

Generalized Assignment Problem

"Strong" Lagrangian Relaxation

In this algorithm, we relax the semi-assignment constraints,

$$\sum_{i=1}^m X_{ij} = 1 \quad \text{for all } j=1,2,\dots,n$$

while maintaining the machine resource constraints,

$$\sum_{j=1}^n A_{ij} X_{ij} \leq b_i \quad \text{for all } i=1,2,\dots,m$$

Sample Generalized Assignment Problem

Costs

Machine	Jobs								
i	1	2	3	4	5	6	7	8	9
1	10	10	18	25	21	23	24	12	18
2	21	20	12	17	17	14	23	25	24
3	10	22	12	22	14	25	25	23	13
4	20	12	21	16	17	19	17	11	15

Resources Used

Machine	Jobs									Available
i	1	2	3	4	5	6	7	8	9	
1	11	7	22	9	15	12	13	10	10	38
2	12	16	20	10	10	23	24	21	20	44
3	9	20	7	23	5	18	10	13	7	19
4	8	23	13	25	13	13	6	14	25	61

Lagrangian Dual of GAP

(Solved via subgradient optimization)

Multiple choice constraints are relaxed.

Lambda = 0.9

Target $Z^* = 117$ (determined previously by a heuristic algorithm!)

At each iteration we take a step g in the subgradient direction,

where

$$\gamma_j = 1 - \sum_{i=1}^m X_{ij}$$

The sum $\sum_{i=1}^m X_{ij}$ is, of course, the number of machines to which job j

has been assigned.

The step is computed by the formula:

$$w' = w + \lambda \frac{\bar{Z} - Z}{\|\gamma\|^2} \gamma$$

where

w is the current vector of Lagrangian multipliers,

λ is the *stepsize parameter*, $0 < \lambda \leq 2$

\bar{Z} is the current *target value* ($\geq Z^*$)

and

Z is the current value of the Lagrangian dual, i.e., the lower bound provided by the Lagrangian relaxation.

Iteration # 1

Current multipliers:

i: 1 2 3 4 5 6 7 8 9
 w[i]: 20 20 18 22 17 23 24 23 18 (second-largest C_{ij})

Lagrangian relaxation coefficients $C_{ij} - w_j$

	job								
	1	2	3	4	5	6	7	8	9
1	-10	-10	0	3	4	0	0	-11	0
2	1	0	-6	-5	0	-9	-1	2	6
3	-10	2	-6	0	-3	2	1	0	-5
4	0	-8	3	-6	0	-4	-7	-12	-3

By the choice of the initial multipliers, at most one coefficient is positive in each column.

Solution of Lagrangian relaxation (knapsack solutions):

Machine #1: items 1 2 8
 Machine #2: items 3 6
 Machine #3: items 1 3
 Machine #4: items 2 6 7 8

*** Dual value is 92 ***

of times jobs are assigned:

i:	1	2	3	4	5	6	7	8	9
#:	2	2	2	0	0	2	1	2	0

Applying heuristic to X:

Feasible solution has cost 117

Subgradient of Dual Objective is $\bar{1} \bar{1} \bar{1} 1 1 \bar{1} 0 \bar{1} 1$ (γ)
 Step size is 2.813

$$\lambda \frac{\bar{Z} - Z}{\|\gamma\|^2} = 0.9 \times \frac{117 - 92}{8} = 2.813$$

Thus, the updated value of the multiplier for job #1 is

$$w'_1 = w_1 + 2.813\gamma_1 = 20 - 2.813 = 17.187$$

Iteration # 2

Current multipliers:

i: 1 2 3 4 5 6 7 8 9
 w[i]: 17.19 17.19 15.19 24.81 19.81 20.19 24 20.19 20.81

Lagrangian relaxation coefficients

	job								
	1	2	3	4	5	6	7	8	9
1	-7.188	-7.188	2.813	0.1875	1.188	2.813	0	-8.188	-2.813
2	3.813	2.813	-3.188	-7.813	-2.813	-6.188	-1	4.813	3.188
3	-7.188	4.813	-3.188	-2.813	-5.813	4.813	1	2.813	-7.813
4	2.813	-5.188	5.813	-8.813	-2.813	-1.188	-7	-9.188	-5.813

Machine #1: items 1 2 8 9
 Machine #2: items 4 5 6
 Machine #3: items 3 5 9
 Machine #4: items 4 5 7 8

*** Dual value is 92.56 *** (Improvement: 0.5625)

of times jobs are assigned:

i:	1	2	3	4	5	6	7	8	9
#:	1	1	1	2	3	1	1	2	2

Applying heuristic to X:

