**Parametric Programming** is the analysis of the variation of the solution of an LP when some element (right-hand-side, objective, etc.) varies.

Consider the optimal value of the LP as a function of the right-hand-side vector, i.e.,

\[
Z^*(b) = \begin{bmatrix}
\min c \times \\
\text{s.t. } Ax \leq b \\
\end{bmatrix}
\] [by duality theory]

\[
\begin{bmatrix}
\max \pi b \\
\text{s.t. } \pi A + c \\
\pi \geq 0
\end{bmatrix}
\]

The function \(Z^*(b)\) is "evaluated" for some particular right-hand-side \(b'\) by solving the LP (either the primal or the dual).

So we can evaluate \(Z^*(b')\) by solving the LP

\[
Z'(b') = \begin{bmatrix}
\max \pi b \\
\text{s.t. } \pi A + c \\
\pi \geq 0
\end{bmatrix}
\]

Notice that the feasible region of the dual LP is the same for every argument \(b'\).

We know that a basic solution is optimal for an LP problem, and that there are a finite (but possibly very large) number of such basic solutions.

Therefore, \(Z^*(b) = \max_{k=1,2,...K} \{ \pi^t b \}\)

That is, \(Z^*(b)\) is the maximum of a family of linear functions, \(\pi^t b, k=1,2,...K\)

which is a piecewise linear convex function.

Let us restrict our analysis of the function \(Z^*(b)\) to a study of its behavior along a line (rather than everywhere in \(m\)-dimensional space).

That is, we assume an initial right-hand-side vector \((b)\) is given, and a direction \((d)\), and study the behavior of \(Z^*(b + \lambda d)\), considered as a function of the scalar parameter \(\lambda\).

Consider the solution of the LP

\[
P_\lambda : \begin{bmatrix}
\max \pi^t (b + \lambda d) \\
\text{s.t. } A x - b + \lambda d \\
\end{bmatrix}
\]

where \(d\) is an \(m\)-vector and \(\lambda\) is a scalar.

\(Z^*(\lambda) = \max_{k=1,2,...K} \{ \pi^t b + \lambda d \}\)
**Example**

\[ z(\lambda) = \min_{x_1, x_2} \quad x_1 - x_2 \]

subject to

\[
\begin{align*}
2x_1 + x_2 & \leq 8 + 2\lambda \\
x_1 + 2x_2 & \leq 7 + 7\lambda \\
x_2 & \leq 3 + 2\lambda \\
x_1 & \geq 0, \quad x_2 \geq 0
\end{align*}
\]

\[ = \max_{x_1, x_2} \quad (2x_1 + 7x_2 + 2x_3) + (2x_1 + 7x_2 + 2x_3)(\lambda) \\
\text{s.t.} \quad 2x_1 + x_2 \leq 1 \\
x_1 + 2x_2 \leq 1 \\
x_1, x_2, x_3 \geq 0 \quad \text{(nonpositive because of direction of s to the primal)}
\]

Of the nine basic solutions, four are dual feasible. Therefore, \( z(\lambda) \) is the maximum of four linear functions:

![Graph showing linear functions](image1)

In this example, with only nine basic dual solutions, it was possible to enumerate all of them, test each for feasibility, and then maximize the corresponding linear functions

\[ z^*(\lambda) = (n^d - 1). \]

However, for most problems, the number of basic dual solutions is astronomical and enumerating them is practically impossible. 

(Usually only a few of these basic dual solutions actually determine \( z(\lambda) \).

![Graph showing linear functions](image2)

Let's consider again the parametric LP \( P_\lambda \):

\[ z(\lambda) = \min_{x_1, x_2} \quad x_1 - x_2 \]

subject to

\[
\begin{align*}
2x_1 + x_2 & \leq 8 + 2\lambda \\
x_1 + 2x_2 & \leq 7 + 7\lambda \\
x_2 & \leq 3 + 2\lambda \\
x_1 & \geq 0, \quad x_2 \geq 0
\end{align*}
\]

Determine the optimal value function \( z(\lambda) \) as well as \( x^*(\lambda) \) and \( x^*_f(\lambda) \) [i.e., the optimal solution] for all values of \( \lambda \in (-\infty, +\infty) \)

The initial tableau:

<table>
<thead>
<tr>
<th>( z )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>B</th>
<th>( \Delta )</th>
</tr>
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<tbody>
<tr>
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Let's start with \( \lambda = 0 \), and investigate the LP as \( \lambda \) increases.

