Parametric Programming on the Right-Hand-Side

Parametric Programming is the analysis of the variation of the solution of an LP when some element (right-hand-side, objective, etc.) varies.

Consider the optimal value of the $\mathbb{LP}$ as a function of the right-hand-side vector, i.e.,

$$Z^*(b) = \left\{\begin{array}{ll} \text{Min} & c\,x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}\right\} \underset{\textit{by duality theory}}{=} \left\{\begin{array}{ll} \text{Max} & \pi b \\ \text{s.t.} & \pi A \leq c \\ & \pi \geq 0 \end{array}\right\}$$

The function z^* is "evaluated" for some particular right-hand-side b' by solving the LP (either the primal or the dual).

So we can evaluate $z^*(b')$ by solving the LP

$$z^*(b') = \left\{\begin{array}{ll} \text{Max} & \pi b' \\ \text{s.t.} & \pi A \leq c \\ & \pi \geq 0 \end{array}\right.$$

Notice that the feasible region of the dual LP is the same for every argument b'.

We know that a basic solution is optimal for an LP problem, and that there are a finite (but possibly very large!) number of such basic solutions.

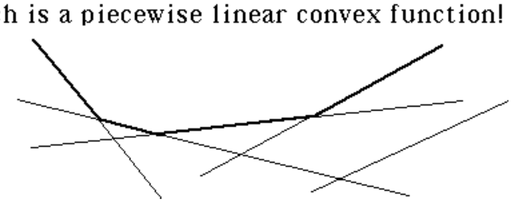
Suppose that we were to number the basic feasible solutions of the dual LP:
$\{\pi^1, \pi^2, \pi^3, \ldots \pi^K\}$
where each $\pi^k = c_{B_k}(A^{B_k})^{-1}$ for some dual-feasible basis B_k
(K is a finite number, no greater than $\binom{n+m}{m}$.)

In theory, then, we could evaluate $z^*(b)$ by enumerating the K basic feasible solutions of the dual, evaluating the dual objective $\pi^k b$ at each, and selecting the maximum such value.

Therefore, $z^*(b) = \underset{k=1,2,\ldots K}{\text{Maximum}} \{\pi^k b\}$

That is, $z^*(b)$ is the maximum of a family of linear functions, $\pi^k b$, $k=1,2,\ldots K$

which is a piecewise linear convex function!



Let us restrict our analysis of the function z(b) to a study of its behavior along a line (rather than everywhere in m-dimensional space!)*

That is, we assume an initial right-hand-side vector (b) is given, and a direction (d), and study the behavior of $z^*(b+\lambda d)$, considered as a function of the scalar parameter λ.

Consider the solution of the LP

$$P_\lambda: \quad z^*(\lambda) = \text{minimum } c\,x \quad \text{s.t. } Ax = b + \lambda d, \quad x \geq 0$$

where d is an m-vector and λ is a scalar.

$$z^*(\lambda) = \underset{k=1,2,\ldots K}{\text{maximum}} \{\pi^k(b+\lambda d)\}$$

$$= \underset{k=1,2,\ldots K}{\text{maximum}} \{\underbrace{\pi^k b}_{\textit{intercept}} + \underbrace{(\pi^k d)}_{\textit{slope}}\lambda\} \leftarrow \textit{linear functions of } \lambda$$

Example

$$z^*(\lambda) = \text{minimum } -x_1 - x_2$$

subject to
$$\begin{cases} 2x_1 + x_2 \le 8 + 2\lambda \\ x_1 + 2x_2 \le 7 + 7\lambda \\ x_2 \le 3 + 2\lambda \\ x_1 \ge 0,\ x_2 \ge 0 \end{cases}$$

$$= \text{maximum } \{ (8\pi_1 + 7\pi_2 + 2\pi_3) + (2\pi_1 + 7\pi_2 + 2\pi_3)\lambda \}$$

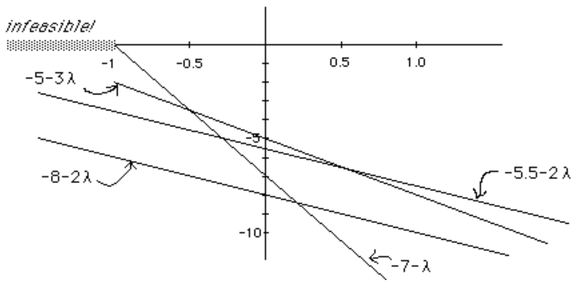
s.t.
$$2\pi_1 + \pi_2 \le -1$$
$$\pi_1 + 2\pi_2 + \pi_3 \le -1$$
$$\pi_1 \le 0, \pi_2 \le 0, \pi_3 \le 0$$

