A fixed sum of money $F$ is to be allocated among $n$ investments, each of which has a known history of returns during the previous $p$ periods.

$$r_{ik} = \text{return per dollar invested in investment #} i \text{ during period } k, \quad i=1,\ldots,n; \quad k=1,\ldots,p$$

$$x_i = \text{amount of money to be allocated to investment #} i$$

Let

$$x_i = \text{amount of money to be allocated to investment #} i$$

Assuming that past history is indicative of future performance, the expected annual return will be

$$E = \frac{1}{p} \sum_{k=1}^{p} r_{ik} x_i$$

where

$$e_i = \frac{1}{p} \sum_{k=1}^{p} r_{ik}$$

If we wish to maximize the expected return, then we would invest the total available funds in investment #3, which has the highest expected return.

Looking at the past history, however, we see a greater variability in the return provided by investment #3:

<table>
<thead>
<tr>
<th>Investment</th>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>10%</td>
<td>4%</td>
<td>12%</td>
<td>13%</td>
<td>6%</td>
<td></td>
</tr>
<tr>
<td>#2</td>
<td>6%</td>
<td>9%</td>
<td>6%</td>
<td>5%</td>
<td>9%</td>
<td></td>
</tr>
<tr>
<td>#3</td>
<td>17%</td>
<td>1%</td>
<td>11%</td>
<td>19%</td>
<td>2%</td>
<td></td>
</tr>
</tbody>
</table>

The expected total annual return from the investments will be

$$E = 0.09 x_1 + 0.07 x_2 + 0.10 x_3$$

The variance of the total annual return, based upon past performance, is

$$V = \frac{1}{p} \sum_{k=1}^{p} \left( \frac{1}{p} \sum_{i=1}^{n} r_{ik} x_i - E \right)^2$$

Substitute $E = \frac{1}{p} \sum_{i=1}^{n} e_i x_i$ into

$$V = \frac{1}{p} \sum_{k=1}^{p} \left( \frac{1}{p} \sum_{i=1}^{n} \left( r_{ik} x_i - e_i x_i \right) \right)^2$$

$$= \frac{1}{p} \left( \frac{1}{p} \sum_{k=1}^{p} \left( \sum_{i=1}^{n} \left( r_{ik} - e_i \right) x_i \right)^2 \right)$$

$$= \frac{1}{p} \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( r_{ik} - e_i \right) \left( r_{jk} - e_j \right) x_i x_j$$

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^2 x_i x_j$$

where

$$\sigma_{ij} = \frac{1}{p} \sum_{k=1}^{p} \left( r_{ik} - e_i \right) \left( r_{jk} - e_j \right)$$

$$= \frac{1}{p} \sum_{k=1}^{p} r_{ik} r_{jk} - \frac{1}{p} \left( \sum_{k=1}^{p} r_{ik} \left( \sum_{k=1}^{p} r_{jk} \right) \right)$$
Thus, the variance of the total return is the quadratic function \( x^T C x \)
where \( C \) is the covariance matrix with entries \( C_{ij}^2 \).

\[
C = \begin{bmatrix}
12 & -5.6 & 23 \\
-5.6 & 2.8 & -12 \\
23 & -23 & 55.2 \\
\end{bmatrix} \times 10^4
\]

If we invest all $10,000 in #3 in order to maximize the expected return, i.e., \( \mathbf{x} = (0,0,10^4) \) the variance of the annual return will be
\[
x^T C x = 55.2 \times 10^4 \times 10^4 \times 10^4 = 55.2 \times 10^4
\]
i.e., the standard deviation will be $743, or 74% of the expected return ($1000)! ... a "risky" investment.

Suppose that we are "risk-averse" and are satisfied with an 8% annual rate of return, but wish to minimize the variance of the total return.

Then we must solve:

**Quadratic Programming Problem**

Minimize \( x^T C x \)
subject to
\[
x_1 + x_2 + x_3 \leq 10000 \\
0.09 x_1 + 0.07 x_2 + 0.10 x_3 \geq 800 \\
x_j \geq 0, j=1,2,3
\]

Optimal solution: \( \mathbf{x} = (5000,5000,0) \)
i.e., invest half of the total in each of investments #1 & #2, and nothing in #3 which yields the greatest expected return!

The expected return will be 8% ($800), with a variance of 0.009\times10^6, i.e., a standard deviation of $94.87.