The optimal tableau for \( \lambda = 0 \):

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<tr>
<th>( z )</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>B</th>
<th>( \Delta )</th>
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<td>0.667</td>
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</tbody>
</table>

(This column updated during each pivot)

The optimal solution of \( P(0) \) is \( z^*(0) = -5 \), at \( x_1^*(0) = 3, x_2^*(0) = 2, x_3^*(0) = 1 \), \( x_4^*(0) = x_5^*(0) = 0 \)
Expressed as functions of $\lambda$, the basic solution is:

\[
\begin{array}{c|cc|c|c}
-z & 1 & 2 & 3 & B & \Delta \\
1 & 0 & 0 & 0 & 5 & 3 & x_1(\lambda) = -5 + 3\lambda \\
0 & 1 & 0 & 0 & 3 & -1 & x_2(\lambda) = 3 - \lambda \\
0 & 0 & 1 & 0 & 2 & 4 & x_3(\lambda) = 2 + 4\lambda \\
0 & 0 & 0 & 1 & 1 & 2 & x_4(\lambda) = 1 - 2\lambda \\
\end{array}
\]

Note that these are linear functions of $\lambda$.

The optimality criterion (reduced cost $\geq 0$) is independent of the parameter $\lambda$, and so the current basis remains optimal so long as the basic variables

\[
\begin{align*}
x_1(\lambda) &= 3 - \lambda \\
x_2(\lambda) &= 2 + 4\lambda \\
x_3(\lambda) &= 1 - 2\lambda \\
x_4(\lambda) - x_4(\lambda) = 0
\end{align*}
\]

remain feasible, i.e., nonnegative.

For what values of $\lambda$ is $x^*(\lambda) \geq 0$?

We solve the inequalities

\[
\begin{align*}
x_1(\lambda) &= 3 - \lambda \geq 0 \\
x_2(\lambda) &= 2 + 4\lambda \geq 0 \\
x_3(\lambda) &= 1 - 2\lambda \geq 0
\end{align*}
\]

for $\lambda$:

\[
\begin{align*}
3 - \lambda &\geq 0 \Rightarrow \lambda \leq 3 \\
2 + 4\lambda &\geq 0 \Rightarrow \lambda \geq -\frac{1}{2} \\
1 - 2\lambda &\geq 0 \Rightarrow \lambda \leq \frac{1}{2}
\end{align*}
\]

That is, the basic solution is feasible for all $\lambda \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ (and, in particular, for $\lambda = 0$).

Let us now increase $\lambda$ from its initial value (0) to the upper limit for which the basis is feasible, i.e., $-\frac{1}{2}$. The basic solution then becomes

\[
\begin{align*}
x_1(+\frac{1}{2}) &= 3 - \frac{1}{2} = 2.5 \\
x_2(+\frac{1}{2}) &= 2 + 4\left(\frac{1}{2}\right) = 4.0 \\
x_3(+\frac{1}{2}) &= 1 - 2\left(\frac{1}{2}\right) = 0
\end{align*}
\]

Any further increase in $\lambda$ would result in infeasible (i.e., negative) values for $x_i$.

In order to increase the parameter $\lambda$ further, $x_5$ must leave the basis, since it would otherwise become negative. In order to remove $x_5$ from the basis, we perform a DUAL SIMPLEX pivot:

\[
\begin{array}{c|cc|c|c|c|c|c|c|c|c|c|c|c|c}
-z & 1 & 2 & 3 & 4 & 5 & B & \Delta \\
1 & 0 & 0 & 0.333 & 0.333 & 0 & 5 & 3 \\
0 & 1 & 0 & 0.667 & 0 & 0 & 3 & -1 \\
0 & 0 & 1 & -0.333 & -0.667 & 0 & 2 & 4 \\
0 & 0 & 0 & 0.333 & 0.667 & 1 & 1 & -2
\end{array}
\]

For this basis, the basic solution is

\[
\begin{array}{c|cc|c|c|c|c|c|c|c|c|c|c|c|c}
-z & 1 & 2 & 3 & 4 & B & \Delta \\
1 & 0 & 0 & 0.5 & 0 & 0 & 5.5 & 2 \\
0 & 1 & 0 & 0.5 & 0 & 0 & 2.5 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 \\
0 & 0 & 0 & 0.5 & 1 & 1.5 & 1.5 & 3
\end{array}
\]

Notice that as $\lambda$ increases, no basic variable decreases. Since the optimality criterion (reduced costs $\geq 0$) does not depend upon $\lambda$, this basis is optimal for all $\lambda \geq 0.5$. 

\[
\begin{align*}
x_1(\lambda) &= 5.5 - 2\lambda \\
x_2(\lambda) &= 2.5 \\
x_3(\lambda) &= 3 + 2\lambda \\
x_4(\lambda) &= -1.5 + 3\lambda
\end{align*}
\]
That is, if we solve the system of inequalities

\[\begin{align*}
x_1^* (\lambda) &= 2.5 \quad 2 \geq 0 \quad \text{(no restriction on } \lambda) \\
x_2^* (\lambda) &= 3 + 2\lambda \quad 2 \geq 0 \quad \lambda \geq -1.5 \\
x_3^* (\lambda) &= -1.5 + 3\lambda \quad 2 \geq 0 \quad \lambda \geq 0.5
\end{align*}\]

we see that it is satisfied for \( \lambda \in [0.5, +\infty) \).