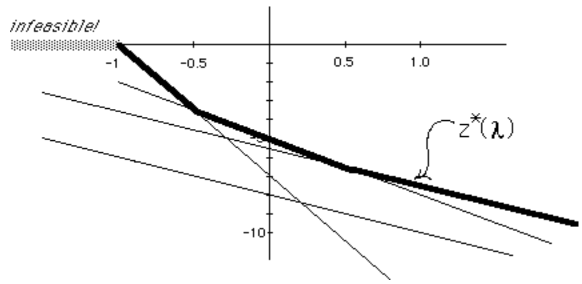
nonpositive because of direction of ≤ in the primal!

π_1	π_2	π_3	intercept $\pi^k b$	slope $\pi^k d$	Basis	
0	0	0	0	0	3 4 5	
-0.5	0	0	-4	-1	1 4 5	
-1	0	0	-8	-2	2 4 5	⇐ feasible
-0.333	-0.333	0	-5	-3	1 2 5	⇐ feasible
0	-1	0	-7	-7	1 3 5	⇐ feasible
0	-0.5	0	-3.5	-3.5	2 3 5	
0	-1	1	-4	-5	1 2 3	
-0.5	0	-0.5	-5.5	-2	1 2 4	⇐ feasible
0	0	-1	-3	-2	2 3 4	

(columns #1,3,4 are dependent & do not form a basis!)

Of the nine basic solutions, four are dual feasible. Therefore, $z^*(\lambda)$ is the maximum of four linear functions:





In this example, with only nine basic dual solutions, it was possible to enumerate all of them, test each for feasibility, and then maximize the corresponding linear functions

$$\pi^k b + (\pi^k d)\lambda$$

However, for most problems, the number of basic dual solutions is astronomical and enumerating them is practically impossible.

(Usually, only a few of these basic dual solutions actually determine z.)*

Let's consider again the parametric LP P_λ:

$$z^*(\lambda) = \text{minimum } -x_1 - x_2$$

subject to
$$\begin{cases} 2x_1 + x_2 \le 8 + 2\lambda \\ x_1 + 2x_2 \le 7 + 7\lambda \\ x_2 \le 3 + 2\lambda \\ x_1 \ge 0,\ x_2 \ge 0 \end{cases}$$

Determine the optimal value function $z^*(\lambda)$ as well as $x_1^*(\lambda)$ and $x_2^*(\lambda)$ [i.e., the optimal solution] for all values of $\lambda \in (-\infty, +\infty)$

The initial tableau:

−z	1	2	3	4	5	B	Δ (change column)
1	−1	−1	0	0	0	0	0
0	2	1	1	0	0	8	2
0	1	2	0	1	0	7	7
0	0	1	0	0	1	3	2

RHS is $B + \lambda\Delta$

Columns 3, 4, 5: *slack variables*

Let's start with $\lambda = 0$, and investigate the LP as λ increases.

The optimal tableau for $\lambda = 0$:

	−z	1	2	3	4	5	B	Δ
(min)	1	0	0	0.333	0.333	0	5	3
	0	1	0	0.667	−0.333	0	3	−1
	0	0	1	−0.333	0.667	0	2	4
	0	0	0	0.333	−0.667	1	1	−2

(this column updated during each pivot)

The optimal solution of $P(0)$ is $z^*(0) = -5$, at $x_1^*(0) = 3$, $x_2^*(0) = 2$, $x_5^*(0) = 1$, $x_3^*(0) = x_4^*(0) = 0$

Expressed as functions of λ, the basic solution is:

$-z$	1	2	...5	B	Δ
1	0	0	0	5	3
0	1	0	0	3	⁻1
0	0	1	0	2	4
0	0	0	1	1	⁻2

$$\Rightarrow \begin{cases} z(\lambda) = -5 + 3\lambda \\ x_1^*(\lambda) = 3 - \lambda \\ x_2^*(\lambda) = 2 + 4\lambda \\ x_5^*(\lambda) = 1 - 2\lambda \\ x_3^*(\lambda) = x_4^*(\lambda) = 0 \end{cases}$$

Note that these are linear functions of λ!

