Let \( X_i \) = state of system at stage \( i \)  
\( Y_i \) = decision at stage \( i \)

**General QC/LD problem**

\[
\text{Minimize } \sum_{i=1}^{N} \left\{ A_iX_i^2 + B_iX_iY_i + C_iY_i^2 + D_iX_i + E_iY_i + F_i \right\} \\
+ A_{N+1}X_{N+1} + D_{N+1}X_{N+1} + F_{N+1}
\]

subject to

\[
X_{i+1} = G_iX_i + H_iY_i + K_i, \quad i = 2, 3, \ldots, N
\]

**Derivation of Solution for Simpler Version**

**Closed-form Solution of General Problem**

**Example APL Output**

**Certainty Equivalence**

---

Let's begin with a simpler version of the problem:

\[
\text{Minimize } \sum_{i=1}^{N} \left\{ A_iX_i^2 + C_iY_i^2 \right\} + A_{N+1}X_{N+1}^2
\]

where

\[
X_{i+1} = G_iX_i + H_iY_i, \quad i = 1, \ldots, N
\]

Assume \( A_i \geq 0 \) and \( C_i \geq 0 \)

**Optimal value function**

\[
V_i(X) = \text{minimum cost of the remaining process} \\
\text{if it starts stage } i \text{ in state } X
\]

\[
= \text{minimum } \left\{ A_iX^2 + C_iY^2 + V_{i+1}(G_iX + H_iY) \right\} \\
\text{min } Y_i \text{ for } i = 1, 2, \ldots, N
\]

\[
V_{N+1}(X) = A_{N+1}X^2
\]

**The problem at the last stage:**

\[
V_N(X) = \min_{Y} \left\{ A_NX^2 + C_NY^2 + Y_{N+1}(G_NX + H_NY) \right\}
\]

\[
= \min_{Y} \left\{ A_NX^2 + C_NY^2 + A_{N+1}(G_NX + H_NY)^2 \right\}
\]

\[
= \min_{Y} \left\{ A_NX^2 + C_NY^2 + A_{N+1}G_N^2X^2 \\
+ 2A_{N+1}G_NH_NXY + A_{N+1}H_N^2Y^2 \right\}
\]
\[ V_N(X) = \min \{ A_N X^2 + C_N Y^2 + A_{N+1} G_N X^2 + 2A_{N+1} G_N H_N X Y + A_{N+1}^2 H_N^2 Y^2 \} \]

Set the partial derivative of the minimand equal to zero:

\[ 2C_N Y + 2A_{N+1} G_N H_N X + 2A_{N+1} H_N^2 Y = 0 \]

\[ Y = -\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2} \]

is a minimizer if the second derivative is positive, i.e., if

\[ C_N + A_{N+1} H_N^2 > 0 \]

which we will assume.

Substituting

\[ Y = -\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2} \]

into

\[ A_N X^2 + C_N Y^2 + A_{N+1} G_N X^2 + 2A_{N+1} G_N H_N X Y + A_{N+1} H_N^2 Y^2 \]

yields

\[ V_N(X) = A_N X^2 + C_N \left( -\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2} \right)^2 + A_{N+1} G_N^2 X^2 + 2A_{N+1} G_N H_N X \left( -\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2} \right) + A_{N+1} H_N^2 \left( -\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2} \right)^2 \]

We can now use \( V_N(X) \) to find \( V_{N-1}(X) \).

The same formulae will result, except that \( P_N \) replaces \( A_{N+1} \), \( A_{N-1} \) replaces \( A_N \), etc.

\[ V_{N-1}(X) = P_{N-1} X^2 \]

where

\[ P_{N-1} = A_{N-1} + \frac{P_{N-1} G_{N-1}^2}{C_{N-1} + P_{N-1} H_{N-1}^2} \]

optimal decision:

\[ Y = -\frac{P_{N-1} G_{N-1} H_{N-1} X}{C_{N-1} + P_{N-1} H_{N-1}^2} \]

