Reliability Model

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

author

Failure Rate

Consider an experiment:
N identical parts are operated until they fail;
At time t, the number of surviving parts is observed:

$N_S(t)$ = # surviving parts

$N_F(t)$ = # failed parts

where $N = N_S(t) + N_F(t)$

Example

N=20 Electronic Components are tested

 # observation · *time* · *# surviving components* · *# failed components*

i	t	NS	NF
1	1.6	19	1
2	2.8	18	2
3	3.5	17	3
4	3.7	16	4
5	4.7	15	5
6	4.9	14	6
7	5.7	13	7
8	6	12	8
9	6.8	11	9
10	7.5	10	10
11	7.6	9	11
12	8.6	8	12
13	9	7	13
14	10.2	6	14
15	10.9	5	15
16	11	4	16
17	11.8	3	17
18	14	2	18
19	17.5	1	19

Example

fraction surviving · *fraction failed*

i	t	NS	NF	FS	FF
1	1.6	19	1	0.95	0.05
2	2.8	18	2	0.9	0.1
3	3.5	17	3	0.85	0.15
4	3.7	16	4	0.8	0.2
5	4.7	15	5	0.75	0.25
6	4.9	14	6	0.7	0.3
7	5.7	13	7	0.65	0.35
8	6	12	8	0.6	0.4
9	6.8	11	9	0.55	0.45
10	7.5	10	10	0.5	0.5
11	7.6	9	11	0.45	0.55
12	8.6	8	12	0.4	0.6
13	9	7	13	0.35	0.65
14	10.2	6	14	0.3	0.7
15	10.9	5	15	0.25	0.75
16	11	4	16	0.2	0.8
17	11.8	3	17	0.15	0.85
18	14	2	18	0.1	0.9
19	17.5	1	19	0.05	0.95

i	t	NS	NF	FS	FF	ΔNF/Δt
1	1.6	19	1	0.95	0.05	0.625
2	2.8	18	2	0.9	0.1	0.83333
3	3.5	17	3	0.85	0.15	1.42857
4	3.7	16	4	0.8	0.2	5
5	4.7	15	5	0.75	0.25	1
6	4.9	14	6	0.7	0.3	5
7	5.7	13	7	0.65	0.35	1.25
8	6	12	8	0.6	0.4	3.33333
9	6.8	11	9	0.55	0.45	1.25
10	7.5	10	10	0.5	0.5	1.42857
11	7.6	9	11	0.45	0.55	10
12	8.6	8	12	0.4	0.6	1
13	9	7	13	0.35	0.65	2.5
14	10.2	6	14	0.3	0.7	0.83333
15	10.9	5	15	0.25	0.75	1.42857
16	11	4	16	0.2	0.8	10
17	11.8	3	17	0.15	0.85	1.25
18	14	2	18	0.1	0.9	0.45454
19	17.5	1	19	0.05	0.95	0.28571

$^1/_{1.6}$ $^1/_{1.2}$ $^1/_{0.7}$

rate of change in # of failures

(failure rate of the entire surviving population)

i	t	NS	NF	FS	FF	ΔNF/Δt	FR
1	1.6	19	1	0.95	0.05	0.625	0.03289
2	2.8	18	2	0.9	0.1	0.83333	0.04629
3	3.5	17	3	0.85	0.15	1.42857	0.08403
4	3.7	16	4	0.8	0.2	5	0.3125
5	4.7	15	5	0.75	0.25	1	0.06666
6	4.9	14	6	0.7	0.3	5	0.35714
7	5.7	13	7	0.65	0.35	1.25	0.09615
8	6	12	8	0.6	0.4	3.33333	0.27777
9	6.8	11	9	0.55	0.45	1.25	0.11363
10	7.5	10	10	0.5	0.5	1.42857	0.14285
11	7.6	9	11	0.45	0.55	10	1.11111
12	8.6	8	12	0.4	0.6	1	0.125
13	9	7	13	0.35	0.65	2.5	0.35714
14	10.2	6	14	0.3	0.7	0.83333	0.13888
15	10.9	5	15	0.25	0.75	1.42857	0.28571
16	11	4	16	0.2	0.8	10	2.5
17	11.8	3	17	0.15	0.85	1.25	0.41666
18	14	2	18	0.1	0.9	0.45454	0.22727
19	17.5	1	19	0.05	0.95	0.28571	0.28571