Let us now investigate \( P(\lambda) \) for \( \lambda < 0.5 \)

Consider the tableau which was optimal for

\[
\begin{array}{c|ccccc|c}
\lambda & 1 & 2 & 3 & 4 & 5 & B \\
\hline
1 & 0 & 0 & 0.333 & 0.333 & 0 & 5 \\
0 & 1 & 0 & 0.667 & -0.333 & 0 & 3 \\
0 & 0 & 1 & -0.333 & 0.667 & 0 & 2 \\
0 & 0 & 0 & 0.333 & -0.667 & 1 & 1 \\
\end{array}
\]

Recall that the lower limit of the parameter, \( \lambda \geq -\frac{1}{2} \), derives from \( x_2^*(\lambda) \geq 0 \), i.e., \( x_2^*(-\frac{1}{2}) = 0 \).

A further decrease in \( \lambda \) requires that \( x_3 \) be removed from the basis (by a dual simplex pivot).

The new basic solution is:

\[
\begin{array}{c|ccccc|c}
\lambda & 1 & 2 & 3 & 4 & 5 & B \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 7 \\
0 & 1 & 0 & 0 & 0 & 1 & 7 \\
0 & 0 & 1 & 0 & 1 & 1 & 6 \\
0 & 0 & 0 & 1 & 0 & 1 & 3 \\
\end{array}
\]

To find the interval for which this basic solution is feasible (and therefore optimal), solve

\[
\begin{align*}
x_1^*(\lambda) &= 7 + 7\lambda \geq 0 \\
x_2^*(\lambda) &= 6 - 12\lambda \geq 0 \\
x_3^*(\lambda) &= 3 + 2\lambda \geq 0
\end{align*}
\]

that is,

\[
\begin{align*}
\lambda &\geq -1 \\
\lambda &\geq -0.5 \\
\lambda &\geq -1.5
\end{align*}
\]

Summary of Parametric Analysis:

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>(-\infty, -1)</th>
<th>[-1, 0.5)</th>
<th>[0.5, +\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1^* )</td>
<td>7 + 7\lambda</td>
<td>3 - \lambda</td>
<td>25</td>
</tr>
<tr>
<td>( x_2^* )</td>
<td>0</td>
<td>2 + 4\lambda</td>
<td>3 + 2\lambda</td>
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<tr>
<td>( x_3^* )</td>
<td>-6 - 12\lambda</td>
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</table>

Plot of \( z \) vs. \( \lambda \):

Plot of \( x_1^* \) vs. \( \lambda \):

Plot of \( x_2^* \) vs. \( \lambda \):

Plot of \( x_3^* \) vs. \( \lambda \):
**Example**

Minimize \( x_1 + x_2 + 7x_3 + 3x_4 + x_5 + 2x_6 \) subject to

\[
\begin{align*}
  x_1 + 2x_2 - x_3 + x_4 + 2x_6 &= 16 - \lambda \\
  -x_1 - 3x_4 + 3x_5 &= -4 + \lambda \\
  x_j &\geq 0, j = 1, 2, \ldots, 6
\end{align*}
\]

### Initial tableau

<table>
<thead>
<tr>
<th>( z )</th>
<th>1</th>
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### Optimal tableau (\( \lambda = 0 \))

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### Deal Simplex pivot

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### New tableau

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### Dual Simplex Pivot

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A dual simplex pivot in row #4 is not possible:

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The LP is infeasible for \( \lambda > 0 \)
Let's return to the optimal tableau for $\lambda = 0$.

<table>
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**Parametric Analysis**

Least Upper Bound (LUB): $[-1.5, 0.5]$  
= Min $(-3, -4)$  
RHS at LUB is $10, 6, 6, 0$

Greatest Lower Bound (GLB): $3$

= Max $0.5$  
RHS at GLB is $16, 18, 9, 6$

Range of parameters $\lambda$ within which basis is feasible:

$[-5, 4]$

Let's return to the optimal tableau for $\lambda = 0$.

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**Parametric Analysis**

Least Upper Bound (LUB): $[-2, 0.5]$  
= Min $(-4, -3)$  
RHS at LUB is $10, 6, 6, 0$

Greatest Lower Bound (GLB): $20$

= Max $0.5$  
RHS at GLB is $70, 52, 6, 0$

Range of parameters $\lambda$ within which basis is feasible:

$[20, 10]$

Let's return to the optimal tableau for $\lambda = 0$.

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</table>

**Parametric Analysis**

Least Upper Bound (LUB): $20$

= Min $[12, 0.667, 2.33, -1.5, 0.333, -0.167]$  
+ Min $[-3, -2, -20]$

RHS at LUB is $10, 6, 6, 0$

No Lower Bound

Range of parameters $\lambda$ within which basis is feasible:

$[1.85308, -\infty]$  

i.e., $\lambda$ is unbounded below.