Clearly, we can repeat this procedure to get \( V_{N-2}(X) \), \( V_{N-3}(X) \), ..., \( V_2(X) \), \( V_1(X) \) where in general,

\[ V_1(X) = P_1 X^2 \]

where

\[ P_1 = A_1 + P_{1+1} G_1^2 - \frac{P_{1+1} G_1^2 H_1^2}{C_1 + P_{1+1} H_1^2} \]

\[ Y = -\frac{P_{1+1} G_1 H_1 X}{C_1 + P_{1+1} H_1^2} \]

**Example**

Given initial state \( X_1 \), select \( Y_1 \), \( Y_2 \), and \( Y_3 \) to minimize

\[ Y_1^2 + 12X_2^2 + 2Y_2^2 + 2X_3^2 + Y_3^2 + \frac{1}{4} X_4^2 \]

where

\[ \begin{align*}
X_2 &= \frac{1}{2} X_1 + \frac{1}{6} Y_1 \\
X_3 &= 3X_2 + \frac{1}{2} Y_2 \\
X_4 &= 4X_3 + 2Y_3
\end{align*} \]
Minimize \( Y_1^2 + 12Y_2^2 + 2Y_3^2 + 3Y_4^2 + \frac{1}{2}Y_5^2 \)

where \[
\begin{align*}
X_0 &= \frac{1}{2}X_1 + \frac{1}{6}Y_1 \\
X_2 &= 2X_1 + Y_2 \\
X_4 &= 4X_3 + 2Y_3 \\
X_1 &= 2
\end{align*}
\]

\[
\Rightarrow \begin{cases}
A_1 = 0, & C_1 = 1, & G_1 = \frac{1}{2}, & H_1 = \frac{1}{6} \\
A_2 = 12, & C_2 = 2, & G_2 = 3, & H_2 = \frac{1}{2} \\
A_3 = 2, & C_3 = 1, & G_3 = 4, & H_3 = 2 \\
A_4 = \frac{1}{4}
\end{cases}
\]

\[
Y_1(X) = P_iX^2
\]

where \[
P_i = A_i + P_{i+1}G_i^2 - \frac{P_{i+1}G_i^2H_i}{C_i + P_{i+1}H_i^2} = \frac{P_{i+1}G_i^2H_i}{C_i + P_{i+1}H_i^2}
\]

\[Y = -\frac{P_{i+1}G_iH_iX}{C_i + P_{i+1}H_i^2}
\]

\[P_3 = 2 + (\frac{1}{4})^2 - \frac{(\frac{1}{4})^4 + 2}{1 + (\frac{1}{4})^2} = 4
\]

\[
Y_3 = -\frac{4(4)(2)}{1 + (\frac{1}{4})^2} X_3 = -X_3
\]

Optimal value: \( Y_1(X) = \frac{9}{2} X^2 \Rightarrow Y_1(2) = \frac{9}{2} \cdot 2^2 = 18 \)

Now perform a "forward computation":

Given \( X_1 = 2\),

\[
\Rightarrow Y_1 = -\frac{3}{2}X_1
\]

\[
\Rightarrow X_2 = \frac{1}{2} X_1 + \frac{1}{6} Y_1 = \frac{1}{2}
\]

\[
\Rightarrow Y_2 = -2X_2 = -1
\]

\[
\Rightarrow X_3 = 3X_2 + \frac{1}{2} Y_2 = 1
\]

\[
\Rightarrow X_4 = 4X_3 + 2Y_3 = 2
\]

Optimal Value Function:

\[
V_d(X) = P_iX^2 + Q_iX_i + R_i
\]

where \[
P_i = A_i + P_{i+1}G_i^2 - \frac{B_i + 2P_{i+1}G_iH_i}{4[C_i + P_{i+1}H_i^2]}
\]

\[
Q_i = D_i + 2P_{i+1}K_iG_i + Q_{i+1}G_i
\]

\[
= \frac{(B_i + 2P_{i+1}G_iH_i)(B_i + 2P_{i+1}G_iH_i + Q_{i+1}H_i)}{2[C_i + P_{i+1}H_i^2]}
\]

\[
R_i = F_i + P_{i+1}K_i^2 + Q_{i+1}K_i + R_{i+1}
\]

\[
= \frac{(B_i + 2P_{i+1}G_iH_i + Q_{i+1}H_i)^2}{4[C_i + P_{i+1}H_i^2]}
\]

\[
P_{i+1} = A_{i+1}
\]

\[
Q_{i+1} = D_{i+1}
\]

\[
R_{i+1} = F_{i+1}
\]