Feasible solution has cost 117

Subgradient of Dual Objective is $0 0 0 \bar{1} \bar{2} 0 0 \bar{1} \bar{1}$
 Step size is 3.142

Iteration # 3

Current multipliers:

i: 1 2 3 4 5 6 7 8 9
w[i]: 17.19 17.19 15.19 21.67 13.53 20.19 24 17.05 17.67

Lagrangian relaxation coefficients

	job								
	1	2	3	4	5	6	7	8	9
1	-7.188	-7.188	2.813	3.329	7.471	2.813	0	-5.046	0.3295
2	3.813	2.813	-3.188	-4.671	3.471	-6.188	-1	7.954	6.329
3	-7.188	4.813	-3.188	0.3295	0.4714	4.813	1	5.954	-4.671
4	2.813	-5.188	5.813	-5.671	3.471	-1.188	-7	-6.046	-2.671

Machine #1: items 1 2 8
Machine #2: items 4 6
Machine #3: items 1 9
Machine #4: items 4 6 7 8

*** Dual value is 101.6 *** (Improvement: 9.063)

of times jobs are assigned:

i:	1	2	3	4	5	6	7	8	9
#:	2	1	0	2	0	2	1	2	1

Applying heuristic to X:

Feasible solution has cost 117

Subgradient of Dual Objective is -1 0 1 -1 1 -1 0 -1 0
Stepsize is 2.306

Iteration # 4

Current multipliers:

i: 1 2 3 4 5 6 7 8 9
w[i]: 14.88 17.19 17.49 19.36 15.83 17.88 24 14.74 17.67

Lagrangian relaxation coefficients

	job								
	1	2	3	4	5	6	7	8	9
1	-4.881	-7.188	0.5063	5.636	5.165	5.119	0	-2.739	0.3295
2	6.119	2.813	-5.494	-2.364	1.165	-3.881	-1	10.26	6.329
3	-4.881	4.813	-5.494	2.636	-1.835	7.119	1	8.261	-4.671
4	5.119	-5.188	3.506	-3.364	1.165	1.119	-7	-3.739	-2.671

Machine #1: items 1 2 8
Machine #2: items 3 6
Machine #3: items 3 5 9
Machine #4: items 2 7 8

*** Dual value is 106.9 *** (Improvement: 5.319)

of times jobs are assigned:

i:	1	2	3	4	5	6	7	8	9
#:	1	2	2	0	1	1	1	2	1

Applying heuristic to X:

Feasible solution has cost 117

Subgradient of Dual Objective is 0 -1 -1 1 0 0 0 -1 0
Stepsize is 2.263

Iteration # 5

Current multipliers:

i: 1 2 3 4 5 6 7 8 9
w[i]: 14.88 14.92 15.23 21.63 15.83 17.88 24 12.48 17.67

Lagrangian relaxation coefficients

	job								
o\	1	2	3	4	5	6	7	8	9
1	-4.881	-4.925	2.769	3.373	5.165	5.119	0	-0.4766	0.3295
2	6.119	5.075	-3.231	-4.627	1.165	-3.881	-1	12.52	6.329
3	-4.881	7.075	-3.231	0.3731	-1.835	7.119	1	10.52	-4.671
4	5.119	-2.925	5.769	-5.627	1.165	1.119	-7	-1.477	-2.671

Machine #1: items 1 2 8
Machine #2: items 4 6
Machine #3: items 3 5 9
Machine #4: items 2 4 7

*** Dual value is 110.4 *** (Improvement: 3.504)

of times jobs are assigned:

i: 1 2 3 4 5 6 7 8 9
#: 1 2 1 2 1 1 1 1 1

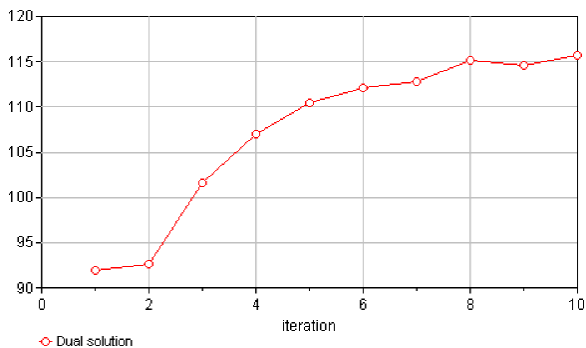
Applying heuristic to X:

Feasible solution has cost 118

Subgradient of Dual Objective is 0 -1 0 -1 0 0 0 0 0

Stepsize is 2.948

... etc.



Note that the dual objective value is not monotonically increasing (although in this case, it is nearly so!)