We will apply the primal simplex method to the minimum-cost network flow problem, but (as was the case with the transportation problem) without pivoting in the full tableau.

Questions to consider:
- How is basis matrix represented?
- How is simplex multiplier vector computed?
- How is change of basis accomplished?

The columns of the node-arc incidence matrix corresponding to the arcs of a cycle are linearly dependent.

Example

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 1
\end{bmatrix}
\]

The sum of the columns is the zero vector!

Example

\[
\begin{align*}
(1,2) & + (2,4) + (3,4) + (1,3) \\
\end{align*}
\]

Send a unit of flow around the cycle ... the coefficient of the column will be +1 if the flow is in direction of arc, and -1 if in opposite direction!

Theorem

A tree containing \(m\) nodes contains \(m-1\) arcs.

Removing a terminal node and its incident arc leaves a tree. \(m-1\) nodes can be so removed (along with \(m-1\) arcs), leaving finally a single node but no arc.

The Node-Arc incidence matrix of a tree is, after rearranging rows \&/or columns, Lower Triangular.
Recall: rank of node-arc incidence matrix of a network is $< m$ (≠ nodes)
rank of node-arc incidence matrix of a spanning tree is $m - 1$

Any basis matrix of the node-arc incidence matrix
is the node-arc incidence matrix of a spanning tree,
plus the column for the artificial variable.

Computing the Basic Solution
(flow in the "rooted" spanning tree)

Begining at the ends of the tree, assign flows until you reach the root.

$x_{13} = 7$ and $x_{32} = 6$

Update "supply" at node 3 and "trim" arcs (1,3) and (3,2)
from the tree.

Node 3 is now an end.

$x_{34} = 4$.

Trim (3,4), leaving node 4 as an end.

Flow in root node is zero.
**Expressing a nonbasic arc as a combination of basic arcs**

To write arc (2,4) as a combination of basic arcs:

Inserting arc (2,4) into the spanning tree creates a cycle.

**Pricing Nonbasic Arcs**

Reduced cost of (i,j) is $C_{ij} - Z_{ij}$, where $Z_{ij}$ = cost of combination of basic arcs which is equivalent to nonbasic arc (i,j).

What is the reduced cost of nonbasic arc (2,4)?

**Pricing Nonbasic Arcs**  
*(An easier approach!)*

Reduced cost of arc (i,j) = $C_{ij} - w^T A^j$

where $w$ is vector of Simplex Multipliers

and $A^j$ is the column of the node-arc incidence matrix for arc (i,j).

How can we compute the Simplex Multipliers?

**Computing Simplex Multipliers**

$w = C_B (A^B)^{-1}$

i.e., $w A^B = C_B$

or $w_i - w_j = C_{ij}$ for each basic arc (i,j)

Because of the fact that the basis matrix is (possibly after rearranging rows &/or columns) lower triangular, these equations are simple to solve for w.

\[
\begin{bmatrix}
w_1 \\ w_2 \\ w_3 \\ w_4
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
w_1 \\ w_2 \\ w_3 \\ w_4
\end{bmatrix} = \begin{bmatrix}2 & 2 & 4 & 0\end{bmatrix}
\]

$w = \begin{bmatrix}w_1 \\ w_2 \\ w_3 \\ w_4\end{bmatrix} \Rightarrow \begin{bmatrix}w_1 \\ w_2 \\ w_3 \\ w_4\end{bmatrix} = \begin{bmatrix}2 \\ 2 \\ 4 \\ 0\end{bmatrix}$

Solve by “back-substitution”:

\[
\begin{align*}
w_1 &= 2 \\
w_2 &= 2 \\
w_3 &= 4 \\
w_4 &= 0
\end{align*}
\]

**Reduced Costs:**

arc (1,2): $1 - (6-2) = -3$

arc (2,3): $3 - (2-4) = +5$

arc (2,4): $3 - (2-0) = +1$

arc (4,1): $1 - (0-6) = 7$

negative reduced cost indicates arc to enter the basis.
Choosing the Arc to Leave the Basis

Suppose that arc $(1,2)$ is to enter the basis, i.e., the tree.

Identify the cycle created by inserting arc $(1,2)$.

Send an amount of flow $\Delta$ around this cycle in direction of $(1,2)$. (Sending flow against direction of an arc will decrease flow on the arc.)

Increase $\Delta$ until the flow in some arc in the cycle drops to zero. Remove this arc from the tree.

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An initial basis (rooted spanning tree):

Basic solution:

Computing the Simplex Multipliers

For each basic arc $(i,j)$, $W_i - W_j = C_{ij}$

Start with "root", assign arbitrary value 0, and work your way to the ends of the branches.

$W_i = C_{ij} + W_j$

Computing Reduced Costs $C_{ij} - (W_i - W_j)$

Adding arc $(5,4)$ to the tree will create a cycle.

Increase flow in $(5,4)$ by an increment $\Delta$.

Adjust other flows around the cycle.

Maximum value for $\Delta$ is $6$, the minimum of flows being decreased. Arc $(6,7)$ leaves the basis.
The new basis (rooted spanning tree)

Thus, one simplex iteration is completed. The algorithm continues until no negative reduced cost remains.

By using a variant of the simplex method known as the "upper bounding technique", it is possible to handle easily the more common network problem in which there are upper &/or lower bounds on the flows in the arcs.