,1	0,2	1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	1,10
0	-1	1	0	0	0	0	0	0	0	0	0	0	0
0,1	0	-1	1	0	0	0	0	0	0	0	0	0	0
0,2	0	0	-1	0	0	1	0	0	0	0	0	0	0
1,1	2	0	0	-3	1	0	0	0	0	0	0	0	0
1,2	0	0	0	2	-3	1	0	0	0	0	0	0	0
1,3	0	0	0	0	2	-3	1	0	0	0	0	0	0
1,4	0	0	0	0	0	2	-3	1	0	0	0	0	0
1,5	0	0	0	0	0	0	2	-3	1	0	0	0	0
1,6	0	0	0	0	0	0	0	2	-3	1	0	0	0
1,7	0	0	0	0	0	0	0	0	2	-3	1	0	0
1,8	0	0	0	0	0	0	0	0	0	2	-3	1	0
1,9	0	0	0	0	0	0	0	0	0	0	2	-3	1
1,10	0	0	0	0	0	0	0	0	0	0	0	2	-2

Solving $\pi \Lambda = 0$ & $\sum_i \pi_i = 1$ yields the steady-state dist'n

Transition Rate Matrix

Steady-State Distribution

States 1, ... Q
represent the queue lengths i=0, 1, ... Q-1
when the server is idle.

Server Idle

i	PI[i]
0	0.166857
1	0.166857
2	0.166857

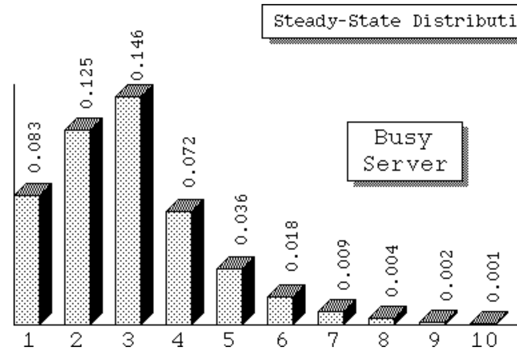
Steady-State Distribution

States Q+1, Q+2,... Q+N
represent the queue lengths i=1,2,... N
when the server is busy.

Server Busy

i	PI[i]
1	0.0834284
2	0.125143
3	0.146
4	0.0729998
5	0.0364999
6	0.01825
7	0.00912498
8	0.00456249
9	0.00228124
10	0.00114062

Steady-State Distribution



Average # in system is 1.98859
$\Longrightarrow$ Holding cost/day = \$1.98859

$\pi_{1,10}$ = 0.00114062
$\Longrightarrow$ Penalty/day for overflow is \$1.14062

What is the average start-up cost/day?

To compute average start-up cost per day, we must find the average cycle time (time between start-ups)

State 0 is visited exactly once per cycle, and the average time spent in this state is the inter-arrival time of the items.

Therefore,

$$\pi_0 = \frac{\text{average interval during which queue is empty}}{\text{average cycle time}}$$

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Average interval during which queue is empty is expected time between arrivals= $1/\lambda$ = 1 day

$\Longrightarrow$ Average cycle time = $\frac{1}{\lambda\,\pi_0}$

Average # cycles per day = $\lambda\,\pi_0$

Frequency of start-up is $\lambda\,\pi_0$ = 0.166857/day

Start-up cost/day is \$125 × 0.166857
= \$20.857125

Holding cost/day =	\$ 1.98859
Penalty/day for overflow is	\$ 1.14062
Start-up cost/day	\$20.857125
Total cost/day	\$23.986335

Q=4

Steady-State Distribution

Frequency of start-ups is 0.125229/unit time
Average number in the system is 2.48257

Server Idle

i	PI[i]
0	0.125229
1	0.125229
2	0.125229
3	0.125229

Server Busy

i	PI[i]
1	0.0626147
2	0.093922
3	0.109576
4	0.117402
5	0.0587012
6	0.0293506
7	0.0146753
8	0.00733765
9	0.00366883
10	0.00183441

Q=4

Holding cost/day:	\$ 2.48257
Overflow penalty/day:	\$ 1.83441
Start-up cost/day:	\$15.653625
Total cost/day:	\$19.970605

Q=5

Steady-State Distribution

Frequency of start-ups is 0.100304/unit time
Average number in the system is 2.97267

Server Idle

i	PI[i]
0	0.100304
1	0.100304
2	0.100304
3	0.100304
4	0.100304

Server Busy

i	PI[i]
1	0.0501518
2	0.0752277
3	0.0877657
4	0.0940347
5	0.0971692
6	0.0485846
7	0.0242923
8	0.0121461
9	0.00607307
10	0.00303654

Q=5

Holding cost/day:	\$ 2.97267
Overflow penalty/day:	\$ 3.03654
Start-up cost/day:	\$12.538
Total cost/day:	\$18.54721

Q=6

Steady-State Distribution

Frequency of start-ups is 0.0837628/unit time
Average number in the system is 3.4562

Server Idle

i	PI[i]
0	0.0837628
1	0.0837628
2	0.0837628
3	0.0837628
4	0.0837628
5	0.0837628

Server Busy

i	PI[i]
1	0.0418814
2	0.0628221
3	0.0732924
4	0.0785276
5	0.0811452
6	0.082454
7	0.041227
8	0.0206135
9	0.0103067
10	0.00515337

Q=6

Holding cost/day:	$	3.4562
Overflow penalty/day:	$	5.15337
Start-up cost/day:	$	10.47035
Total cost/day:	$	19.07992

Q=7

Steady-State Distribution

Frequency of start-ups is 0.072067/unit time
Average number in the system is 3.9285

Server Idle

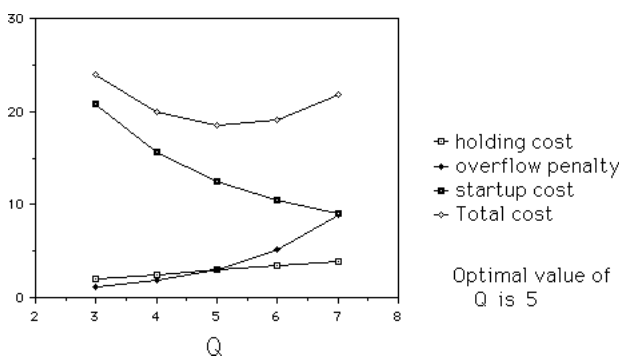
i	PI[i]
0	0.072067
1	0.072067
2	0.072067
3	0.072067
4	0.072067
5	0.072067
6	0.072067

Server Busy

i	PI[i]
1	0.0360335
2	0.0540502
3	0.0630586
4	0.0675628
5	0.0698149
6	0.070941
7	0.071504
8	0.035752
9	0.017876
10	0.008938

Q=7

Holding cost/day:	$	3.9285
Overflow penalty/day:	$	8.938
Start-up cost/day:	$	9.008375
Total cost/day:	$	21.874875



Optimal value of Q is 5