$\frac{\Delta NF/\Delta t}{NS}$ $\quad \frac{0.625}{19}$ $\quad \frac{0.8333}{18}$

failure rate per unit

instantaneous failure rate

$$Z(t) = \frac{\frac{d}{dt}N_F(t)}{N_S(t)}$$

a.k.a. "hazard" rate

We wish to express Z(t) in terms of the distribution function F:

reliability $R(t) = \dfrac{N_S(t)}{N} = \dfrac{N - N_F(t)}{N}$

$$\Longrightarrow \frac{d}{dt}R(t) = -\frac{1}{N}\frac{d}{dt}N_F(t) \Longrightarrow \frac{d}{dt}N_F(t) = -N\frac{d}{dt}R(t)$$

Since

$$Z(t) = \frac{\frac{d}{dt}N_F(t)}{N_S(t)} \quad \text{and} \quad \frac{d}{dt}N_F(t) = -N\frac{d}{dt}R(t)$$

therefore

$$Z(t) = \frac{-N\frac{d}{dt}R(t)}{N_S(t)} = \frac{N}{N_S(t)}\left(-\frac{d}{dt}R(t)\right)$$

Recall that

$R(t) = 1 - F(t)$

$$\Longrightarrow -\frac{d}{dt}R(t) = \frac{d}{dt}F(t)$$

$$Z(t) = \frac{1}{R(t)}\frac{d}{dt}F(t)$$

$$\Longrightarrow \boxed{Z(t) = \frac{f(t)}{R(t)}}$$

Another derivation of this relationship:

If T is the time of failure of a part,
with distribution F(t)
and density function f(t),

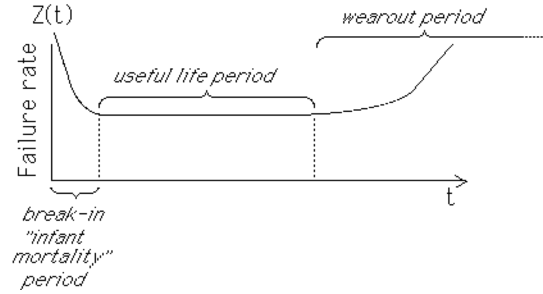
$$Z(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\left\{ \begin{array}{l} \text{part fails in } [t, t+\Delta t], \text{ given} \\ \text{that it has survived to time } t \end{array} \right\}$$

$$Z(t) = \lim_{\Delta t \to 0} \frac{P\{T \le t+\Delta t \mid T \ge t\}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{f(t)\Delta t}{1-F(t)} = \frac{f(t)}{1-F(t)} = \frac{f(t)}{R(t)}$$

"bathtub" curve

Failure rate is initially high, due to manufacturing defects, then levels off (random failures), and finally begins to increase due to wearout



Lifetime with Weibull Dist'n

If there are many possible causes of failure of a system or a component, the lifetime may be considered to be the minimum of a large number of nonnegative random variables, which in the limit is the ***Weibull*** distribution.

Weibull Dist'n

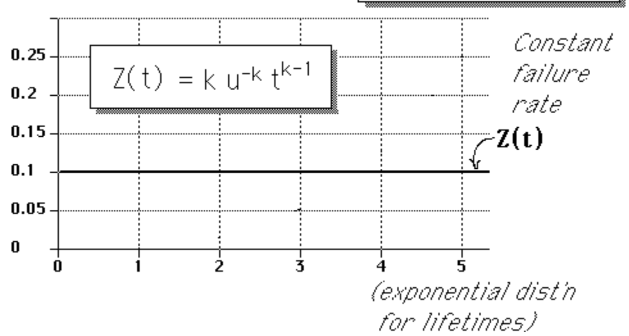
$$\begin{cases} F(t) = 1 - e^{-(t/u)^k} \\ f(t) = k u^{-k} t^{k-1} e^{-(t/u)^k} \end{cases}$$

$$\Rightarrow Z(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1-F(t)} = k\, u^{-k} t^{k-1}$$