The optimality criterion (reduced cost ≥ 0) is independent of the parameter λ, and so the current basis remains optimal so long as the basic variables

$$x_1^*(\lambda) = 3 - \lambda$$
$$x_2^*(\lambda) = 2 + 4\lambda$$
$$x_5^*(\lambda) = 1 - 2\lambda$$

remain feasible, i.e., nonnegative.

For what values of λ is $x^*(\lambda) \geq 0$?

We solve the inequalities

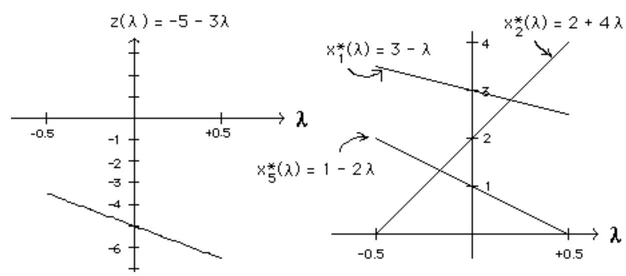
$$x_1^*(\lambda) = 3 - \lambda \geq 0$$
$$x_2^*(\lambda) = 2 + 4\lambda \geq 0$$
$$x_5^*(\lambda) = 1 - 2\lambda \geq 0$$

for λ:

$$3 - \lambda \geq 0 \Rightarrow \lambda \leq 3$$
$$2 + 4\lambda \geq 0 \Rightarrow \lambda \geq -\tfrac{1}{2}$$
$$1 - 2\lambda \geq 0 \Rightarrow \lambda \leq +\tfrac{1}{2}$$

That is, the basic solution is feasible for all $\lambda \in [-\tfrac{1}{2}, +\tfrac{1}{2}]$ (and, in particular, for $\lambda = 0$).

Plot of $z(\lambda)$ & $x_j^*(\lambda)$ over the interval [-0.5, +0.5]:



Let us now increase λ from its initial value (0) to the upper limit for which the basis is feasible, i.e., $+\tfrac{1}{2}$. The basic solution then becomes

$$x_1^*(+\tfrac{1}{2}) = 3 - \tfrac{1}{2} = 2.5$$
$$x_2^*(+\tfrac{1}{2}) = 2 + 4(\tfrac{1}{2}) = 4.0$$
$$x_5^*(+\tfrac{1}{2}) = 1 - 2(\tfrac{1}{2}) = 0$$

Any further increase in λ would result in infeasible (i.e., negative) values for x_5^*!

In order to increase the parameter λ further, x_5 must leave the basis, since it would otherwise become negative. In order to remove x_5 from the basis, we perform a DUAL SIMPLEX pivot:

$-z$	1	2	3	4	5	B	Δ
1	0	0	0.333	0.333	0	5	3
0	1	0	0.667	⁻0.333	0	3	⁻1
0	0	1	⁻0.333	0.667	0	2	4
0	0	0	0.333	⁻0.667	1	1	⁻2 ← = 0 when λ = 1/2

pivot in this row!

$-z$	1	2	3	4	5	B	Δ
1	0	0	0.333	0.333	0	5	3
0	1	0	0.667	⁻0.333	0	3	⁻1
0	0	1	⁻0.333	0.667	0	2	4
0	0	0	0.333	⁻0.667	1	1	⁻2

pivot

$-z$	1	2	3	4	5	B	Δ
1	0	0	0.5	0	0.5	5.5	2
0	1	0	0.5	0	⁻0.5	2.5	0
0	0	1	0	0	1	3	2
0	0	0	⁻0.5	1	⁻1.5	⁻1.5	3

For this basis, the basic solution is

$-z$	1	2	...4...	B	Δ
1	0	0	0	5.5	2
0	1	0	0	2.5	0
0	0	1	0	3	2
0	0	0	1	⁻1.5	3

$$\Rightarrow \begin{cases} z(\lambda) = -5.5 - 2\lambda \\ x_1^*(\lambda) = 2.5 \\ x_2^*(\lambda) = 3 + 2\lambda \\ x_4^*(\lambda) = -1.5 + 3\lambda \end{cases}$$

Notice that as λ increases, no basic variable decreases. Since the optimality criterion (reduced costs ≥ 0) does not depend upon λ, this basis is optimal for all $\lambda \geq 0.5$