Optimal Decisions:

\[
Y_i = \frac{(B_i + 2P_{i+1}G_iH_i)X + E_i + 2P_{i+1}H_iK_i + Q_{i+1}H_i}{2[C_i + P_{i+1}H_i^2]}
\]

General QC/LD problem:

Minimize \[
\sum_{i=1}^{N} \left\{ A_iX_i^2 + B_iX_iY_i + C_iY_i^2 + D_iX_i + E_iY_i + F_i \right\}
\]

subject to \[
X_{i+1} = G_iX_i + H_iY_i + K_i, \quad i=3,6,\ldots,N
\]

Using the same method as before, we can derive closed-form expressions for the optimal value and decisions.
APL output

\[ \begin{array}{l|cccccc}
 & A & B & C & D & E & F \\
\hline
0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
\end{array} \]

\[ \begin{array}{l|cccccc}
 & G & H & K & L & M & N \\
\hline
0 & 0.5 & 0.1666666667 & 0 & 0 & 0 & 0 \\
1 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\
2 & 0.4 & 0.2 & 0.2 & 0 & 0 & 0 \\
\end{array} \]

where

\[ A(t) = \text{coefficient of } x(t)^2 \]
\[ B(t) = \text{coefficient of } x(t) \]
\[ C(t) = \text{coefficient of } x(t) \]
\[ D(t) = \text{coefficient of } y(t) \]
\[ E(t) = \text{coefficient of } y(t) \]
\[ F(t) = \text{constant} \]

Cost of final stage: \( 0.25 \times x(N)^2 + 0 \times x(N) + 0 \)

\[ \]

Optimal Decision

\[ y(t) = (E(t) + x(t)) + Y(t) \]

Optimal Value

\[ v(t) = (F(t) + x(t) + E(t) + x(t)) + R(t) \]

\[ \]

Certainty Equivalence

Consider the following stochastic version of the QC/LD problem, with a random additive term \( Z_i \) in the linear dynamics (transition equation)

\[ \text{Minimize } \sum_{t=0}^{N} \left\{ A(t)X(t)^2 + B(t)Y(t) + C(t)Y(t) + D(t)X(t) + E(t)Y(t) + F(t) \right\} \]

subject to \( X(t) = Q(t)X(t) + H(t)Y(t) + Z(t), \quad t = 2, 3, \ldots, N \)

Define an optimal value function:

\[ W(t) = \text{minimum expected cost of the remaining process, if we start stage } t \text{ in state } X, \text{ and have not yet learned the value of } Z_t \]

\[ \]

Transition Equations

\[ X(t+1) = QX(t) + H(t)Y(t) + Z(t) \]

where \( Z(t), t = 1, 2, \ldots, N \) are independent random variables, with

\[ \text{E}(Z(t)) = \mu_t \]
\[ \text{Var}(Z(t)) = \sigma_t^2 \]

We assume that \( Y(t) \) must be selected before the random variable \( Z(t) \) is observed.

\[ \]

Define an optimal value function:

\[ W(t) = P(t)X(t)^2 + QX(t) + R(t) \]

where

\[ P(t) = A(t) + B(t) + C(t) \]
\[ Q(t) = D(t) + 2E(t) + H(t) \]
\[ R(t) = F(t) + H(t) \]

\[ \]

Solution

\[ \]

Optimal Decision

\[ Y(t) = -\frac{2P(t)Q(t)X(t) + 2P(t)H(t)X(t) + Q(t)H(t)}{2(C(t) + P(t)H(t))} \]
The closed-form solution to this stochastic problem is identical to the deterministic version of the problem with $K_i - E[|Z_i|] - \mu_z$ except that the equation for $R_i$ differs by a term $p_{i-1} \sigma_i^2$.

The value of $R_i$ does not enter into the computation of the optimal decision $Y_i$, however.

**Certainty Equivalence**

The optimal policy for the stochastic problem is the same as that of the deterministic problem, with the random variable replaced by its expected value.

(The cost of the optimal policy is increased, due to the different formula for $R_i$, reflecting the cost due to randomness.)