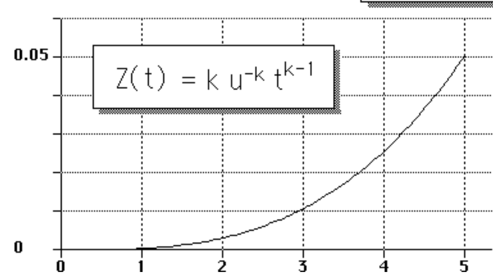
$$\boxed{Z(t) = k\, u^{-k} t^{k-1}}$$ ***instantaneous failure rate***

Z(t) is increasing or decreasing, depending upon k

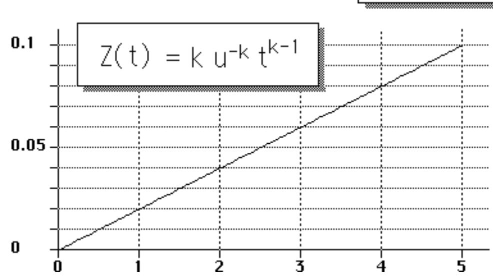
u=10, k=**1** **Weibull Failure Rate Z(t)**



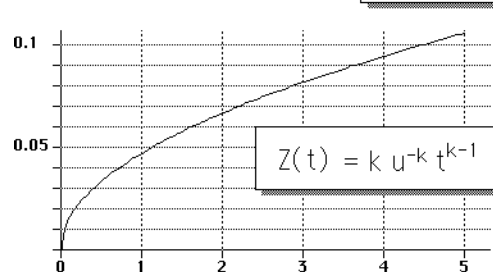
u=10, k=**4** **Weibull Failure Rate Z(t)**



u=10, k=**2** **Weibull Failure Rate Z(t)**

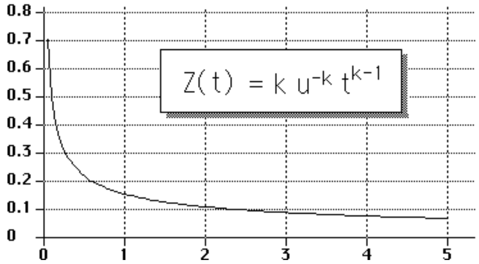


u=10, k=**1.5** **Weibull Failure Rate Z(t)**



Weibull Failure Rate Z(t)

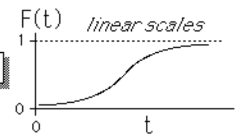
u=10, k= **0.5**



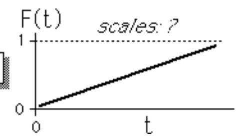
decreasing failure rate

Weibull Probability Paper

CDF



CDF



We would like to change the scales on the F and/or t axes, so that the plot of the CDF of the Weibull distribution is a ***straight line!***

CDF $\quad F(t) = 1 - e^{-(t/u)^k} \implies 1 - F(t) = e^{-(t/u)^k}$

$$\implies \ln \frac{1}{1-F(t)} = \left(\frac{t}{u}\right)^k$$ *take log of both sides*

$$\implies \ln \ln \frac{1}{1-F(t)} = k \ln t - k \ln u$$ *take logarithms again*

transformation of coordinates:

$$\begin{cases} x = \ln t \\ y = \ln \ln \dfrac{1}{1-F(t)} \end{cases}$$

$$\implies \boxed{y = kx - k \ln u}$$ *a line, with slope k and y-intercept −k ln u*

y=0 ⟹ x=ln u (x-intercept)

Given the mean and standard deviation of the lifetimes of all the parts in a sample, we could estimate the parameters of the Weibull distribution:

solve for k: $\quad \dfrac{\sigma_Y}{\mu_Y} = \sqrt{\dfrac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)} - 1}$

then find u: $\quad u = \dfrac{\mu_Y}{\Gamma\left(1+\frac{1}{k}\right)}$

The difficulty is, however, that the parts be tested until ***every*** part has failed, which might require an excessive amount of time!