That is, if we solve the system of inequalities

$x_1^*(\lambda) = 2.5 \geq 0 \implies$ *(no restriction on λ.)*
$x_2^*(\lambda) = 3 + 2\lambda \geq 0 \implies \lambda \geq -1.5$
$x_4^*(\lambda) = -1.5+3\lambda \geq 0 \implies \lambda \geq 0.5$

we see that it is satisfied for $\lambda \in [0.5, +\infty)$

Let us now investigate P(λ) for $\lambda < -0.5$
Consider the tableau which was optimal for $-0.5 \leq \lambda \leq +0.5$

-z	1	2	3	4	5	B	Δ
1	0	0	0.333	0.333	0	5	3
0	1	0	0.667	-0.333	0	3	-1
0	0	1	-0.333	0.667	0	2	4
0	0	0	0.333	-0.667	1	1	-2

Recall that the lower limit of the parameter, $\lambda \geq -\frac{1}{2}$, derives from $x_2^*(\lambda) \geq 0$, i.e., $x_2^*(-\frac{1}{2}) = 0$

A further decrease in λ requires that X_2 be removed from the basis (by a dual simplex pivot)

-z	1	2	3	4	5	B	Δ	
1	0	0	0.333	0.333	0	5	3	
0	1	0	0.667	-0.333	0	3	-1	
0	0	1	-0.333	0.667	0	2	4	← *pivot row*
0	0	0	0.333	-0.667	1	1	-2	

pivot here!

the dual simplex pivot yields ⟹

-z	1	2	3	4	5	B	Δ
1	0	1	0	1	0	7	7
0	1	2	0	1	0	7	7
0	0	-3	1	-2	0	-6	-12
0	0	1	0	0	1	3	2

The new basic solution is:

-z	1	3	5	B	Δ
1	0	0	0	7	7
0	1	0	0	7	7
0	0	1	0	-6	-12
0	0	0	1	3	2

$$\Rightarrow \begin{cases} z(\lambda) = -7 - 7\lambda \\ x_1^*(\lambda) = 7 + 7\lambda \\ x_3^*(\lambda) = -6 - 12\lambda \\ x_5^*(\lambda) = 3 + 2\lambda \end{cases}$$

To find the interval for which this basic solution is feasible (& therefore optimal), solve

$$\begin{aligned} x_1^*(\lambda) &= 7 + 7\lambda \geq 0 \\ x_3^*(\lambda) &= -6 - 12\lambda \geq 0 \\ x_5^*(\lambda) &= 3 + 2\lambda \geq 0 \end{aligned} \implies \begin{cases} \lambda \geq -1 \\ \lambda \leq -0.5 \\ \lambda \geq -1.5 \end{cases}$$

that is, $\lambda \in [-1.0, -0.5]$

When λ decreases to -1.0, $x_1^*(\lambda)$ decreases to 0 and must be removed from the basis to allow any further decrease in the parameter. We therefore attempt another dual simplex pivot:

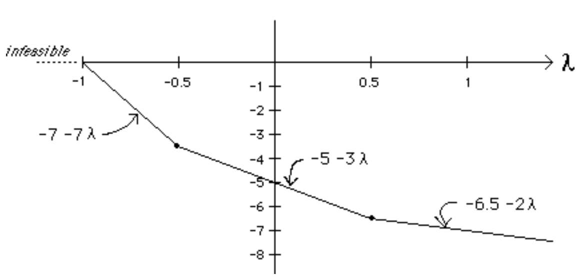
-z	1	2	3	4	5	B	Δ	
1	0	1	0	1	0	7	7	
0	1	2	0	1	0	7	7	← *pivot row*
0	0	-3	1	-2	0	-6	-12	
0	0	1	0	0	1	3	2	

Because there is no negative element in the pivot row, x_1 cannot be removed from the basis, and it is evident that P(λ) is infeasible for $\lambda < -1.0$

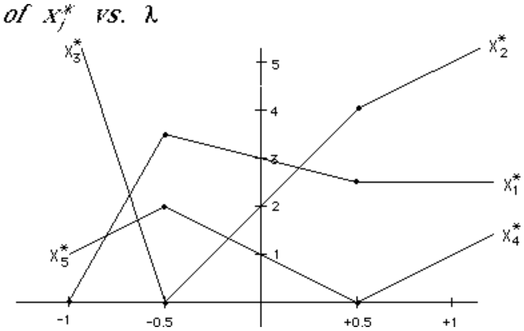
Summary of Parametric Analysis:

