**Example: Revised Simplex Method**

Consider the LP:

Minimize \( z = 3x_1 + 2x_2 + 6x_3 \)
subject to
\[
\begin{align*}
4x_1 + 8x_2 - x_3 & \leq 5 \\
7x_1 - 2x_2 + 2x_3 & \geq 4 \\
x_1 & \geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]

By introducing slack and surplus variables, the problem is rewritten with equality constraints as

Minimize \( cx \) subject to \( Ax=b, x \geq 0 \)

where
\[
\begin{align*}
c &= [3, 2, 6, 0, 0], \\
b &= [5, 4] \text{ and } \\
A &= \begin{bmatrix} 4 & 8 & -1 & 1 & 0 \\
7 & -2 & 2 & 0 & -1 \end{bmatrix}.
\end{align*}
\]

Although \( x_4 \) (the slack variable in 1st constraint) can be used as a basic variable in the first row, the choice of a basic variable in 2nd constraint is not obvious, requiring solution of a “Phase One” problem with artificial variables introduced.

Suppose that “Phase One” has found the initial basis \( B = \{1,2\} \) for the constraints, i.e., basic variables \( x_1 \) and \( x_2 \).

We begin the first iteration of the revised simplex method (RSM) by computing the basis inverse matrix:

\[
B=\{1,2\} \quad \Rightarrow A^B = \begin{bmatrix} 4 & 8 \\
7 & -2 \end{bmatrix}
\]

\[
\Rightarrow (A^B)^{-1} = \begin{bmatrix} 0.03125 & 0.125 \\
0.10937 & -0.0625 \end{bmatrix}
\]
Using the basis inverse matrix, we compute the values of the current basic variables,

\[
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} = (A^B)^{-1} b
\]

\[
= \begin{bmatrix}
    0.03125 & 0.125 \\
    0.109375 & -0.0625
\end{bmatrix} \begin{bmatrix}
    5 \\
    4
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    0.65625 \\
    0.296875
\end{bmatrix}
\]

Next we compute the simplex multiplier vector \( \pi \), to be used in “pricing” the nonbasic columns:

\[
\pi = c_B (A^B)^{-1}
\]

\[
= \begin{bmatrix}
    3 & 2 \\
\end{bmatrix} \begin{bmatrix}
    0.03125 & 0.125 \\
    0.109375 & -0.0625
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    0.3125 & 0.25
\end{bmatrix}
\]

Use the simplex multiplier vector \( \pi \) to compute the reduced cost of the nonbasic variables (\( x_3, x_4, \& x_5 \)), starting with \( x_3 \):

\[
\bar{c}_3 = c_3 - \pi A^l
\]

\[
= 0 - \begin{bmatrix}
    0.3125, 0.25 \\
\end{bmatrix} \begin{bmatrix}
    -1 \\
    2
\end{bmatrix}
\]

\[
= 5.1875 > 0
\]

Since this reduced cost is **positive**, increasing \( x_3 \) would **increase** the cost.

So \( x_3 \) is **rejected** as a pivot variable.

*(If we had been maximizing rather than minimizing, of course, then increasing \( x_3 \) would benefit the objective!)*

We now proceed to the next nonbasic variable, \( x_4 \).

Use the simplex multiplier vector \( \pi \) to compute the reduced cost of the nonbasic variable \( x_4 \):

\[
\bar{c}_4 = c_4 - \pi A^l
\]

\[
= 0 - \begin{bmatrix}
    0.3125, 0.25 \\
\end{bmatrix} \begin{bmatrix}
    1 \\
    0
\end{bmatrix}
\]

\[
= -0.3125 < 0
\]

Increasing \( x_4 \) will improve (i.e. **lower**) the solution, since its reduced cost is **negative**!
Rather than continuing to “price” the remaining nonbasic variables (in this case, only \(x_5\)), we will proceed by entering \(x_4\) into the basis!

For the **minimum ratio test**, we need the **substitution rates** of \(x_4\) for the basic variables:

\[
\alpha = (A^B)^{-1}A^I = \begin{bmatrix} 0.03125 & 0.125 & \frac{1}{0} \\ 0.10937 & -0.0625 & \frac{1}{0} \end{bmatrix}
\]

\[
= \begin{bmatrix} 0.03125 \\ 0.10937 \end{bmatrix}
\]

That is, one unit of \(x_4\) will substitute for 0.03125 units of the first basic variable and 0.10937 of the second.

Perform the **minimum ratio test** to determine which variable leaves the basis.

\[
\min \left\{ \frac{x_B}{\alpha_B} : \alpha_B > 0 \right\} = \min \left\{ \frac{0.6562}{0.03125}, \frac{0.29687}{0.10937} \right\} = \min \{21, 2.7143\} = 2.7143
\]

Since the **second** ratio is minimum, the second basic variable (i.e., \(x_2\)) is replaced by the entering variable \(x_4\) (which will be 2.7143 in the new basic solution), and the **new basis** is \(B = \{1, 4\}\).

Update the basis inverse matrix with a **pivot**:

\[
(A^B)^{-1} \alpha = \begin{bmatrix} 0.03125 & 0.125 & 0.03125 \\ 0.10937 & -0.0625 & 0.109375 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0.03125 & 0.1428 & 0 \\ 0.10937 & -0.5714 & 1 \end{bmatrix}
\]

\[
\Rightarrow (A^B)^{-1} = \begin{bmatrix} 0 & 0.1428 \\ 1 & -0.5714 \end{bmatrix}
\]

For the new basis \(B=\{1,4\}\),

\[
c_B = [3, 0], A^B = \begin{bmatrix} 4 & 1 \\ 7 & 0 \end{bmatrix}, (A^B)^{-1} = \begin{bmatrix} 0 & 0.142857 \\ 1 & -0.571429 \end{bmatrix}
\]

The **basic variables** are \([x_I, x_J]\):

\[
x_B = (A^B)^{-1}b = \begin{bmatrix} 0.571429 & 2.71429 \end{bmatrix}
\]

and the new **simplex multipliers** are

\[
\pi = c_B (A^B)^{-1} = \begin{bmatrix} 0 & 0.428571 \end{bmatrix}
\]
The **reduced costs** of the nonbasic variables \{2, 3, 5\} are now:

\[
\begin{align*}
\bar{c}_2 &= c_2 - \pi A^2 = 2 - \begin{bmatrix} 0 & 0.428571 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \end{bmatrix} = 2.85714 > 0 \\
\bar{c}_3 &= c_3 - \pi A^3 = 6 - \begin{bmatrix} 0 & 0.428571 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 5.14286 > 0 \\
\bar{c}_5 &= c_5 - \pi A^5 = 0 - \begin{bmatrix} 0 & 0.428571 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0.428571 > 0
\end{align*}
\]

Since the reduced costs are all **positive**, the current solution \([x_1 \ x_4] = [0.571429 \ 2.71429]\) is **optimal**!