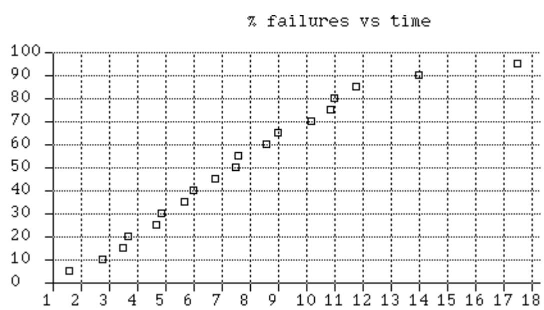
Consider again the example lifetest experiment

(Note that experiment was terminated before all components had failed!)

We will estimate the Weibull parameters by fitting a line to the data on Weibull Probability Paper

i	t	NF	FF
1	1.6	1	0.05
2	2.8	2	0.1
3	3.5	3	0.15
4	3.7	4	0.2
5	4.7	5	0.25
6	4.9	6	0.3
7	5.7	7	0.35
8	6	8	0.4
9	6.8	9	0.45
10	7.5	10	0.5
11	7.6	11	0.55
12	8.6	12	0.6
13	9	13	0.65
14	10.2	14	0.7
15	10.9	15	0.75
16	11	16	0.8
17	11.8	17	0.85
18	14	18	0.9
19	17.5	19	0.95

% failures vs time



Cricket Graph

File Edit Data Graph Curve Fit Goodies Formats

Lifetest Data

	1	2	3	4	5	6
	t	Ns	ln t	R	ln 1/R	ln ln 1/R
1	1.6	19	0.470	.95	0.052	-2.957
2	2.8	18	1.030	.90	0.105	-2.254
3	3.5	17	1.253	.85	0.162	-1.820
4	3.7	16	1.308	.80	0.223	-1.501
5	4.7	15	1.548	.75	0.287	-1.248
6	4.9	14	1.589	.70	0.357	-1.030
7	5.7	13	1.740	.65	0.430	-0.844
8	6	12	1.792	.60	0.511	-0.671
9	6.8	11	1.917	.55	0.598	-0.514
10	7.5	10	2.015	.50	0.693	-0.367
11	7.6	9	2.028	.45	0.798	-0.226
12	8.6	8	2.152	.40	0.916	-0.088
13	9	7	2.197	.35	1.050	0.049
14	10.2	6	2.322	.30	1.204	0.186
15	10.9	5	2.389	.25	1.386	0.326
16	11	4	2.398	.20	1.609	0.476
17	11.8	3	2.468	.15	1.897	0.640
18	14	2	2.639	.10	2.303	0.834
19	17.5	1	2.862	.05	2.996	1.097