λ	(-∞, -1]	[-1, -0.5]	[-0.5, +0.5]	[+0.5, +∞)
x_1^*	*infeasible*	7 + 7λ	3 - λ	2.5
x_2^*		0	2 + 4λ	3 + 2λ
x_3^*		-6 - 12λ	0	0
x_4^*		0	0	-1.5 + 3λ
x_5^*		3 + 2λ	1 - 2λ	0

Plot of z vs λ :



Plot of x_j^* vs. λ



Example

Minimize $x_1 + x_2 + 7x_3 + 3x_4 + x_5 + 2x_6$
subject to

$$x_1 + 2x_2 - x_3 - x_4 + x_5 + 2x_6 = 16 - \lambda$$
$$x_2 - 3x_4 - x_5 + 3x_6 = 2 + \lambda$$
$$-x_1 - 3x_3 + 3x_4 + x_5 = -4 + \lambda$$
$$x_j \geq 0,\ j=1,2,\ldots 6$$

Initial tableau:

-z	1	2	3	4	5	6	B	Δ
1	1	1	7	3	1	2	0	0
0	1	2	-1	-1	1	2	16	-1
0	0	1	0	-3	-1	3	2	1
0	-1	0	-3	3	1	0	-4	1

Optimal tableau (λ = 0)

-z	1	2	3	4	5	6	B	Δ
1	0	0	7	3	0	2	-12	1.5
0	1	0	2	-1	0	-1	6	-1.5
0	0	1	-1	-1	0	2	4	0.5
0	0	0	-1	2	1	-1	2	-0.5

-z	1	2	3	4	5	6	B	Δ
1	0	0	7	3	0	2	-12	1.5
0	1	0	2	-1	0	-1	6	-1.5
0	0	1	-1	-1	0	2	4	0.5
0	0	0	-1	2	1	-1	2	-0.5

Parametric Analysis

```
Least Upper Bound (LUB): 4
           = Min{-6 -2 ÷ -1.5 -0.5} = Min {4 4}
          RHS at LUB is -10 0 6 0
Greatest Lower Bound (GLB): -8
           = Max{-4 ÷ 0.5} = Max {-8}
          RHS at GLB is -16 18 0 6
Range of parameters LAMBDA within which basis is feasible:
          [ -8 , 4 ]
```


Dual Simplex pivot

-z	1	2	3	4	5	6	B	Δ
1	0	0	7	3	0	2	-12	1.5
0	1	0	2	-1	0	-1	6	-1.5 ←
0	0	1	-1	-1	0	2	4	0.5
0	0	0	-1	2	1	-1	2	-0.5

New tableau

-z	1	2	3	4	5	6	B	Δ
1	2	0	11	1	0	0	0	-1.5
0	-1	0	-2	1	0	1	-6	1.5
0	2	1	3	-3	0	0	16	-2.5
0	-1	0	-3	3	1	0	-4	1

-z	1	2	3	4	5	6	B	Δ
1	2	0	11	1	0	0	0	-1.5
0	-1	0	-2	1	0	1	-6	1.5
0	2	1	3	-3	0	0	16	-2.5
0	-1	0	-3	3	1	0	-4	1

Parametric Analysis

```
Least Upper Bound (LUB): 6.4
           = Min{-16 ÷ -2.5} = Min {6.4}
          RHS at LUB is -16 3.6 0 2.4
Greatest Lower Bound (GLB): 4
           = Max{6 4 ÷ 1.5 1} = Max {4 4}
          RHS at GLB is -10 0 6 0
Range of parameters LAMBDA within which basis is feasible:
          [ 4 , 6.4 ]
```


Dual Simplex Pivot

-z	1	2	3	4	5	6	B	Δ
1	2	0	11	1	0	0	0	-1.5
0	-1	0	-2	1	0	1	-6	1.5
0	2	1	3	-3	0	0	16	-2.5 ←
0	-1	0	-3	3	1	0	-4	1

-z	1	2	3	4	5	6	B	Δ
1	2.67	0.333	12	0	0	0	5.33	-2.33
0	-0.333	0.333	-1	0	0	1	-0.667	0.667
0	-0.667	-0.333	-1	1	0	0	-5.33	0.833
0	1	1	0	0	1	0	12	-1.5

-z	1	2	3	4	5	6	B	Δ
1	2.67	0.333	12	0	0	0	5.33	-2.33
0	-0.333	0.333	-1	0	0	1	-0.667	0.667
0	-0.667	-0.333	-1	1	0	0	-5.33	0.833
0	1	1	0	0	1	0	12	-1.5