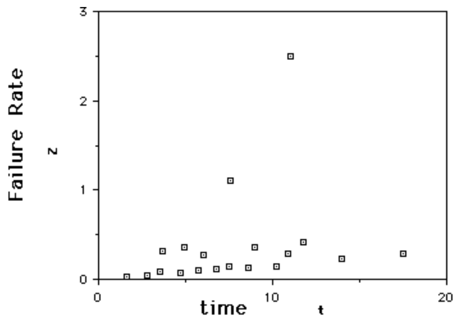
Cricket Graph

File Edit Data Graph Curve Fit Goodies Formats

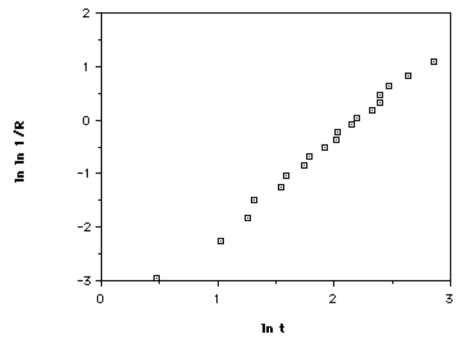
Lifetest Data

	6	7	8	9	10	11
	ln ln 1/R	DNf	lag t	Dt	DNf/Dt	Z
1	-2.957	1	0	1.600	0.625	0.033
2	-2.254	1	1.600	1.200	0.833	0.046
3	-1.820	1	2.800	0.700	1.429	0.084
4	-1.501	1	3.500	0.200	5.000	0.312
5	-1.248	1	3.700	1.000	1.000	0.067
6	-1.030	1	4.700	0.200	5.000	0.357
7	-0.844	1	4.900	0.800	1.250	0.096
8	-0.671	1	5.700	0.300	3.333	0.278
9	-0.514	1	6.000	0.800	1.250	0.114
10	-0.367	1	6.800	0.700	1.429	0.143
11	-0.226	1	7.500	0.100	10.000	1.111
12	-0.088	1	7.600	1.000	1.000	0.125
13	0.049	1	8.600	0.400	2.500	0.357
14	0.186	1	9.000	1.200	0.833	0.139
15	0.326	1	10.200	0.700	1.429	0.286
16	0.476	1	10.900	0.100	10.000	2.500
17	0.640	1	11.000	0.800	1.250	0.417
18	0.834	1	11.800	2.200	0.455	0.228
19	1.097	1	14.000	3.500	0.286	0.286
20			17.500			

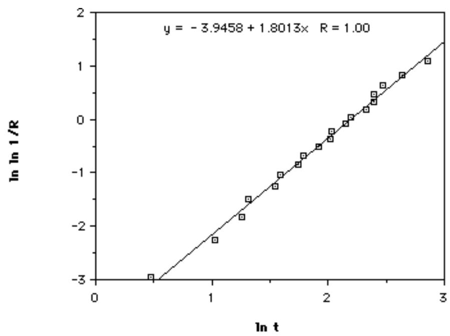
Cricket Graph



Cricket Graph



Cricket Graph



Cricket Graph

Fitting a straight line to the data

"least-squares" regression

Weibull Distribution Parameters

fitted line: $y = -3.9458 + 1.8013x$

slope $= \boxed{k = 1.8013}$ *"shape parameter"*

y-intercept $= -k \ln u = -3.9458$

$\Longrightarrow \ln u = \dfrac{3.9458}{1.8013} = 2.19053$

$\Longrightarrow u = \exp\{2.19053\}$

$\boxed{u = 8.9399}$ *"scale parameter"*

Results of the Analysis

(Note: Reliability is computed as (NS+0.5)÷N,
rather than NS÷N)

A least-squares linear regression was performed on the transformed variables

X = ln T, Y = ln ln 1÷R

to obtain:

Shape parameter K = 1.9834707,
Scale parameter U = 9.3412548

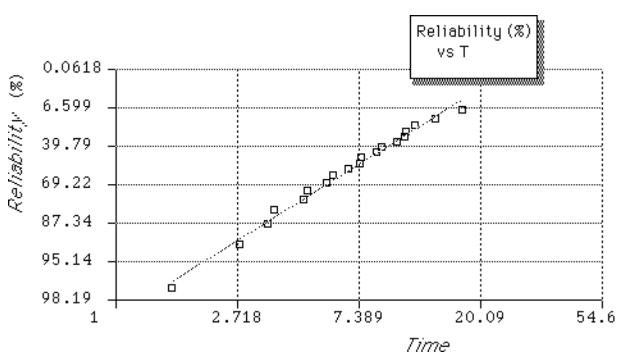
Mean failure time = 8.2797962
Std. deviation of failure time = 4.3604985