Parametric Analysis

```
Least Upper Bound (LUB): 8
           = Min{-12 ÷ -1.5} = Min {8}
          RHS at LUB is -21.3 4.67 1.33 0
Greatest Lower Bound (GLB): 6.4
           = Max{0.667 5.33 ÷ 0.667 0.833} = Max {1 6.4}
          RHS at GLB is -16 3.6 0 2.4
Range of parameters LAMBDA within which basis is feasible:
          [ 6.4 , 8 ]
```


A dual simplex pivot in row #4 is not possible:

-z	1	2	3	4	5	6	B	Δ
1	2.67	0.333	12	0	0	0	5.33	-2.33
0	-0.333	0.333	-1	0	0	1	-0.667	0.667
0	-0.667	-0.333	-1	1	0	0	-5.33	0.833
0	1	1	0	0	1	0	12	-1.5

The LP is infeasible for $\lambda > 8$

Let's return to the optimal tableau for $\lambda = 0$:

-z	1	2	3	4	5	6	B	Δ
1	0	0	7	3	0	2	-12	1.5
0	1	0	2	-1	0	-1	6	-1.5
0	0	1	-1	-1	0	2	4	0.5
0	0	0	-1	2	1	-1	2	-0.5

Parametric Analysis

```
Least Upper Bound (LUB): 4
          = Min{-6 -2 ÷ -1.5 -0.5} = Min {4 4}
         RHS at LUB is -10 0 6 0
Greatest Lower Bound (GLB): -8
          = Max{-4 ÷ 0.5} = Max {-8}
         RHS at GLB is -16 18 0 6
Range of parameters LAMBDA within which basis is feasible:
         [ -8 , 4 ]
```


Dual Simplex Pivot

-z	1	2	3	4	5	6	B	Δ
1	0	0	7	3	0	2	-12	1.5
0	1	0	2	-1	0	-1	6	-1.5
0	0	1	-1	-1	0	2	4	0.5 ⟵
0	0	0	-1	2	1	-1	2	-0.5

New tableau

-z	1	2	3	4	5	6	B	Δ
1	0	3	4	0	0	8	0	3
0	1	-1	3	0	0	-3	2	-2
0	0	-1	1	1	0	-2	-4	-0.5
0	0	2	-3	0	1	3	10	0.5

-z	1	2	3	4	5	6	B	Δ
1	0	3	4	0	0	8	0	3
0	1	-1	3	0	0	-3	2	-2
0	0	-1	1	1	0	-2	-4	-0.5
0	0	2	-3	0	1	3	10	0.5

Parametric Analysis

```
Least Upper Bound (LUB): -8
          = Min{-2 4 ÷ -2 -0.5} = Min {1 -8}
         RHS at LUB is -16 18 0 6
Greatest Lower Bound (GLB): -20
          = Max{-10 ÷ 0.5} = Max {-20}
         RHS at GLB is -40 42 6 0
Range of parameters LAMBDA within which basis is feasible:
         [ -20 , -8 ]
```


Dual Simplex Pivot

-z	1	2	3	4	5	6	B	Δ
1	0	3	4	0	0	8	0	3
0	1	-1	3	0	0	-3	2	-2
0	0	-1	1	1	0	-2	-4	-0.5
0	0	2	-3	0	1	3	10	0.5 ⟵

New tableau

-z	1	2	3	4	5	6	B	Δ
1	0	5.67	0	0	1.33	12	13.3	3.67
0	1	1	0	0	1	0	12	-1.5
0	0	-0.333	0	1	0.333	-1	-0.667	-0.333
0	0	-0.667	1	0	-0.333	-1	-3.33	-0.167

-z	1	2	3	4	5	6	B	Δ
1	0	5.67	0	0	1.33	12	13.3	3.67
0	1	1	0	0	1	0	12	-1.5
0	0	-0.333	0	1	0.333	-1	-0.667	-0.333
0	0	-0.667	1	0	-0.333	-1	-3.33	-0.167

Parametric Analysis

```
Least Upper Bound (LUB): -20
          = Min{-12 0.667 3.33 ÷ -1.5 -0.333 -0.167}
          = Min {  8 -2 -20}
         RHS at LUB is -60 42 6 0

         No Lower Bound
Range of parameters LAMBDA within which basis is feasible:
         [ -1.8E308 , -20 ]
```

i.e., $-\infty$