In the table below, Z = Hazard Rate
Y' and R' are the fitted values for Y and R

T	f	R	X	Y	Y'	Z	R'
1.6	1	0.975	0.47	-3.6762	-3.4997	0.037446	0.97025
2.8	2	0.925	1.0296	-2.5515	-2.3897	0.064927	0.91242
3.5	3	0.875	1.2528	-2.0134	-1.9471	0.080859	0.86703
3.7	4	0.825	1.3083	-1.6483	-1.8369	0.085401	0.85273
4.7	5	0.775	1.5476	-1.3669	-1.3624	0.10805	0.77411
4.9	6	0.725	1.5892	-1.1345	-1.2797	0.11258	0.75722
5.7	7	0.675	1.7405	-0.93384	-0.97978	0.13063	0.68702
6	8	0.625	1.7918	-0.75501	-0.87805	0.13739	0.65995
6.8	9	0.575	1.9169	-0.5917	-0.62979	0.15538	0.58701
7.5	10	0.525	2.0149	-0.4395	-0.43545	0.1711	0.52363
7.6	11	0.475	2.0281	-0.29512	-0.40917	0.17334	0.51469
8.6	12	0.425	2.1518	-0.15588	-0.16399	0.19575	0.42795
9	13	0.375	2.1972	-0.019357	-0.073817	0.2047	0.39501
10.2	14	0.325	2.3224	0.11683	0.17444	0.23152	0.30404
10.9	15	0.275	2.3888	0.2554	0.30609	0.24713	0.25715
11	16	0.225	2.3979	0.39989	0.32421	0.24936	0.25084
11.8	17	0.175	2.4681	0.55559	0.46346	0.26719	0.20402
14	18	0.125	2.6391	0.7321	0.80255	0.31611	0.1074
17.5	19	0.075	2.8622	0.95176	1.2451	0.39368	0.03101

X = ln T, Y = ln ln 1÷R

fitted values



Example: Germanium Power Transistor

N = sample size = 75

time(hrs)	total # failures
250	17
500	25
750	26
1000	27
2000	27
3000	32
4000	35
5000	39
6000	42
7000	44

A lot of transistors is tested, and the number of failures are counted at certain times.

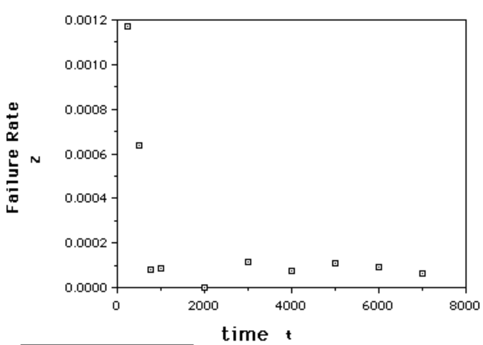
Lifetest Data

	1	2	3	4	5	6
	t	Nf	Ns	R	ln t	ln ln 1/R
1	250	17	58.000	0.773	5.521	-1.355
2	500	25	50.000	0.667	6.215	-0.904
3	750	26	49.000	0.653	6.620	-0.853
4	1000	27	48.000	0.640	6.908	-0.807
5	2000	27	48.000	0.640	7.601	-0.807
6	3000	32	43.000	0.573	8.006	-0.585
7	4000	35	40.000	0.533	8.294	-0.464
8	5000	39	36.000	0.480	8.517	-0.309
9	6000	42	33.000	0.440	8.700	-0.197
10	7000	44	31.000	0.413	8.854	-0.123

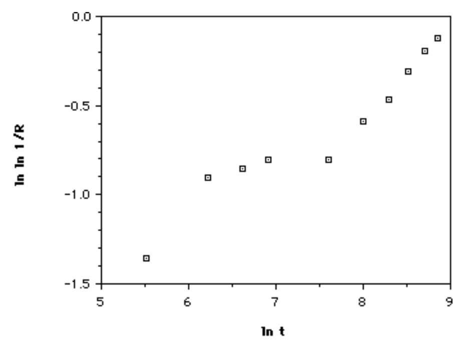
reliability.2

	7	8	9	10	11	12
	Lag Nf	DNf	Lag t	Dt	DNf/Dt	Z
1	0	17.000	0	250.000	0.068	1.172e-3
2	17.000	8.000	250.000	250.000	0.032	6.400e-4
3	25.000	1.000	500.000	250.000	4.000e-3	8.163e-5
4	26.000	1.000	750.000	250.000	4.000e-3	8.333e-5
5	27.000	0.000	1000.000	1000.000	0.000	0.000
6	27.000	5.000	2000.000	1000.000	5.000e-3	1.163e-4
7	32.000	3.000	3000.000	1000.000	3.000e-3	7.500e-5
8	35.000	4.000	4000.000	1000.000	4.000e-3	1.111e-4
9	39.000	3.000	5000.000	1000.000	3.000e-3	9.091e-5
10	42.000	2.000	6000.000	1000.000	2.000e-3	6.452e-5
11	44.000		7000.000			

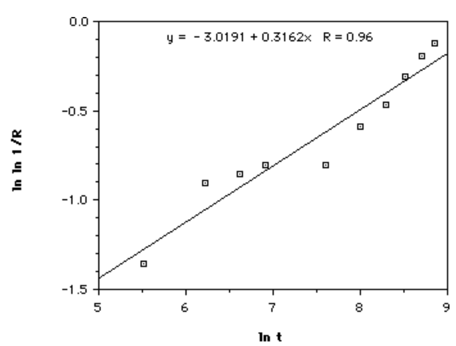
Cricket Graph



Cricket Graph



Cricket Graph



Cricket Graph

Cricket Graph

Weibull Distribution Parameters

fitted line: $y = -3.0191 + 0.3162x$

slope $= \boxed{k = 0.3162}$ *"shape parameter"*

y-intercept $= -k \ln u = -3.0191$

$\Longrightarrow \ln u = \dfrac{3.0191}{0.3162} = 9.5481$

$\Longrightarrow u = \exp\{9.5481\}$

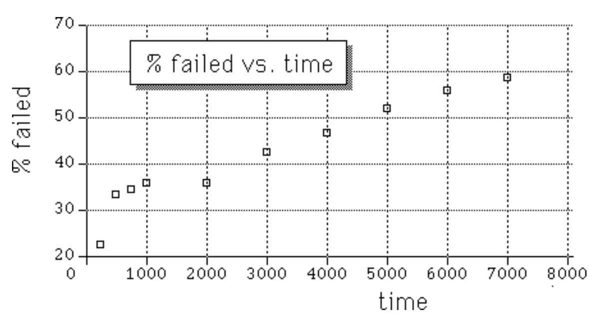
$\boxed{u = 14017.6}$ *"scale parameter"*

i	t	NS	NF	FS	FF	ΔNF/Δ	FR
1	250	58	17	0.77333	0.22667	0.068	0.0011724
2	500	50	25	0.66667	0.33333	0.032	0.00064
3	750	49	26	0.65333	0.34667	0.004	0.000081633
4	1000	48	27	0.64	0.36	0.004	0.000083333
5	2000	48	27	0.64	0.36	0	0
6	3000	43	32	0.57333	0.42667	0.005	0.00011628
7	4000	40	35	0.53333	0.46667	0.003	0.000075
8	5000	36	39	0.48	0.52	0.004	0.00011111
9	6000	33	42	0.44	0.56	0.003	0.000090909
10	7000	31	44	0.41333	0.58667	0.002	0.000064516

NS = # survivors FS = fraction survived (~ reliability)

NF = # failures FF = fraction failed

ΔNF/Δt = failure rate of population
FR=(ΔNF/Δt)/NS = failure rate of individuals



Results of the Analysis

(Note: Reliability is computed as (NS+0.5)÷N, rather than NS÷N)

A least-squares linear regression was performed on the transformed variables

X = ln T, Y = ln ln 1÷R

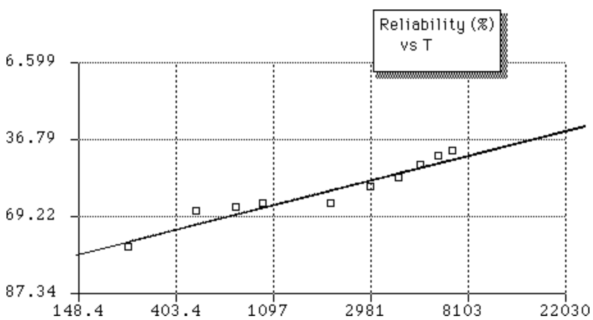
to obtain:

Shape parameter K = 0.32022,
Scale parameter U = 14707

Mean failure time = 103180
Std. deviation of failure time = 487020

In the table below,
X = ln T,
Y = ln ln 1÷R
Z = Hazard Rate (×10E¯3)/unit time
Y' and R' are the fitted values for Y and R

T	f	R	X	Y	Y'	Z	R'
250	17	0.78	5.5215	¯1.3925	¯1.3048	0.34742	0.76244
500	25	0.67333	6.2146	¯0.92757	¯1.0828	0.21688	0.71274
750	26	0.66	6.6201	¯0.87824	¯0.95298	0.16463	0.68005
1000	27	0.64667	6.9078	¯0.83029	¯0.86086	0.13539	0.65521
2000	27	0.64667	7.6009	¯0.83029	¯0.6389	0.084519	0.58986
3000	32	0.58	8.0064	¯0.60747	¯0.50906	0.064158	0.54823
4000	35	0.54	8.294	¯0.48421	¯0.41694	0.052762	0.51734
5000	39	0.48667	8.5172	¯0.32826	¯0.34548	0.045336	0.49269
6000	42	0.44667	8.6995	¯0.21574	¯0.2871	0.040051	0.47216
7000	44	0.42	8.8537	¯0.14214	¯0.23774	0.036067	0.45457

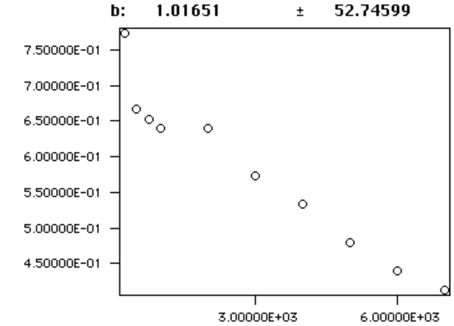


X Coord.	Y Coord.	Plot	Fit
2.50000E+02	7.73000E-01	☒	☒
5.00000E+02	6.67000E-01	☒	☒
7.50000E+02	6.53000E-01	☒	☒
1.00000E+03	6.40000E-01	☒	☒
2.00000E+03	6.40000E-01	☒	☒
3.00000E+03	5.73000E-01	☒	☒
4.00000E+03	5.33000E-01	☒	☒
5.00000E+03	4.80000E-01	☒	☒
6.00000E+03	4.40000E-01	☒	☒
7.00000E+03	4.13000E-01	☒	☒

Estimated Errors in Coefficients:

a:	**112.12051**	±	**27678.54271**
b:	**1.01651**	±	**52.74599**

f(x) = 2.1718281828^((x/a)^b)



X Coord.	Y Coord.	Plot	Fit
2.50000E+02	7.73000E-01	☒	☐
5.00000E+02	6.67000E-01	☒	☒
7.50000E+02	6.53000E-01	☒	☒
1.00000E+03	6.40000E-01	☒	☒
2.00000E+03	6.40000E-01	☒	☒
3.00000E+03	5.73000E-01	☒	☒
4.00000E+03	5.33000E-01	☒	☒
5.00000E+03	4.80000E-01	☒	☒
6.00000E+03	4.40000E-01	☒	☒
7.00000E+03	4.13000E-01	☒	☒

Estimated Errors in Coefficients:

a:	**123.21391**	±	**26268.33996**
b:	**1.02913**	±	**47.52466**